Algorithmic aspects of Galois theory in recent times

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Calculating Galois groups ... in theory
Calculating Galois groups . . . in theory

Calculating Galois groups . . . in practice
Calculating Galois groups . . . in theory
Calculating Galois groups . . . in practice
Calculating Differential Galois groups
Calculating Galois groups in theory
Calculating Galois groups in theory

Can we calculate Galois groups?
Calculating Galois groups in theory

Can we calculate Galois groups?

Yes we can, but ....
An Algorithm
(Kronecker, *Gründzuge* . . ., Crelle, 92, 1882)
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1) Let \( R(Y, X_1, \ldots, X_n) = \prod_{\sigma \in S_n} (Y - (\alpha_{\sigma(1)}X_1 + \ldots + \alpha_{\sigma(n)}X_n)) \)
\[ = \prod_{\sigma \in S_n} (Y - (\alpha_1 X_{\sigma^{-1}(1)} + \ldots + \alpha_n X_{\sigma^{-1}(n)})) \]
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2) Factor \( R(Y, X_1, \ldots, X_n) = R_1(Y, X_1, \ldots, X_n) \cdots R_t(Y, X_1, \ldots, X_n) \)
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High Complexity!
Polynomial Time Algorithms

Question: Is there an algorithm to compute the Galois group of $f(x) \in \mathbb{Z}[x]$ whose running time is given as a polynomial in the size of $f$?
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For \( m \in \mathbb{Z} \), \( \text{size}(m) = \) number of digits \( \approx \log(m) \)

For \( f(x) = a_n x^n + \ldots + a_0 \in \mathbb{Z}[x] \), \( \text{size}(f(x)) = n \cdot \max_i \{ \text{size}(a_i) \} \).
Question: Is there an algorithm to compute the Galois group of \( f(x) \in \mathbb{Z}[x] \) whose **running time** is given as a polynomial in the **size** of \( f \)?

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**Running Time** = number of +, ×, −, ÷
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Revised Question: Is there an algorithm to compute generators of the Galois group of $f(x) \in \mathbb{Z}[x]$ whose running time is given as a polynomial in the size of $f$?
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Revised Question: Is there an algorithm to compute generators of the Galois group of \( f(x) \in \mathbb{Z}[x] \) whose running time is given as a polynomial in the size of \( f \)?

We do not know!
Galois (*Discours préliminaire*):

“Si maintenant vous me donnez une équation que vous aurez choisie à votre gré et que vous desiriez connaître si elle est ou non soluble par radicaux, je n’aurai rien à y faire que de vous indiquer le moyen de répondre à votre question, sans vouloir charger ni moi ni personne de le faire. En un mot les calculs sont impraticables.”

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**Question:** Is there an algorithm to decide if \( f(x) \in \mathbb{Z}[x] \) is solvable by radicals whose running time is given as a polynomial in the size of \( f \)?

**Landau and Miller (**Solvability by radicals in polynomial time**, J. Comp. Sys. Sci., 1985):**

“If now you give us a polynomial which you have chosen at your pleasure, and if you want to know if it is or is not solvable by radicals, we have presented techniques to answer that question in polynomial time.”
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Ingredients of the Landau-Miller Algorithm

• Lenstra-Lenstra-Lovász (1982): Polynomial time algorithm to factor $f(x) \in \mathbb{Q}[x]$.

⇒ Landau (1985): Construct splitting field and Galois group $G$ in time polynomial in $|G|$ and size $(f)$.

⋄ Adjoin root of $f \Rightarrow \mathbb{Q}(\alpha_1)$, factor $f$ over $\mathbb{Q}(\alpha_1)$.

⇒ $f = f_1 f_2 \cdots f_t$.

⋄ Adjoin root of $f_1 \Rightarrow \mathbb{Q}(\alpha_1, \alpha_2)$, factor $f$ over $\mathbb{Q}(\alpha_1, \alpha_2)$.

⋄ Stop when $f$ factors completely over $K = \mathbb{Q}(\alpha_1, \ldots, \alpha_\ell)$.

⋄ $K = \mathbb{Q}(\beta)$, $\beta = r_1 \alpha_1 + \ldots + r_n \alpha_n$, $g(x) =$ min. poly $\beta$ over $\mathbb{Q}$.

⋄ Galois group $= \{ \sigma \in S_n | g(r_1 \alpha_\sigma(1) + \ldots + r_n \alpha_\sigma(n)) = 0 \}$.
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\[ \diamond \text{ Adjoin root of } f \Rightarrow \mathbb{Q}(\alpha_1), \text{ factor } f \text{ over } \mathbb{Q}(\alpha_1) \Rightarrow f = f_1 f_2 \cdots f_t \]
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\[ \diamond K = \mathbb{Q}(\beta), \beta = r_1 \alpha_1 + \ldots + r_n \alpha_n, \ g(x) = \text{ min. poly } \beta \text{ over } \mathbb{Q} \]
\[ \diamond \text{ Galois group } = \{ \sigma \in S_n \mid g(r_1 \alpha_{\sigma(1)} + \ldots + r_n \alpha_{\sigma(n)}) = 0 \} \]
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  - Landau (1985): Construct splitting field and Galois group $G$ in time polynomial in $|G|$ and size($f$)

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Calculating Galois groups in practice

Things that work
Mod p techniques

\[ f(x) \in \mathbb{Z}[x], \text{monic, deg } f = n \quad \Delta(f) = \prod (\alpha_i - \alpha_j)^2 \in \mathbb{Z} \]
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\[ = (R_{1,1} \cdots R_{1,m_1}) \cdots (R_{t,1} \cdots R_{t,m_t}) \mod p \]
Mod \( p \) techniques

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Theorem of Frobenius

Let \( n = n_1 + \ldots + n_t, \quad n_1 \geq n_2 \ldots \geq n_t \)

Density of \( \{ p \mid p \nmid \Delta(f), \quad f = f_1 \cdots f_t \, (\text{mod } p), \quad \deg(f_i) = n_i \} \)

\[ \frac{1}{|G|} \cdot |\{ \sigma \in G \mid \sigma = \tau_1 \cdots \tau_t, \quad \tau_i \text{ a cycle of length } n_i \}| \]
Mod $p$ techniques

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Advantages:

- Easy to factor mod $p$ (Berlekamp, 1967)
- Gives a good probabilistic test for $S_n, A_n$; good evidence for other groups.
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**Advantages:**
- Easy to factor mod p (Berlekamp, 1967)
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**Disadvantages:**
- Asymptotic result
- Groups not determined by distribution of cycle patterns - already in deg. 8
Invariant Theoretic Techniques

Example:

\[ f(x) = x^3 + bx + c \in \mathbb{Q}[x] \]

\[ \text{Gal}(f) \subset S_3 \]

\[ f(x) = (x + \alpha)(x + \beta x + \gamma), \alpha, \beta, \gamma \in \mathbb{Q} \]

\[ \Rightarrow \text{Gal}(f) = S_2 \text{ or } \{\text{id}\} \]

\[ f(x) \text{ irreducible} \Rightarrow \text{Gal}(f) \text{ acts transitively on the roots } \alpha_1, \alpha_2, \alpha_3 \]

Group Theory

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Let

\[ F(z) = z^2 + 4b^3 + 27c^2 = (z + \delta)(z - \delta) \]

\[ \delta = (\alpha_1 - \alpha_2)(\alpha_2 - \alpha_3)(\alpha_3 - \alpha_1) \]

\[ \delta \text{ is an invariant of } A_3 \text{ but not of } S_3 \text{ so } \]

\[ \text{Gal}(f) = A_3 \iff F(z) \text{ factors over } \mathbb{Q}. \]

Reduce calculation of Galois groups to factorization of associated polynomials.
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Reduce calculation of Galois groups to factorization of associated polynomials
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Why does this work?

I. A finite group is determined by its permutation representations.

Given $H \subseteq G$, $\exists \rho : G \rightarrow S_N$ such that $G$ acts transitively but $H$ does not.

II. One can find permutation representations of $G = \text{Gal}(K/k)$ in $K$.

Let $\rho : G \rightarrow S_N$, then $\exists \beta_1, \ldots, \beta_N \in K$ such that

$$\sigma(\alpha_i) = \alpha_{\rho(\sigma)(i)}, \text{ for all } \sigma \in G$$
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Given \( \text{Gal}(K/k) \subset G \), to show \( \text{Gal}(K/k) = G \):

- For each maximal subgroup \( H \subset G \), find a representation as in I.
- Find \( \beta_1, \ldots, \beta_N \in K \) as in II.
- Form \( F_H(z) = \prod (z - \beta_i) \in k[z] \).
- If \( F_H(z) \) is irreducible for each \( H \), then \( \text{Gal}(K/k) = G \).
Differential Galois Groups
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What are Differential Galois Groups and what do they measure?
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Calculating Differential Galois Groups ... in practice
Picard-Vessiot Theory

Consider a linear differential equation

\[ L(y) = d^n y dz^n + a_{n-1}(z) d^{n-1} y dz^{n-1} + \ldots + a_0 y = 0, \]

where \( a_i(z) \in \mathbb{C}(z) \) for \( i = 0, 1, \ldots, n \). At a nonsingular point, \( \exists \) solutions \( y_1, \ldots, y_n \) analytic near \( z_0 \), linearly independent over \( \mathbb{C} \).

PV-extension \( K = \mathbb{C}(z)(y_1, \ldots, y_n, y'_1, \ldots, y'_n, \ldots, y^{(n-1)}, \ldots) \).

PV-group \( DGal(K/k) = \{ \sigma: K \to K | \sigma \) is a \( \mathbb{C}(z) \)-diff. autom. of \( K \} \).

\( DGal(K/k) \) leaves \( Soln(L) \) invariant \( \Rightarrow DGal(K/k) \subset GL_n(\mathbb{C}) \).

\( DGal(K/k) \) is Zariski-closed.

Galois Correspondence: \( H \subset DGal(K/k) \) Zariski closed \( \Leftrightarrow F \subset C(z) \subset F \subset K \).
Consider a linear differential equation

$$L(y) = \frac{d^n y}{dz^n} + a_{n-1}(z) \frac{d^{n-1} y}{dz^{n-1}} + \ldots + a_0 y = 0, \quad a_i(z) \in \mathbb{C}(z)$$
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PV-extension \(K = \mathbb{C}(z)(y_1, \ldots, y_n, y'_1, \ldots, y'_n, \ldots, y_{(n-1)}^{(n-1)}, \ldots, y_n^{(n-1)})\).
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Galois Correspondence: \( H^\text{Zariski closed} \subset \text{DGal}(K/\mathbb{C}(z)) \iff F^\text{Diff. field, } \mathbb{C}(z) \subset F \subset K \)
What do Differential Galois Groups Measure?
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- Algebraic Dependence: $K$ - a PV-extension of $\mathbb{C}(z)$ with PV-group $G$

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- **Algebraic Dependence:** $K$ - a PV-extension of $\mathbb{C}(z)$ with PV-group $G$

  $$\text{tr. deg.}_{\mathbb{C}(z)} K = \dim_{\mathbb{C}} G$$

Example:

$$L(y) = y'' + \frac{1}{z} + \left(1 - \frac{\lambda^2}{z^2}\right)y = 0, \quad \lambda - \frac{1}{2} \notin \mathbb{Z}$$

$$\Rightarrow \text{DGal} = \text{SL}_n(\mathbb{C})$$

$$\Rightarrow \text{tr. deg.}_{\mathbb{C}(z)} \mathbb{C}(z)(J_\lambda, Y_\lambda, J'_\lambda, Y'_\lambda) = 3$$
Solvability

L(y) = 0 is solvable in terms of Liouvillian functions if there exists a tower of fields $C(z) = K_0 \subset \ldots \subset K_n$ such that $K_{i+1} = K_i(t_i)$ with $t_i$ algebraic over $K_i$, or $t'_i \in K_i$, i.e., $t_i = Ru_i$, $u_i \in K_i$, or $t'_i/t_i \in K_i$, i.e., $t_i = e^{Ru_i}$, $u_i \in K_i$.

with $K \subset K_n$, where $K$ is the PV-extension associated with $L(y) = 0$.

Example:

$L(y) = \frac{d^2y}{dx^2} - \frac{1}{2}x \frac{dy}{dx} - xy$,

$K_0 = C(x) \subset K_1 = K_0(\sqrt{x}) \subset K_2 = K_1(e^{R\sqrt{x}})$

$\{e^{R\sqrt{x}}, e^{-R\sqrt{x}}\}$ is a basis for $\text{Soln}(L(y) = 0)$

Thm:

$L(y) = 0$ solvable in terms of Liouvillian functions $\iff \text{DGal}$ contains a solvable subgroup of finite index.
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Calculating Differential Galois Groups . . . in theory

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- One can compute the Galois group. (Hrushovksy)
Calculating Differential Galois Groups . . . in practice
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Ideas based on Tannakian philosophy:
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    (iii) \quad & \text{Sym}^m(V) = \{y_1 \cdot \cdots y_m \mid y_i \in V\} = \text{Soln}(L^{\otimes m}(y))
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$\text{DGal} = \text{Valentiner Group } A^\text{SL}_6 \text{ of order 1080}$
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$\text{DGal} = \text{Valentiner Group } A_6^{SL_3}$ of order 1080

$\iff \quad L^{\otimes 2}$ and $L^{\otimes 3}$ are irreducible

$L^{\otimes 4} = L_9 \circ L_6$, $L_9, L_6$ irreducible.