Galois theory of parameterized differential equations and linear differential algebraic groups

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Abstract. We present a Galois theory of parameterized linear differential equations where the Galois groups are linear differential algebraic groups, that is, groups of matrices whose entries are functions of the parameters and satisfy a set of differential equations with respect to these parameters. We present the basic constructions and results, give examples, discuss how isomonodromic families fit into this theory and show how results from the theory of linear differential algebraic groups may be used to classify systems of second order linear differential equations.

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1 Introduction

We will describe a Galois theory of differential equations of the form

$$\frac{\partial Y}{\partial x} = A(x, t_1, \ldots, t_n)Y$$

where $A(x, t_1, \ldots, t_n)$ is an $m \times m$ matrix with entries that are functions of the principal variable x and parameters $t_1, \ldots, t_n$. The Galois groups in this theory are linear differential algebraic groups, that is, groups of $m \times m$ matrices $(f_{i,j}(t_1, \ldots, t_n))$ whose entries satisfy a fixed set of differential equations. For example, in this theory, the equation

$$\frac{\partial y}{\partial x} = \frac{t}{x} y$$

has Galois group

$$G = \left\{(f(t)) \mid f \neq 0 \text{ and } f\frac{d^2 f}{dt^2} - \left(\frac{df}{dt}\right)^2 = 0\right\}.$$

In the process, we will give an introduction to the theory of linear differential algebraic groups and show how one can use properties of the structure of these groups to deduce results concerning parameterized linear differential equations.

Various differential Galois theories now exist that go beyond the eponymous theory of linear differential equations pioneered by Picard and Vessiot at the end of the 19th century and made rigorous and expanded by Kolchin in the middle of the 20th century. These include theories developed by B. Malgrange, A. Pillay, H. Umemura and one presently being developed by P. Landesman. In many ways the Galois theory presented here is a special case of the results of Pillay and Landesman yet we hope that the explicit nature of our presentation and the applications we give justify our exposition. We will give a comparison with these theories in the final comments.

The rest of the paper is organized as follows. In Section 2 we review the Picard–Vessiot theory of integrable systems of linear partial differential equations. In Section 3 we introduce and give the basic definitions and results for the Galois theory of parameterized linear differential equations ending with a statement of the Fundamental Theorem of this Galois theory as well as a characterization of parameterized equations that are solvable in terms of parameterized liouvillian functions. In Section 4 we describe the basic results concerning linear differential algebraic groups and give many examples. In Section 5 we show that, in the regular singular case, isomonodromic families of linear differential equations are precisely the parameterized linear differential equations whose parameterized Galois theory reduces to the usual Picard–Vessiot theory. In Section 6 we apply a classification of 2×2 linear differential algebraic groups to show that any parameterized system of linear differential equations with regular singular points is equivalent to a system that is generic (in a suitable sense) or isomonodromic or solvable in terms of parameterized liouvillian functions. Section 7 gives two simple examples illustrating the subtleties of the inverse problem in our setting. In Section 8 we discuss the relationship between the theory presented here and other differential Galois theories and give some directions for future research. The Appendices contain proofs of the results of Section 3.

2 Review of Picard–Vessiot theory

In the usual Galois theory of polynomial equations, the Galois group is the collection of transformations of the roots that preserve all algebraic relations among these roots. To be more formal, given a field k and a polynomial $p(y)$ with coefficients in k, one forms the *splitting field* K of $p(y)$ by adjoining all the roots of $p(y)$ to k. The *Galois group* is then the group of all automorphisms of K that leave each element of k fixed. The structure of the Galois group is well known to reflect the algebraic properties of the roots of $p(y)$. In this section we will review the Galois theory of linear differential equations. Proofs (and other references) can be found in [40].

One can proceed in an analogous fashion with integrable systems of linear differential equations and define a Galois group that is a collection of transformations of solutions of a linear differential system that preserve all the algebraic relations among the solutions *and their derivatives*. Let k be a differential field[1], that is, a field k together with a set of commuting derivations $\Delta = \{\partial_1, \dots, \partial_m\}$. To emphasize the role of Δ, we shall refer to such a field as a Δ*-field*. Examples of such fields are the field $\mathbb{C}(x_1, \dots, x_m)$ of rational functions in m variables, the quotient field $\mathbb{C}((x_1, \dots, x_m))$ of the ring of formal power series in m variables and the quotient field $\mathbb{C}(\{x_1, \dots, x_m\})$ of the ring of convergent power series, all with the derivations $\Delta = \left\{\frac{\partial}{\partial x_1}, \dots, \frac{\partial}{\partial x_m}\right\}$. If k is a Δ-field and $\Delta' \subset \Delta$, the field $C_k^{\Delta'} = \{c \in k \mid \partial c = 0 \text{ for all } \partial \in \Delta'\}$ is called the subfield of Δ'-constants of k. When $\Delta' = \Delta$ we shall write C_k for C_k^{Δ} and refer to this latter field as the field of constants of k. An *integrable system of linear differential equations* is a set of equations

$$\begin{aligned}
\partial_1 Y &= A_1 Y \\
\partial_2 Y &= A_2 Y \\
&\;\vdots \\
\partial_m Y &= A_m Y
\end{aligned} \tag{2.1}$$

where the $A_i \in \mathrm{gl}_n(k)$, the set of $n \times n$ matrices with entries in k, such that

$$\partial_i A_j - \partial_j A_i = [A_i, A_j] \tag{2.2}$$

for all i, j. These latter equations are referred to as the *integrability conditions*. Note that if $m = 1$, these conditions are trivially satisfied.

[1] All fields in this paper will be of characteristic zero

The role of a splitting field is assumed by the *Picard–Vessiot extension* associated with the integrable system (2.1). This is a Δ-extension field $K = k(z_{1,1}, \dots, z_{n,n})$ where

(1) the $z_{i,j}$ are entries of a matrix $Z \in \mathrm{GL}_n(K)$ satisfying $\partial_i Z = A_i Z$ for $i = 1, \dots, m$, and

(2) $C_K = C_k = C$, *i.e.,* the Δ-constants of K coincide with the Δ-constants of k.

Note that condition (1) defines uniquely the actions on K of the derivations ∂_i and that the integrability conditions (2.2) must be satisfied since these derivations commute. We refer to the Z above as a *fundamental solution matrix* and we shall denote K by $k(Z)$. If $k = \mathbb{C}(x_1, \dots, x_m)$ with the obvious derivations, one can easily show the existence of Picard–Vessiot extensions. If we let $\vec{a} = (a_1, \dots, a_n)$ be a point of $\mathbb{C}^n$ where the denominators of all entries of the A_i are holomorphic, then the Frobenius Theorem ([55],Ch. 1.3) implies that, in a neighborhood of $\vec{a}$, there exist n linearly independent analytic solutions $(z_{1,1}, \dots, z_{n,1})^T, \dots, (z_{1,n}, \dots, z_{n,n})^T$ of the equations (2.1). The field $k(z_{1,1}, \dots, z_{n,n})$ with the obvious derivations satisfies the conditions defining a Picard–Vessiot extension. In general, if k is an arbitrary Δ-field with C_k algebraically closed, then there always exists a Picard–Vessiot extension K for the integrable system (2.1) and K is unique up to k-differential isomorphism. We shall refer to K as the *PV-extension associated with* (2.1).

Let K be a PV-extension associated with (2.1) and let $K = k(Z)$ with Z a fundamental solution matrix. If U is another fundamental solution matrix then an easy calculation shows that $\partial_i(U^{-1}Z) = 0$ for all i and so $U^{-1}Z \in \mathrm{GL}_n(C_k)$. We define the Δ-*Galois group* $\mathrm{Gal}_\Delta(K/k)$ of K over k (or of the system (2.1)) to be

$$\mathrm{Gal}_\Delta(K/k) = \{\sigma : K \to K \mid \sigma \text{ is a } k\text{-automorphism of } K \text{ and } \\ \partial_i \sigma = \sigma \partial_i, \text{ for } i = 1, \dots, m\}.$$

Note that a k-automorphism σ of K such that $\partial_i \sigma = \sigma \partial_i$ is called a *k-differential automorphism*. For any $\sigma \in \mathrm{Gal}_\Delta(K/k)$, we have that $\sigma(Z)$ is again a fundamental solution matrix so the above discussion implies that $\sigma(Z) = ZA_\sigma$ for some $A_\sigma \in \mathrm{GL}_n(C_k)$. This yields a representation $\mathrm{Gal}_\Delta(K/k) \to \mathrm{GL}_n(C_k)$. Note that different fundamental solution matrices yield conjugate representations. A fundamental fact is that the image of $\mathrm{Gal}_\Delta(K/k)$ in $\mathrm{GL}_n(C_k)$ is Zariski-closed, that is, it is defined by a set of polynomial equations involving the entries of the matrices and so has the structure of an linear algebraic group. If G is a linear algebraic group defined over F (that is, defined by equations having coefficients in the field F) and E is any field containing F, we will use the notation $G(E)$ to denote the set of points of G having entries in E.

These facts lead to a rich Galois theory, originally due to E. Picard and E. Vessiot and given rigor and greatly expanded by E. R. Kolchin. We summarize the fundamental result in the following

Theorem 2.1. *Let k be a Δ-field with algebraically closed field of constants C and* (2.1) *be an integrable system of linear differential equations over k.*

(1) *There exists a PV-extension K of k associated with* (2.1) *and this extension is unique up to a differential k-isomorphism.*

(2) *The Δ-Galois group* $\mathrm{Gal}_\Delta(K/k)$ *may be identified with $G(C)$, where G is a linear algebraic group defined over C.*

(3) *The map that sends any differential subfield $F, k \subset F \subset K$, to the group* $\mathrm{Gal}_\Delta(K/F)$ *is a bijection between the set of differential subfields of K containing k and the set of algebraic subgroups of* $\mathrm{Gal}_\Delta(K/k)$. *Its inverse is given by the map that sends a Zariski closed group H to the field $K^H = \{z \in K \mid \sigma(z) = z$ for all $\sigma \in H\}$.*

(4) *A Zariski-closed subgroup H of* $\mathrm{Gal}_\Delta(K/k)$ *is a normal subgroup of* $\mathrm{Gal}_\Delta(K/k)$ *if and only if the field K^H is left set-wise invariant by* $\mathrm{Gal}_\Delta(K/k)$. *If this is the case, the map* $\mathrm{Gal}_\Delta(K/k) \to \mathrm{Gal}_\Delta(K^H/k)$ *is surjective with kernel H and K^H is a PV-extension of k with PV-group isomorphic to* $\mathrm{Gal}_\Delta(K/k)/H$. *Conversely, if F is a differential subfield of K containing k and F is a PV-extension of k, then* $\mathrm{Gal}_\Delta(F/K)$ *is a normal subgroup of* $\mathrm{Gal}_\Delta(K/k)$.

Remarks 2.2. 1. The assumption that C is algebraically closed is necessary for the existence of PV-extensions (*cf.,* [42]) as well as necessary to guarantee that there are enough automorphisms so that (3) is correct. Kolchin's development in [21] of the Galois correspondence for PV-extensions does not make this assumption and he replaced automorphisms of the PV-extension with embeddings of the PV-extension into a large field (a *universal differential field*). Another approach to studying linear differential equations over fields whose constants are not algebraically closed is given in [1] (see in particular Corollaire 3.4.2.4 and Exemple 3.4.2.6). One can also study linear differential equations over fields whose fields of constants are not algebraically closed using descent techniques (see [17]).

2. Theorem 2.1 is usually stated and proven for the case when $m = 1$, the ordinary differential case, although it is proven in this generality in [21]. The usual proofs in the ordinary differential case do however usually generalize to this case as well. In the appendix of [40], the authors also discuss the case of $m > 1$ and show how the Galois theory may be developed in this case. We will give a proof of a more general theorem in the appendix from which Theorem 2.1 follows as well.

3. Theorem 2.1 is a manifestation of a deeper result. If $K = k(Z)$ is a PV-extension then the ring $k\big[Z, \frac{1}{\det Z}\big]$ is the coordinate ring of a torsor (principal homogeneous space) V defined over k for the group $\mathrm{Gal}_\Delta(K/k)$, that is, there is a morphism $V \times G \to V$ denoted by $(v, g) \mapsto vg$ and defined over k such that $v1 = v$ and $(vg_1)g_2 = v(g_1 g_2)$ and such that the morphism $V \times G \to V \times V$ given by $(v, g) \mapsto (v, vg)$ is an isomorphism. The path to the Galois theory given by first establishing this fact is presented in [15], [25] and [40] (although Kolchin was well aware of this fact as well, *cf.,* [21], Ch. VI.8 and the references there to the original papers.) This approach allows one to give an intrinsic definition of the linear algebraic group structure on the Galois group as well.

We end this section with a simple example that will also illuminate the Galois theory of parameterized equations.

Example 2.3. Let $k = \mathbb{C}(x)$ be the ordinary differential field with derivation $\frac{d}{dx}$ and consider the differential equation

$$\frac{dy}{dx} = \frac{t}{x}y$$

where $t \in \mathbb{C}$. The associated Picard–Vessiot extension is $k(x^t)$. The Galois group will be identified with a Zariski-closed subgroup of $\mathrm{GL}_1(\mathbb{C})$. When $t \in \mathbb{Q}$, one has that x^t is an algebraic function and when $t \notin \mathbb{Q}$, x^t is transcendental. It is therefore not surprising that one can show that

$$\mathrm{Gal}_\Delta(K/k) = \begin{cases} \mathbb{C}^* = \mathrm{GL}_1(\mathbb{C}) & \text{if } t \notin \mathbb{Q}, \\ \mathbb{Z}/q\mathbb{Z} & \text{if } t = \frac{p}{q}, (p, q) = 1. \end{cases} \qquad \square$$

3 Parameterized Picard–Vessiot theory

In this section we will consider differential equations of the form

$$\frac{\partial Y}{\partial x} = A(x, t_1, \dots, t_m)Y$$

where A is an $n \times n$ matrix whose entries are functions of x and parameters $t_1, \dots, t_m$ and we will define a Galois group of transformations that preserves the algebraic relations among a set of solutions and their derivatives *with respect to all the variables*. Before we make things precise, let us consider an example.

Example 3.1. Let $k = \mathbb{C}(x, t)$ be the differential field with derivations $\Delta = \left\{\partial_x = \frac{\partial}{\partial x}, \partial_t = \frac{\partial}{\partial t}\right\}$. Consider the differential equation

$$\partial_x y = \frac{t}{x}y.$$

In the usual Picard–Vessiot theory, one forms the differential field generated by the entries of a fundamental solution matrix and all their derivatives (in fact, because the matrix satisfies the differential equation, we get the derivatives for free). We will proceed in a similar fashion here. The function

$$y = x^t$$

is a solution of the above equation. Although all derivatives with respect to x lie in the field $k(x^t)$, this is not true for $\partial_t(x^t) = (\log x)x^t$. Nonetheless, this is all that is missing and the derivations Δ naturally extend to the field

$$K = k(x^t, \log x),$$

the field gotten by adjoining to k a fundamental solution and its derivatives (of all orders) with respect to all the variables.

Let us now calculate the group $\mathrm{Gal}_\Delta(K/k)$ of k-automorphisms of K commuting with both ∂_x and ∂_t. Let $\sigma \in \mathrm{Gal}_\Delta(K/k)$. We first note that $\partial_x(\sigma(x^t)(x^t)^{-1}) = 0$ so $\sigma(x^t) = a_\sigma x^t$ for some $a_\sigma \in K$ with $\partial_x a_\sigma = 0$, *i.e.,* $a_\sigma \in C_K^{\{\partial_x\}} = C_k^{\{\partial_x\}} = \mathbb{C}(t)$. Next, a calculation shows that $\partial_x(\sigma(\log x) - \log x) = 0 = \partial_t(\sigma(\log x) - \log x)$ so we have that $\sigma(\log x) = \log x + c_\sigma$ for some $c_\sigma \in \mathbb{C}$. Finally, a calculation shows that

$$0 = \partial_t(\sigma(x^t)) - \sigma(\partial_t(x^t)) = (\partial_t a_\sigma - a_\sigma c_\sigma)x^t$$

so we have that

$$\partial_t\left(\frac{\partial_t a_\sigma}{a_\sigma}\right) = 0. \tag{3.1}$$

Conversely, one can show that for any a such that $\partial_x a = 0$ and equation (3.1) holds, the map defined by $x^t \mapsto ax^t,\ \log x \mapsto \log x + \frac{\partial_t a}{a}$ defines a differential k-automorphism of K so we have

$$\mathrm{Gal}_\Delta(K/k) = \left\{a \in C_K^{(\frac{\partial}{\partial x})} = C_k^{(\frac{\partial}{\partial x})} \mid a \neq 0 \text{ and } \partial_t\left(\tfrac{\partial_t a}{a}\right) = 0\right\}.$$

□

This example illustrates two facts. The first is that the Galois group of a parameterized linear differential equation is a group of $n \times n$ matrices (here $n = 1$) whose entries are functions of the parameters (in this case, t) satisfying certain differential equations; such a group is called a linear differential algebraic group (see Definition 3.3 below). In general, the Galois group of a parameterized linear differential equation will be such a group.

The second fact is that in this example $\mathrm{Gal}_\Delta(K/k)$ does not contain enough elements to give a Galois correspondence. Expressing an element of $\mathbb{C}(t)$ as $a = a_0 \prod (t - b_i)^{n_i},\ a_0, b_i \in \mathbb{C}, n_i \in \mathbb{Z}$, one can show that if $a \in \mathrm{Gal}_\Delta(K/k)$ then $a \in \mathbb{C}$, that is $\mathrm{Gal}_\Delta(K/k) = \mathbb{C}^*$. If $\sigma \in \mathrm{Gal}_\Delta(K/k)$ and $\sigma(x^t) = ax^t$ with $a \in \mathbb{C}$, then

$$\sigma(\log x) = \sigma\left(\frac{\partial_t x^t}{x^t}\right) = \frac{\partial_t(ax^t)}{ax^t} = \log x.$$

Therefore $\log x$ is fixed by the Galois group and so there cannot be a Galois correspondence. The problem is that we do not have an element $a \in \mathbb{C}(t)$ such that $\partial_t\left(\frac{\partial_t a}{a}\right) = 0$ and $\partial_t a \neq 0$.

In the Picard–Vessiot theory, one avoids a similar problem by insisting that the constant subfield is large enough, *i.e.,* algebraically closed. This insures that any consistent set of polynomial equations with constant coefficients will have a solution in the field. In the parameterized Picard–Vessiot theory that we will develop, we will need to insure that any consistent system of differential equations (with respect to the parametric variables) has a solution. This motivates the following definition.

Let k be a Δ-field with derivations $\Delta = \{\partial_1, \dots, \partial_m\}$. The Δ-*ring* $k\{y_1, \dots, y_n\}_\Delta$ *of differential polynomials in n variables over k* is the usual polynomial ring in the infinite set of variables

$$\{\partial_1^{n_1}\partial_2^{n_2}\dots\partial_m^{n_m}y_j\}_{j=1,\dots,n}^{n_i\in\mathbb{N}}$$

with derivations extending those in Δ on k and defined by

$$\partial_i(\partial_1^{n_1}\dots\partial_i^{n_i}\dots\partial_m^{n_m}y_j)=\partial_1^{n_1}\dots\partial_i^{n_i+1}\dots\partial_m^{n_m}y_j.$$

Definition 3.2. We say that a Δ-field k is *differentially closed* if for any n and any set $\{P_1(y_1,\dots,y_n),\dots,P_r(y_1,\dots,y_n),Q(y_1,\dots,y_n)\}\subset k\{y_1,\dots,y_n\}_\Delta$, if the system

$$\{P_1(y_1,\dots,y_n)=0,\dots,P_r(y_1,\dots,y_n)=0,Q(y_1,\dots,y_n)\neq 0\}$$

has a solution in some Δ-field K containing k, then it has a solution in k

This notion was introduced by A. Robinson [41] and extensively developed by L. Blum [5] (in the ordinary differential case) and E. R. Kolchin [20] (who referred to these as constrainedly closed differential fields). More recent discussions can be found in [30] and [32]. A fundamental fact is that any Δ-field k is contained in a differentially closed differential field. In fact, for any such k there is a differentially closed Δ-field $\bar{k}$ containing k such that for any differentially closed Δ-field K containing k, there is a differential k-isomorphism of $\bar{k}$ into K. Differentially closed fields have many of the same properties with respect to differential fields as algebraically closed fields have with respect to fields but there are some striking differences. For example, the differential closure of a field has proper subfields that are again differentially closed. For more information, the reader is referred to the above papers.

Example 3.1 (bis). Let k be a $\Delta=\{\partial_x,\partial_t\}$-field and let $k_0=C_k^{\partial_x}$. Assume that k_0 is a differentially closed ∂_t-field and that $k=k_0(x)$ where $\partial_x x=1$ and $\partial_t x=0$. We again consider the differential equation

$$\partial_x y=\frac{t}{x}y$$

and let $K=k(x^t,\log x)$ where x^t, $\log x$ are considered formally as algebraically independent elements satisfying $\partial_t(x^t)=(\log x)x^t$, $\partial_x(x^t)=\frac{t}{x}x^t$, $\partial_t(\log x)=0$, $\partial_x(\log x)=\frac{1}{x}$. One can show that $C_K^{\{\partial_x\}}=k_0$ and that the Galois group is again

$$\mathrm{Gal}_\Delta(K/k)=\left\{a\in k_o^*\mid \partial_t\left(\tfrac{\partial_t(a)}{a}\right)=0\right\}.$$

Note that $\mathrm{Gal}_\Delta(K/k)$ contains an element a such that $\partial_t a\neq 0$ and $\partial_t\left(\frac{\partial_t(a)}{a}\right)=0$. To see this, note that the $\{\partial_t\}$-field $k_0(u)$, where u is transcendental over k_0 and $\partial_t u=u$ is a $\{\partial_t\}$-extension of k_0 containing such an element (*e.g.,* u). The definition of differentially closed ensures that k_0 also contains such an element. This implies that $\log x$ is not left fixed by $\mathrm{Gal}_\Delta(K/k)$. In fact, we will show in Section 4 that the following is a complete list of differential algebraic subgroups of $\mathrm{Gal}_\Delta(K/k)$ and the corresponding Δ-subfields of K:

Field	Group
$k((x^t)^n, \log x), n \in \mathbb{N}_{>0}$	$\{a \in k_0^* \mid a^n = 1\} = \mathbb{Z}/n\mathbb{Z}$
$k(\log x)$	$\{a \in k_0^* \mid \partial_t a = 0\}$
k	$\{a \in k_0^* \mid \partial_t(\partial_t(a)/a) = 0\}$

□

We now turn to stating the Fundamental Theorem in the Galois theory of parameterized linear differential equations. We need to give a formal definition of the kinds of groups that can occur and also of what takes the place of a Picard–Vessiot extension. This is done in the next two definitions.

Definition 3.3. Let k be a differentially closed Δ-differential field.

(1) A subset $X \subset k^n$ is said to be *Kolchin-closed* if there exists a set $\{f_1, \dots, f_r\}$ of differential polynomials in n variables such that $X = \{a \in k^n \mid f_1(a) = \cdots = f_r(a) = 0\}$.

(2) A subgroup $G \subset \mathrm{GL}_n(k) \subset k^{n^2}$ is a *linear differential algebraic group* if $G = X \cap \mathrm{GL}_n(k)$ for some Kolchin-closed subset of k^{n^2}.

In the previous example, the Galois group was exhibited as a linear differential algebraic subgroup of $\mathrm{GL}_1(k_0)$. For any linear algebraic group G, the group $G(k)$ is a linear differential algebraic group. Furthermore, the group $G(C_k^\Delta)$ of constant points of G is also a linear differential algebraic group since it is defined by the (algebraic) equations defining G as well as the (differential) equations stating that the entries of the matrices are constants. We will give more examples in the next section

In the next definition, we will use the following conventions. If F is a $\Delta = \{\partial_0, \partial_1, \dots, \partial_m\}$-field, we denote by C_F^0 the ∂_0 constants of F, that is, $C_F^0 = C_F^{\{\partial_0\}} = \{c \in F \mid \partial_0 c = 0\}$. One sees that C_F^0 is a $\Pi = \{\partial_1, \dots, \partial_m\}$-field. We will use the notation $k\langle z_1, \dots, z_r\rangle_\Delta$ to denote a Δ-field containing k and elements $z_1, \dots, z_r$ such that no proper Δ-field has this property, *i.e.,*$k\langle z_1, \dots, z_r\rangle_\Delta$ is the field generated over k by $z_1, \dots, z_n$ and their higher derivatives.

Definition 3.4. Let k be a $\Delta = \{\partial_0, \partial_1, \dots, \partial_m\}$-field and let

$$\partial_0 Y = AY$$

be a differential equation with $A \in \mathrm{gl}_n(k)$.

(1) A Δ-extension K of k is a *parameterized Picard–Vessiot extension of* k (or, more compactly, a PPV-extension of k) if $K = k\langle z_{1,1}, \dots, z_{n,n}\rangle_\Delta$ where

 (a) the $z_{i,j}$ are entries of a matrix $Z \in \mathrm{GL}_n(K)$ satisfying $\partial_0 Z = AZ$, and

 (b) $C_K^0 = C_k^0$, *i.e.,* the ∂_0-constants of K coincide with the ∂_0-constants of k.

(2) The group $\mathrm{Gal}_\Delta(K/k) = \{\sigma : K \to K \mid \sigma$ is a k-automorphism such that $\sigma\partial = \partial\sigma$ for all $\partial \in \Delta\}$ is called the *parameterized Picard–Vessiot group* (PPV-group) associated with the PPV-extension K of k.

We note that if K is a PPV-extension of k and Z is as above then for any $\sigma \in \mathrm{Gal}_\Delta(K/k)$ one has that $\partial_0(\sigma(Z)Z^{-1}) = 0$. Therefore we can identify each $\sigma \in \mathrm{Gal}_\Delta(K/k)$ with a matrix in $GL_n(C_k^0)$. We can now state the Fundamental Theorem of parameterized Picard–Vessiot extensions

Theorem 3.5. *Let k be a $\Delta = \{\partial_0, \partial_1, \ldots, \partial_m\}$-field and assume that C_k^0 is a differentially closed $\Pi = \{\partial_1, \ldots, \partial_m\}$-field. Let*

$$\partial_0 Y = AY \tag{3.2}$$

be a differential equation with $A \in \mathrm{gl}_n(k)$.

(1) *There exists a PPV-extension K of k associated with* (3.2) *and this is unique up to differential k-isomorphism.*
(2) *The PPV-group* $\mathrm{Gal}_\Delta(K/k)$ *may be identified with $G(C_k^0)$, where G is a linear differential algebraic group defined over C_k^0.*
(3) *The map that sends any Δ-subfield F, $k \subset F \subset K$, to the group* $\mathrm{Gal}_\Delta(K/F)$ *is a bijection between differential subfields of K containing k and Kolchin-closed subgroups of* $\mathrm{Gal}_\Delta(K/k)$. *Its inverse is given by the map that sends a Kolchin-closed group H to the field $K^H = \{z \in K \mid \sigma(z) = z$ for all $\sigma \in H\}$.*
(4) *A Kolchin-closed subgroup H of* $\mathrm{Gal}_\Delta(K/k)$ *is a normal subgroup of* $\mathrm{Gal}_\Delta(K/k)$ *if and only if the field K^H is left set-wise invariant by* $\mathrm{Gal}_\Delta(K/k)$. *If this is the case, the map* $\mathrm{Gal}_\Delta(K/k) \to \mathrm{Gal}_\Delta(K^H/k)$ *is surjective with kernel H and K^H is a PPV-extension of k with PPV-group isomorphic to* $\mathrm{Gal}_\Delta(K/k)/H$. *Conversely, if F is a differential subfield of K containing k and F is a PPV-extension of k, then* $\mathrm{Gal}_\Delta(F/K)$ *is a normal subgroup of* $\mathrm{Gal}_\Delta(K/k)$.

The proof of this result is virtually the same as for the corresponding result of Picard–Vessiot theory. We give the details in Appendices 9.1–9.4.

We will give two simple applications of this theorem. For the first, let K be a PPV-extension of k corresponding to the equation $\partial_0 Y = AY$ and let $K = k\langle Z\rangle_\Delta$, where $Z \in \mathrm{GL}_n(K)$ and $\partial_0 Z = AZ$. We now consider the field $K_A^{\mathrm{PV}} = k(Z) \subset K$. Note that K_A^{PV} is not necessarily a Δ-field but it is a $\{\partial_0\}$-field. One can easily see that it is a PV-extension for the equation $\partial_0 Y = AY$ and that the PPV-group leaves it invariant and acts as $\{\partial_0\}$-automorphisms. We therefore have an injective homomorphism of $\mathrm{Gal}_\Delta(K/k) \to \mathrm{Gal}_{\{\partial_0\}}(K_A^{\mathrm{PV}}/k)$, defined by restriction. We then have the following result

Proposition 3.6. *Let k, C_k^0, K and K_A^{PV} be as above. Then:*

(1) *When considered as ordinary $\{\partial_0\}$-fields, K_A^{PV} is a PV-extension of k with algebraically closed field C_k^0 of ∂_0-constants.*

(2) *If* $\mathrm{Gal}_{\{\partial_0\}}(K_A^{\mathrm{PV}}/k) \subset \mathrm{GL}_n(C_k^0)$ *is the Galois group of the ordinary differential field* K_A^{PV} *over* k*, then the Zariski closure of the Galois group* $\mathrm{Gal}_\Delta(K/k)$ *in* $\mathrm{GL}_n(C_k^0)$ *equals* $\mathrm{Gal}_{\{\partial_0\}}(K_A^{\mathrm{PV}}/k)$.

Proof. Since a differentially closed field is algebraically closed, we have already justified the first statement. Clearly, $\mathrm{Gal}_\Delta(K/k) \subset \mathrm{Gal}_{\{\partial_0\}}(K/k)$. Since $\mathrm{Gal}_\Delta(K/k)$ and $\mathrm{Gal}_{\{\partial_0\}}(K_A^{\mathrm{PV}}/k)$ have the same fixed field k, the second statement follows. $\square$

Remark 3.7. Fix a PPV-extension K of k and let $K = k\langle z_{1,1}, \ldots, z_{n,n}\rangle_\Delta$ where the $z_{i,j}$ are entries of a matrix $Z \in \mathrm{GL}_n(K)$ satisfying $\partial_0 Z = AZ$ with $A \in \mathrm{gl}_n(k)$. One sees that the field K_A^{PV} defined above is independent of the particular invertible solution Z of $\partial_0 Y = AY$ used to generate K (although the Galois groups are only determined up to conjugacy). On the other hand, K_A^{PV} does depend on the equation $\partial_0 Y = AY$ and not just on the field K, that is if K is a PPV-extension of k for two different equations $\partial_0 Y = A_1 Y$ and $\partial_0 Y = A_2 Y$ with solutions Z_1 and Z_2 respectively, the fields $K_{A_1}^{\mathrm{PV}}$ and $K_{A_2}^{\mathrm{PV}}$ (and their respective PV-groups) may be very different. We will give an example of this in Remark 7.3.

Our second application is to characterize those equations $\partial_0 Y = AY$ whose PPV-groups are the set of Δ-constant points of a linear algebraic group. We first make the following definition.

Definition 3.8. Let k be a Δ-differential field and let $A \in \mathrm{gl}_n(k)$. We say that $\partial_0 Y = AY$ is *completely integrable* if there exist $A_i \in \mathrm{gl}_n(k)$, $i = 0, \ldots, n$ with $A_0 = A$ such that

$$\partial_j A_i - \partial_i A_j = A_j A_i - A_i A_j \quad \text{for all } i, j = 0, \ldots \text{ n.}$$

The nomenclature is motivated by the fact that the latter conditions on the A_i are the usual integrability conditions and the system of differential equations $\partial_i Y = A_i Y$, $i = 0, \ldots, n$ are as in equations (2.1).

Proposition 3.9. *Let* k *be a* Δ*-differential field and assume that* k_0 *is a* Π*-differentially closed* Π*-field. Let* $A \in \mathrm{gl}_n(k)$ *and let* K *be a PPV-extension of* k *for* $\partial_0 Y = AY$*. Finally, let* $C = C_k^\Delta$.

(1) *There exists a linear algebraic group* G *defined over* C *such that* $\mathrm{Gal}_\Delta(K/k)$ *is conjugate to* $G(C)$ *if and only if* $\partial_0 Y = AY$ *is completely integrable. If this is the case, then* K *is a PV-extension of* k *corresponding to this integrable system.*

(2) *If* $A \in \mathrm{gl}_n(C_k^\Pi)$*, then* $\mathrm{Gal}_\Delta(K/k)$ *is conjugate to* $G(C)$ *for some linear algebraic group defined over* C.

Proof. (1) Let $K = k\langle Z\rangle_\Delta$ where $Z \in \mathrm{GL}_n(K)$ satisfies $\partial_0 Z = AZ$. If the PPV-group is as described, then there exists a $B \in \mathrm{GL}_n(C_k^0)$ such that $B\mathrm{Gal}_\Delta(K/k)B^{-1} = G(C)$, G an algebraic subgroup of $\mathrm{GL}_n(C_k^0)$, defined over C. Set $W = ZB^{-1}$. One sees that

$\partial_0 W = AW$ and $K = k\langle W\rangle_\Delta$. A simple calculation shows that for any $i = 0, \ldots, n$, $\partial_i W \cdot W^{-1}$ is left fixed by all elements of the PPV-group. Therefore $\partial_i W = A_i W$ for some $A_i \in \mathrm{gl}_n(k)$. Since the ∂_i commute, one sees that the A_i satisfy the integrability conditions.

Now assume that there exist $A_i \in \mathrm{gl}_n(k)$ as above satisfying the integrability conditions. Let K be a PV-extension of k for the corresponding integrable system. From Lemma 9.9 in the Appendix, we know that K is also a PPV-extension of k for $\partial_0 Y = AY$. Let $\sigma \in \mathrm{Gal}_\Delta(K/k)$ and let $\sigma(Z) = ZD$ for some $D \in \mathrm{GL}_n(C_k^0)$. Since $\partial_i Z \cdot Z^{-1} = A_i \in \mathrm{GL}_n(k)$, we have that $\sigma(\partial_i Z \cdot Z^{-1}) = \partial_i Z \cdot Z^{-1}$. A calculation then shows that $\partial_i(D) = 0$. Therefore $D \in \mathrm{GL}_n(C_K^\Delta)$. We now need to show that $C_K^\Delta = C_k^\Delta$. This is clear since $C_K^\Delta \subset C_K^0 = C_k^0$. The final claim of Part (1) is now clear.

(2) Under the assumptions, the matrices $A_0 = A, A_1 = 0, \ldots, A_n = 0$ satisfy the integrability conditions, so the conclusion follows from Part (1) above. □

If A has entries that are analytic functions of $x, t_1, \ldots, t_m$, the fact that $\mathrm{Gal}_\Delta(K/k) = G(C)$ for some linear algebraic group does not imply that, for some open set of values $\vec{\tau} = (\tau_1, \ldots, \tau_m)$ of $(t_1, \ldots, t_m)$, the Galois group $G_{\vec{\tau}}$ of the ordinary differential equation $\partial_x Y = A(x, \tau_1, \ldots, \tau_m)Y$ is independent of the choice of $\vec{\tau}$. We shall see in Section 5 that for equations with regular singular points we do have a constant Galois group (on some open set of parameters) if the PPV-group is $G(C)$ for some linear algebraic group but the following shows that this is not true in general.

Example 3.10. Let $\Pi = \{\partial_1 = \frac{\partial}{\partial t_1}, \partial_2 = \frac{\partial}{\partial t_2}\}$ and k_0 be a differentially closed Π-field containing $\mathbb{C}$. Let $k = k_0(x)$ be a $\Delta = \{\partial_0 = \frac{\partial}{\partial x}, \partial_1, \partial_2\}$-field where $\partial_0|_{k_0} = 0$, $\partial_0(x) = 1$, and ∂_1, ∂_2 extend the derivations on k_0 and satisfy $\partial_1(x) = \partial_2(x) = 0$. The equation

$$\frac{\partial Y}{\partial x} = A(x, t_1, t_2)Y = \begin{pmatrix} t_1 & 0 \\ 0 & t_2 \end{pmatrix} Y$$

has solution

$$Y = \begin{pmatrix} e^{t_1 x} & 0 \\ 0 & e^{t_2 x} \end{pmatrix}$$

One easily checks that

$$A_1 = \frac{\partial Y}{\partial t_1} Y^{-1} \in \mathrm{gl}_2(k) \quad \text{and} \quad A_2 = \frac{\partial Y}{\partial t_2} Y^{-1} \in \mathrm{gl}_2(k)$$

so the Galois group associated to this equation is conjugate to $G(C)$ for some linear algebraic group G (in fact $G(C) = C^* \times C^*$). Nonetheless, for fixed values $\vec{\tau} = (\tau_1, \tau_2) \in C^2$, the Galois group of $\partial_0 Y = A(x, \tau_1, \tau_2)Y$ is $G(C)$ if and only if τ_1 and τ_2 are linearly independent over the rational numbers. □

For more information on how a differential Galois group can vary in a family of linear differential equations see [1] §3.3, [2], [3], [4], [18], and [46].

We end this section with a result concerning solving parameterized linear differential equations in "finite terms". The statement of the result is the same *mutatis mutandi* as the corresponding result in the usual Picard–Vessiot theory (*cf.,* [40], Ch. 1.5) and will be proved in the Appendix.

Definition 3.11. Let k be a $\Delta = \{\partial_0, \dots \partial_m\}$-field. We say that a Δ-field L is a *parameterized liouvillian extension* of k if $C_L^0 = C_k^0$ and there exist a tower of Δ-fields $k = L_0 \subset L_1 \subset \cdots \subset L_r$ such that $L \subset L_r$ and $L_i = L_{i-1}\langle t_i\rangle_\Delta$ for $i = 1 \dots r$, where either

(1) $\partial_0 t_i \in L_{i-1}$, that is t_i is a *parameterized integral* (*of an element of* L_{i-1}), or

(2) $t_i \neq 0$ and $\partial_0 t_i/t_i \in L_{i-1}$, that is t_i is a *parameterized exponential* (*of an integral of an element in* L_{i-1}), or

(3) t_i is algebraic over L_{i-1}.

In Section 9.5 we shall prove a result (Lemma 9.14) that implies that a parmeterized liouvillian extension is an inductive limit of ∂_0-liouvillian extension (in the usual sense, *cf.,* Ch. 1.5, [40]). We will use this to prove the following result

Theorem 3.12. *Let k be a Δ-field and assume that C_k^0 is a differentially closed $\Pi = \{\partial_1, \dots \partial_m\}$-field. Let K be a PPV-extension of k with PPV-group G. The following are equivalent*

(1) *G contains a solvable subgroup* (*in the sense of abstract groups*) *H of finite index.*

(2) *K is a parameterized liouvillian extension of k.*

(3) *K is contained in a parameterized liouvillian extension of k.*

4 Linear differential algebraic groups

In this section we review some known facts concerning linear differential algebraic groups and give some examples of these groups. The theory of linear differential algebraic groups was initiated by P. Cassidy in [9] and further developed in [10]–[14]. The topic has also been addressed in [7], [22], [34], [36], [23], [47], and [48]. For a general overview see [8].

Let k_0 be a differentially closed $\Pi = \{\partial_1, \dots, \partial_m\}$-field and let $C = C_{k_0}^{\Pi}$. As we have already defined, a linear differential algebraic group is a Kolchin-closed subgroup of $\mathrm{GL}_n(k_0)$. Although the definition is a natural generalization of the definition of a linear algebraic group there are many points at which the theories diverge. The first is that an affine differential algebraic group (a Kolchin-closed subset X of k_0^m with group operations defined by everywhere defined rational differential functions) need not be a linear differential algebraic group although affine differential algebraic groups whose group laws are given by differential polynomial maps are linear differential algebraic groups [9]. Other distinguishing phenomena will emerge as we examine some examples.

Differential algebraic subgroups of $\boldsymbol{G_a^n}$. The group $\boldsymbol{G_a} = (k_0, +)$ is naturally isomorphic to $\left\{ \left(\begin{smallmatrix} 1 & a \\ 0 & 1 \end{smallmatrix}\right) \mid a \in k_0 \right\}$ and, as such, has the structure of a linear differential algebraic group. Nonetheless we will continue to identify this group with k_0. The set $\boldsymbol{G_a^n} = (k_0^n, +)$ can also be seen to be a linear differential algebraic group. In ([9], Lemma 11), Cassidy shows that a subgroup H of $\boldsymbol{G_a^n}$ is a linear differential algebraic group if and only if H is the set of zeros of a set of linear homogeneous differential polynomials in $k_0\{y_1, \dots, y_n\}$. In particular, when $m = n = 1$, $\Pi = \{\partial\}$, the linear differential algebraic subgroups of $\boldsymbol{G_a}$ are all of the form

$$\boldsymbol{G_a^L}(k_0) = \{a \in \boldsymbol{G_a}(k_0) \mid L(a) = 0\}$$

where L is a linear differential operator (*i.e.,* an element of the ring $k_0[\partial]$ whose multiplication is given by $\partial \cdot a = a\partial + \partial(a)$). The lattice structure of these subgroups is given by

$$\boldsymbol{G_a^{L_1}}(k_0) \subset \boldsymbol{G_a^{L_2}}(k_0) \Leftrightarrow L_2 = L_3 L_1 \text{ for some } L_3 \in k_0[\partial].$$

Differential algebraic subgroups of $\boldsymbol{G_m^n}$. These have been classified by Cassidy ([9], Ch.IV).We shall restrict ourselves to the case $n = m = 1$, $\Pi = \{\partial\}$, that is, differential algebraic subgroups of $\boldsymbol{G_m}(k_0) = \mathrm{GL}_1(k_0) = k_0^*$. Any such group is either

(1) finite and cyclic, or

(2) $\boldsymbol{G_m^L} = \left\{a \in \boldsymbol{G_m}(k_0) \mid L\left(\frac{\partial a}{a}\right) = 0\right\}$ for some $L \in k_0[\partial]\big\}$.

For example, if $L = \partial$, the group

$$\boldsymbol{G_m^\partial}(k) = \left\{a \in k_0^* \mid \partial\left(\tfrac{\partial a}{a}\right) = 0\right\}$$

is the PPV-group of the parameterized linear differential equation $\partial_x y = \frac{t}{x} y$ where $\partial = \partial_t$. Notice that the only proper differential algebraic subgroup of $\{a \in k_0 \,|\, \partial a = 0\}$ is $\{0\}$. Therefore the only proper differential algebraic subgroups of $\boldsymbol{G_m^\partial}$ are either the finite cyclic groups, or $\boldsymbol{G_m}(C)$. This justifies the left column in the table given in Example 3.1 (bis). The right column follows by calculation.

The proof that the groups of (1) and (2) are the only possibilities proceeds in two steps. The first is to show that if the group is not connected (in the Kolchin topology where closed sets are Kolchin-closed sets), it must be finite (and therefore cyclic). The second step involves the *logarithmic derivative map* $l\partial\colon \boldsymbol{G_m}(k_0) \to \boldsymbol{G_a}(k_0)$ defined by

$$l\partial(a) = \frac{\partial a}{a}.$$

This map is a differential rational map (*i.e.,* the quotient of differential polynomials) and is a homomorphism. Furthermore, it can be shown that the following sequence is exact:

$$(1) \longrightarrow \boldsymbol{G_m}(C) \longrightarrow \boldsymbol{G_m}(k_0) \longrightarrow \boldsymbol{G_a}(k_0) \longrightarrow (0)$$
$$a \longmapsto \frac{\partial a}{a}.$$

The result then follows from the classification of differential subgroups of $\boldsymbol{G_a}(k_0)$. Note that in the usual theory of linear algebraic groups, there are no nontrivial rational homomorphisms from $\boldsymbol{G_m}$ to $\boldsymbol{G_a}$.

Semisimple differential algebraic groups. These groups have been classified by Cassidy in [14]. Buium [7] and Pillay [36] have given simplified proofs in the ordinary case (*i.e.,* $m = 1$). Buium's proof is geometric using the notion of jet groups and Pillay's proof is model theoretic and assumes from the start that the groups are finite dimensional (of finite Morely rank).

We say that a connected differential algebraic group is semisimple if it has no nontrivial normal Kolchin-connected, commutative subgroups. Let us start by considering semisimple differential algebraic subgroups G of $\mathrm{SL}_2(k_0)$. Let H be the Zariski-closure of such a group. If $H \neq \mathrm{SL}_2(k_0)$, then H is solvable (*cf.,* [40], p. 127) and so the same would be true of G. Therefore G must be Zariski-dense in $\mathrm{SL}_2(k_0)$. In [9] Proposition 42, Cassidy classified the Zariski-dense differential algebraic subgroups of $\mathrm{SL}_n(k_0)$. Let $\mathbb{D}$ be the k_0-vector space of derivations spanned by Π.

Proposition 4.1. *Let G be a proper Zariski-dense differential algebraic subgroup of* $\mathrm{SL}_n(k_0)$*. Then there exists a finite set $\Delta_1 \subset \mathbb{D}$ of commuting derivations such that G is conjugate to* $\mathrm{SL}_n(C_{k_0}^{\Delta_1})$*, the Δ_1-constant points of* $\mathrm{SL}_n(k_0)$.

Note that in the ordinary case $m = 1$, we can restate this more simply: *A proper Zariski-dense subgroup of* $\mathrm{SL}_n(k_0)$ *is conjugate to* $\mathrm{SL}_n(C)$. A complete classification of differential subgroups of SL_2 is given in [48]. The complete classification of semisimple differential algebraic groups is given by the following result (see [14], Theorem 18). By a Chevalley group, we mean a connected simple $\mathbb{Q}$-group containing a maximal torus diagonalizable over $\mathbb{Q}$.

Proposition 4.2. *Let G be a Kolchin-connected semisimple linear[2] differential algebraic group. Then there exist finite subsets of commuting derivations $\Delta_1, \ldots, \Delta_r$ of $\mathbb{D}$, Chevalley groups $H_1, \ldots, H_r$ and a differential isogeny $\sigma : H_1(C_{k_0}^{\Delta_1}) \times \cdots \times H_r(C_{k_0}^{\Delta_r}) \to G$.*

[2] One need not assume that G is linear since Pillay [34] showed that a semisimple differential algebraic group is differentially isomorphic to a linear differential algebraic group.

5 Isomonodromic families

In this section we shall describe how isomonodromic families of linear differential equations fit into this theory of parameterized linear differential equations. We begin[3] with some definitions and follow the exposition of Sibuya [45], Appendix 5. Let $\mathcal{D}$ be an open subset of the Riemann sphere (for simplicity, we assume that the point at infinity is not in $\mathcal{D}$) and let $\mathcal{D}(\vec{\tau},\vec{r}\,) = \prod_{h=1}^{p} D(\tau_h,\rho_h)$ where $\vec{r} = (\rho_1,\ldots,\rho_p)$ is a p-tuple of positive numbers, $\vec{\tau} = (\tau_1,\ldots,\tau_p) \in \mathbb{C}^p$ and $D(\tau_h,\rho_h)$ is the open disk in $\mathbb{C}$ of radius ρ_h centered at the point τ_h. We denote by $\mathcal{O}(\mathcal{D}\times\mathcal{D}(\vec{\tau},\vec{r}\,))$ the ring of functions $f(x,\vec{t}\,)$ holomorphic on $\mathcal{D}\times\mathcal{D}(\vec{\tau},\vec{r}\,)$. Let $A(x,\vec{t}\,) \in \mathrm{gl}_n(\mathcal{O}(\mathcal{D}\times\mathcal{D}(\vec{\tau},\vec{r}\,))$ and consider the differential equation

$$\frac{\partial W}{\partial x} = A(x,\vec{t}\,)W \tag{5.1}$$

Definition 5.1. A *system of fundamental solutions* of (5.1) is a collection of pairs $\{D(x_j,\vec{r}_j), W_j(x,\vec{t}\,)\}$ such that

(1) the disks $D(x_j,\vec{r}_j)$ cover $\mathcal{D}$ and

(2) for each $\vec{t} \in \mathcal{D}(\vec{\tau},\vec{r}\,)$ the $W_j(x,\vec{t}\,) \in \mathrm{GL}_n(\mathcal{O}(D(x_j,\vec{r}_j)\times\mathcal{D}(\vec{\tau},\vec{r}\,))$ are solutions of (5.1).

We define $C_{i,j}(\vec{t}\,) = W_i(x,\vec{t}\,)^{-1}W_j(x,\vec{t}\,)$ whenever $D(x_i,\vec{r}_i)\cap D(x_j,\vec{r}_j) \neq \emptyset$ and refer to these as the *connection matrices* of the system of fundamental solutions.

Definition 5.2. The differential equation (5.1) is *isomonodromic on* $\mathcal{D}\times\mathcal{D}(\vec{\tau},\vec{r}\,)$ if there exists a system $\{D(x_j,\vec{r}_j), W_j(x,\vec{t}\,)\}$ of fundamental solutions such that the connection matrices $C_{i,j}(\vec{t}\,)$ are independent of t.

We note that for a differential equation that is isomonodromic in the above sense, the monodromy around any path is independent of $\vec{t}$ as well. To see this let γ be a path in $\mathcal{D}$ beginning and ending at x_0 and let $D(x_1,\vec{r}_1),\ldots,D(x_s,\vec{r}_s), D(x_1,\vec{r}_1)$ be a sequence of disks covering the path so that $D(x_i,\vec{r}_i)\cap D(x_{i+1},\vec{r}_{i+1}) \neq \emptyset$ and $x_0 \in D(x_1,\vec{r}_1)$. If one continues $W_1(x_1,\vec{t}\,)$ around the path then the resulting matrix $\tilde{W} = W_1(x_1,\vec{t}\,)C_{1,s}C_{s,s-1}\ldots C_{2,1}$. By assumption, the monodromy matrix $C_{1,s}C_{s,s-1}\ldots C_{2,1}$ is independent of $\vec{t}$.

For equations with regular singular points, the monodromy group is Zariski dense in the PV-group. The above comments therefore imply that for an isomonodromic family, there is a nonempty open set of parameters such that for these values the monodromy (and therefore the PV-group) is constant as the parameters vary in this set. Conversely, fix $x_0 \in \mathcal{D}$ and fix a generating set S for $\Pi_1(\mathcal{D},x_0)$. Assume that, for each value $\vec{t} \in \mathcal{D}(\vec{t},\vec{r}\,)$, we can select a basis of the solution space of (5.1) such that the monodromy matrices corresponding to S with respect to this basis are independent

[3] The presentation clearly could be cast in the language of vector bundles (see [3], [4], [26], [27]) but the approach presented here is more in the spirit of the rest of the paper.

of $\vec{t}$. Bolibruch (Proposition 1, [6]) has shown that under these assumptions (5.1) is isomonodromic in the above sense[4].

With these definitions, Sibuya shows ([45], Theorem A.5.2.3)

Proposition 5.3. *The differential equation* (5.1) *is isomonodromic on* $\mathcal{D} \times \mathcal{D}(\vec{\tau}, \vec{r})$ *if and only if there exist p matrices* $B_h(x, \vec{t}) \in \mathrm{gl}_n(\mathcal{O}(\mathcal{D} \times \mathcal{D}(\vec{\tau}, \vec{r}))),\ h = 1, \ldots, p$ *such that the system*

$$\begin{aligned} \frac{\partial W}{\partial x} &= A(x, \vec{t})W \\ \frac{\partial W}{\partial t_h} &= B_h(x, \vec{t})W \quad (h = 1, \ldots, p) \end{aligned} \tag{5.2}$$

is completely integrable.

Some authors use the existence of matrices B_i as in Proposition 5.3 as the definition of isomonodromic (*cf.,* [26]). Sibuya goes on to note that if $A(x, \vec{t})$ is rational in x and if the differential equation has only regular singular points, then the $B_h(x, \vec{t})$ will be rational in x as well (without the assumption of regular singular points one cannot conclude that the B_i will be rational in x.) This observation leads to the next proposition.

For any open set $\mathcal{U} \subset \mathbb{C}^p$, let $\mathcal{M}(\mathcal{U})$ be the field of functions meromorphic on $\mathcal{U}$. Note that $\mathcal{M}(\mathcal{U})$ is a $\Pi = \{\frac{\partial}{\partial t_1}, \ldots, \frac{\partial}{\partial t_p}\}$-field. If $\mathcal{U}' \subset \mathcal{U}$ then there is a natural injection of $\mathrm{res}_{\mathcal{U},\mathcal{U}'} \colon \mathcal{M}(\mathcal{U}) \to \mathcal{M}(\mathcal{U}')$. We shall need the following result of Seidenberg [43], [44]: *Let $\mathcal{U}$ be an open subset of $\mathbb{C}^p$ and let F be a Π-subfield of $\mathcal{M}(\mathcal{U})$ containing $\mathbb{C}$. If E is Π-field containing F and finitely generated* (*as a Π-field*) *over $\mathbb{Q}$, then there exists a nonempty open set $\mathcal{U}' \subset \mathcal{U}$ and an isomorphism $\phi \colon E \to \mathcal{M}(\mathcal{U})$ such that $\phi|_F = \mathrm{res}_{\mathcal{U},\mathcal{U}'}$.*

Let $A(x, \vec{t})$ be as above, assume the entries of A are rational in x and let F be the Π-field generated by the coefficients of powers of x that appear in the entries of A. Let k_0 be the differential closure of F. We consider $k = k_0(x)$ to be a $\Delta = \{\frac{\partial}{\partial x}, \frac{\partial}{\partial t_1}, \ldots, \frac{\partial}{\partial t_p}\}$-field in the obvious way. Given open subsets $U_1 \subset U_2$ of the Riemann sphere, we say that U_1 is a *punctured subset of* U_2 if there exist a finite number of disjoint closed disks $D_1, \ldots, D_r \subset U_2$ such that $U_1 = U_2 \backslash \left(\bigcup_{i=1}^r D_i\right)$.

Proposition 5.4. *Let* $A(x, \vec{t}), k_0$ *and* k *be as above. Assume that the differential equation*

$$\frac{\partial W}{\partial x} = A(x, \vec{t})W \tag{5.3}$$

has only regular singular points. Then this equation is isomonodromic on $\mathcal{D}' \times \mathcal{U}$*, for some nonempty, open* $\mathcal{U} \subset \mathcal{D}(\vec{\tau}, \vec{r})$ *and* $\mathcal{D}'$ *a punctured subset of D if and only if*

[4] Throughout [6], Bolibruch assumes that $A(x, \vec{t}) = \sum_{i=1}^s \frac{A_i(\vec{t})}{x - t_i}$ but his proof of this result works *mutatis mutandi* for any equation with regular singular points.

the PPV-group of this equation over k is conjugate to $G(\mathbb{C})$ for some linear algebraic group G defined over $\mathbb{C}$. In this case, the monodromy group of (5.3) *is independent of $\vec{t} \in U$.*

Proof. Assume that (5.3) is isomonodromic. Proposition 5.3 and the comments after it ensure that we can complete (5.3) to a completely integrable system (5.2) where the $B_i(x, \vec{t}\,)$ are rational in x. The fact that this is a completely integrable system is equivalent to the coefficients of the powers of x appearing in the entries of the B_i satisfying a system $\mathcal{S}$ of Π-differential equations with coefficients in k_0. Since this system has a solution and k_0 is differentially closed, the system must have a solution in k_0. Therefore we may assume that the $B_i \in \mathrm{gl}_n(k)$. An application of Proposition 3.9 (1) yields the conclusion.

Now assume that the PPV-group is conjugate to $G(\mathbb{C})$ for some linear algebraic group G. Proposition 3.9 (1) implies that we can complete (5.3) to a completely integrable system (5.2) where the $B_i(x, \vec{t}\,)$ are in $\mathrm{gl}_n(k)$. Let E be the Π-field generated by the coefficients of powers of x appearing in the entries of A and the B_i. By the result of Seidenberg referred to above, there is a nonempty open set $\mathcal{U} \subset \mathcal{D}(\vec{\tau}, \vec{r}\,)$ such that these coefficients can be assumed to be analytic on $\mathcal{U}$.The matrices B_i have entries that are rational in x and so may have poles (depending on $\vec{t}$) in D. By shrinking $\mathcal{U}$ if necessary and replacing D with a punctured subset D' of D, we can assume that A and the B_i have entries that are holomorphic in $D' \times \mathcal{U}$. We now apply Proposition 5.3 to reach the conclusion. □

6 Second order systems

In this section we will apply the results of the previous four sections to give a classification of parameterized second order systems of linear differential equations. We will first consider the case of second order parameterized linear equations depending on only one parameter.

Proposition 6.1. *Let k be a $\Delta = \{\partial_0, \partial_1\}$-field, assume that $k_0 = C_k^0$ is a differentially closed $\Pi = \{\partial_1\}$-field and let $C = C_k^\Delta$. Let $A \in \mathrm{sl}_2(k)$ and let K be the PPV-extension corresponding to the differential equation*

$$\partial_0 Y = AY. \tag{6.1}$$

Then, either

(1) $\mathrm{Gal}_\Delta(K/k)$ *equals* $\mathrm{SL}_2(k_0)$*, or*

(2) $\mathrm{Gal}_\Delta(K/k)$ *contains a solvable subgroup of finite index and K is a parameterized liouvillian extension of k, or*

(3) $\mathrm{Gal}_\Delta(K/k)$ *is conjugate to* $\mathrm{SL}_2(C)$ *and there exist* $B_1 \in \mathrm{sl}_2(k)$ *such that the system*

$$\partial_0 Y = AY$$
$$\partial_1 Y = B_1 Y$$

is an integrable system.

Proof. Let $Z \in \mathrm{GL}_2(k)$ be a fundamental solution matrix of (6.1) and let $z = \det Z$. We have that $\partial_0 z = (\mathrm{trace} A)z$ ([40], Exercise 1.14.5), so $z \in k_0$. For any $\sigma \in \mathrm{Gal}_\Delta(K/k)$, $z = \sigma(z) = \det\sigma \cdot z$ so $\det\sigma = 1$. Therefore, $\mathrm{Gal}_\Delta \subset \mathrm{SL}_2(k_0)$. Let G be the Zariski-closure of $\mathrm{Gal}_\Delta(K/k)$. If $G \neq \mathrm{SL}_2(k_0)$, then G has a solvable subgroup of finite index and so the same holds for $\mathrm{Gal}_\Delta(K/k)$. Therefore, (2) holds. If $G = \mathrm{Gal}_\Delta(K/k) = \mathrm{SL}_2(k_0)$, then (1) holds. If $G = \mathrm{SL}_2(k_0)$ and $G \neq \mathrm{Gal}_\Delta(K/k)$, then Proposition 4.1 and the discussion immediately following it imply that there is a $B \in \mathrm{SL}_2(k_0)$ such that $B\mathrm{Gal}_\Delta(K/k)B^{-1} = \mathrm{SL}_2(C)$. Proposition 3.9 then implies that the parameterized equation $\partial_0 Y = AY$ is completely integrable, yielding conclusion (3). □

If the entries of A are functions of x and t, analytic in some domain and rational in x, we can combine the above proposition with Proposition 5.4 to yield the next corollary. Let $\mathcal{D}$ be an open region on the Riemann sphere and $D(\tau_0, \rho_0)$ be the open disk of radius ρ_0 centered at τ_0 in $\mathbb{C}$. Let $\mathcal{O}(\mathcal{D} \times D(\tau_0, \rho_0))$ be the ring of functions holomorphic in $\mathcal{D} \times D(\tau_0, \rho_0)$ and let $A(t, x) \in \mathrm{sl}_2(\mathcal{O}(\mathcal{D} \times D(\tau_0, \rho_0))$ and assume that $A(x, t)$ is rational in x. Let $\Delta = \left\{\partial_0 = \frac{\partial}{\partial x}, \partial_1 = \frac{\partial}{\partial t}\right\}$ and $\Pi = \{\partial_1\}$. Let k_0 be a differentially closed Π-field containing the coefficients of powers of x appearing in the entries of A and let $k = k_0(x)$ be the Δ-field gotten by extending ∂_1 via $\partial_1(x) = 0$ and defining ∂_0 to be zero on k_0 and $\partial_1(x) = 1$.

Corollary 6.2. *Let* $k_0, k, A(t, x)$ *be as above and let* K *be the PV-extension associated with*

$$\frac{\partial Y}{\partial x} = A(x, t)Y. \tag{6.2}$$

Then, either

(1) $\mathrm{Gal}_\Delta(K/k) = \mathrm{SL}_2(k_0)$*, or*

(2) $\mathrm{Gal}_\Delta(K/k)$ *contains a solvable subgroup of finite index and* K *is a parameterized liouvillian extension of* k*, or*

(3) *equation* (6.2) *is isomonodromic on* $D' \times \mathcal{U}$ *where* D' *is a punctured subset of* D *and* U *is an open subset of* $D(\tau_0, \rho_0)$.

We can also state a result similar to Proposition 6.1 for parameterized linear equations having more than one parameter. We recall that if k_0 is a $\Pi = \{\partial_1, \ldots \partial_m\}$-field, we denote by $\mathbb{D}$ the k_0-vector space of derivations spanned by Π.

Proposition 6.3. *Let k be a $\Delta = \{\partial_0, \dots \partial_m\}$-field, assume that $k_0 = C_k^0$ is a differentially closed $\Pi = \{\partial_1, \dots \partial_m\}$-field. Let $A \in \mathrm{sl}_2(k)$ and let K be the PPV-extension corresponding to the differential equation*

$$\partial_0 Y = AY.$$

Then, either

(1) $\mathrm{Gal}_\Delta(K/k) = \mathrm{SL}_2(k_0)$*, or*

(2) $\mathrm{Gal}_\Delta(K/k)$ *contains a solvable subgroup of finite index and K is a parameterized liouvillian extension of k, or*

(3) $\mathrm{Gal}_\Delta(K/k)$ *is a proper Zariski-dense subgroup of* $\mathrm{SL}_2(k_0)$ *and there exist*

(a) *a commuting k_0-basis $\{\partial'_1, \dots, \partial'_m\}$ of $\mathbb{D}$, and*

(b) *an integer r, $1 \le r \le m$ and elements $B_i \in \mathrm{gl}_2(k)$, $i = 1, \dots r$,*

such that the system

$$\begin{aligned} \partial_0 Y &= AY \\ \partial'_1 Y &= B_1 Y \\ &\vdots \\ \partial'_r Y &= B_r Y \end{aligned}$$

is an integrable system.

Proof. The proof begins in the same way as that for Proposition 6.1 and Cases (1) and (2) are the same. If neither of these hold, then $\mathrm{Gal}_\Delta(K/k)$ is a proper Zariski-dense subgroup of $\mathrm{SL}_2(k_0)$ and so by Proposition 4.1, there exist commuting derivations $\Delta' = \{\partial'_1, \dots, \partial'_r\} \subset \mathbb{D}$ such that $\mathrm{Gal}_\Delta(K/k)$ is conjugate to $\mathrm{SL}_2(C_k^\Pi)$. We may assume that the ∂'_i are k_0 independent. Proposition 7 of Chapter 0 of [22] implies that we can extend Δ' to a commuting basis of $\mathbb{D}$. After conjugation by an element $B \in \mathrm{GL}_2(k)$, we can assume that the PPV-group is $\mathrm{SL}_2(C_k^{\Delta'})$. A calculation shows that $(\partial'_i Y)Y^{-1}$ is left invariant by this group for $i = 1, \dots, r$ and the conclusion follows. □

The third case of the previous proposition can be stated informally as: *After a change of variables in the parameter space, the parameterized differential equation is completely integrable with respect to x and a subset of the new parameters.*

7 Inverse problem

The general inverse problem can be stated as: *Given a differential field, which groups can occur as Galois groups of PPV-extensions of this field?* We have no definitive results but will give two examples in this section.

Example 7.1. Let k be a $\Delta = \{\partial_0, \partial_1\}$-field with $k_0 = C_k^{\{\partial_0\}}$ differentially closed and $k = k_0(x)$, $\partial_0(x) = 1$, $\partial_1(x) = 0$. We wish to know: *Which subgroups G of $\boldsymbol{G_a}(k_0)$ are Galois groups of PPV-extensions of k?* The answer is that all proper differential algebraic subgroups of $\boldsymbol{G_a}(k_0)$ appear in this way but $\boldsymbol{G_a}(k_0)$ itself cannot be the Galois group of a PPV-extension K of k.

We begin by showing that $\boldsymbol{G_a}(k_0)$ cannot be the Galois group of a PPV-extension K of k. In Section 9.4, we show that K is the differential function field of a G-torsor. If $G = \boldsymbol{G_a}(k_0)$, then the Corollary to Theorem 4 of Chapter VII.3 of [22] implies that this torsor is trivial and so $K = k\langle z\rangle_\Delta$ where $\sigma(z) = z + c_\sigma$ for all $\sigma \in \boldsymbol{G_a}(k_0)$. This further implies that $\partial_0(z) = a$ for some $a \in k$. Since $k = k_0(x)$ and k_0 is algebraically closed, we may write

$$a = P(x) + \sum_{i=1}^{r}\left(\sum_{j=1}^{s_i}\frac{b_{i,j}}{(x-c_i)^j}\right)$$

where $P(x)$ is a polynomial with coefficients in k_0 and the $b_{i,j}, c_i \in k_0$. Furthermore, there exists an element $R(x) \in k$ such that

$$\partial_0(z - R(x)) = \sum_{i=1}^{r}\frac{b_{i,1}}{(x-c_i)}$$

so after such a change, we may assume that

$$a = \sum_{i=1}^{r}\frac{b_i}{x-c_i}$$

for some $b_i, c_i \in k_0$.

We shall show that the Galois group of K over k is

$$\{z \in k_0 \mid L(z) = 0\}$$

where L is the linear differential equation in $k[\partial_1]$ whose solution space is spanned (over C) by $\{b_1, b_2, \ldots, b_r\}$. In particular, *the group $\boldsymbol{G_a}(k_0)$ is not a Galois group of a PPV-extension of k.*

To do this form a new PPV-extension $F = k\langle z_1, \ldots, z_r\rangle_\Delta$ where $\partial_0 z_i = \frac{1}{x-c_i}$. Clearly, there exists an element $w = \sum_{i=1}^r b_i z_i \in F$ such that $\partial_0 w = a$. Therefore we can consider K as a subfield of F. A calculation shows that $\partial_0\big(\partial_1 z_i + \frac{\partial_1 c_i}{x-c_i}\big) = 0$ so $\partial_1 z_i \in k$. Therefore Proposition 3.9 implies that the PPV-group $\mathrm{Gal}_\Delta(F/k)$ is of the form $G(C)$ for some linear algebraic group G and that F is a PV-extension of k. The Kolchin–Ostrowski Theorem ([21], p. 407) implies that the elements z_i are algebraically independent over k. The PPV-group $\mathrm{Gal}_\Delta(F/k)$ is clearly a subgroup of $\boldsymbol{G_a}(C)^r$ and since the transcendence degree of F over k must equal the dimension of this group, we have $\mathrm{Gal}_\Delta(F/k) = \boldsymbol{G_a}(C)^r$.

For $\sigma = (d_1, \ldots, d_r) \in \boldsymbol{G_a}(C)^r = \mathrm{Gal}_\Delta(F/k)$, $\sigma(w) = w + \sum_{i=1}^r d_i b_i$. The Galois theory implies that restricting elements of $\mathrm{Gal}_\Delta(F/k)$ to K yields a surjective homomorphism onto $\mathrm{Gal}_\Delta(K/k)$, so we can identify $\mathrm{Gal}_\Delta(K/k)$ with the C-span of the b_i. Therefore $\mathrm{Gal}_\Delta(K/k)$ has the desired form.

We now show that any proper differential algebraic subgroup H of $\boldsymbol{G_a}(k_0)$ is the PPV-group of a PPV-extension of k. As stated in Section 4. $H = \{a \in \boldsymbol{G_a}(k_0) \mid L(a) = 0\}$ for some linear differential operator L with coefficients in k_0. Since k_0 is differentially closed, there exist $b_1, \dots, b_m \in k_0$ linearly independent over $C = C_k^\Delta$ that span the solution space of $L(Y) = 0$. Let

$$a = \sum_{i=1}^{m} \frac{b_i}{x-i}.$$

The calculation above shows that the PPV-group of the PPV-extension of k for $\partial_0 y = a$ is H. □

The previous example leads to the question: Find a Δ-field k such that $\boldsymbol{G_a}(k_0)$ is a Galois group of a PPV-extension of k. We do this in the next example.

Example 7.2. Let k be a $\Delta = \{\partial_0, \partial_1\}$-field with $k_0 = C_k^{\{\partial_0\}}$ differentially closed and $k = k_0(x, \log x, x^{t-1}e^{-x})$, $\partial_0(x) = 1$, $\partial_0(\log x) = \frac{1}{x}$, $\partial_0(x^{t-1}e^{-x}) = \frac{t-x-1}{x}x^{t-1}e^{-x}$, $\partial_1(x) = \partial_1(\log x) = 0$, $\partial_1(x^{t-1}e^{-x}) = (\log x)x^{t-1}e^{-x}$. Consider the differential equation

$$\partial_0 y = x^{t-1}e^{-x}$$

and let K be the PPV-extension of k for this equation. We may write $K = k\langle\gamma\rangle_\Delta$, where γ satisfies the above equation (γ is known as the *incomplete Gamma function*). We have that $K = k(\gamma, \partial_1\gamma, \partial_1^2\gamma, \dots)$. In [19], the authors show that $\gamma, \partial_1\gamma, \partial_1^2\gamma, \dots$ are algebraically independent over k. Therefore, for any $c \in \boldsymbol{G_a}(k_0)$, $\partial_1^i\gamma \mapsto \partial_1^i\gamma + \partial_1^i c$, $i = 0, 1, \dots$ defines an element of $\mathrm{Gal}_\Delta(K/k)$. Therefore $\mathrm{Gal}_\Delta(K/k) = \boldsymbol{G_a}(k_0)$.

Over $k_0(x)$, γ satisfies

$$\frac{\partial^2\gamma}{\partial x^2} - \frac{t-1-x}{x}\,\frac{\partial\gamma}{\partial x} = 0$$

and one can furthermore show that the PPV-group over $k_0(x)$ of this latter equation is

$$\begin{aligned} H &= \left\{ \left(\begin{smallmatrix} 1 & a \\ 0 & b \end{smallmatrix}\right) \mid a \in k_0,\ b \in k_0^*,\ \partial_1\left(\tfrac{\partial_1 b}{b}\right) = 0 \right\} \\ &= \boldsymbol{G_a}(k_0) \rtimes \boldsymbol{G_m}^{\partial_1}, \end{aligned}$$

where $\boldsymbol{G_m}^{\partial_1} = \left\{ b \in k_0^* \mid \partial_1\left(\frac{\partial_1 b}{b}\right) = 0 \right\}$. □

Remark 7.3. We can use the previous example to exhibit two equations $\partial_0 Y = A_1 Y$ and $\partial_0 Y = A_2 Y$ having the same PPV-extension K of k but such that $K_{A_1}^{\mathrm{PV}} \neq K_{A_2}^{\mathrm{PV}}$ and that these latter PV-extensions have different PV-groups (*cf.*, Remark 3.7). Let k

and γ be as in the above example and let

$$A_1 = \begin{pmatrix} 0 & x^{t-1}e^{-x} \\ 0 & 0 \end{pmatrix} \qquad A_2 = \begin{pmatrix} 0 & x^{t-1}e^{-x} & (\log x)x^{t-1}e^{-x} \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

We have that

$$Z_1 = \begin{pmatrix} 1 & \gamma \\ 0 & 1 \end{pmatrix} \qquad Z_2 = \begin{pmatrix} 1 & \gamma & \partial_1(\gamma) \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

satisfy $\partial_0 Z_1 = A_1 Z_1$ and $\partial_0 Z_2 = A_2 Z_2$. K is the PPV-extension associated with either equation and the Galois group $\mathrm{Gal}_\Delta(K/k)$ is $\boldsymbol{G_a}(k_0)$. We have that $K^{\mathrm{PV}}_{A_1} = k(\gamma) \neq K^{\mathrm{PV}}_{A_2} = k(\gamma, \partial_1\gamma)$ since γ and $\partial_1\gamma$ are algebraically independent over k. With respect to the first equation, $\mathrm{Gal}_\Delta(K/k)$ is represented in $\mathrm{GL}_2(k_0)$ as

$$\left\{ \begin{pmatrix} 1 & c \\ 0 & 1 \end{pmatrix} \;\middle|\; c \in k_0 \right\}$$

and with respect to the second equation $\mathrm{Gal}_\Delta(K/k)$ is represented in $\mathrm{GL}_3(k_0)$ as

$$\left\{ \begin{pmatrix} 1 & c & \partial_1(c) \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \;\middle|\; c \in k_0 \right\}.$$

The image of $\boldsymbol{G_a}(k_0)$ in $\mathrm{GL}_2(k_0)$ is Zariski-closed while the Zariski closure of the image of $\boldsymbol{G_a}(k_0)$ in $\mathrm{GL}_3(k_0)$ is

$$\left\{ \begin{pmatrix} 1 & c & d \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \;\middle|\; c, d \in k_0 \right\}.$$

As algebraic groups, the first group is just $\boldsymbol{G_a}(k_0)$ and the second is $\boldsymbol{G_a}(k_0) \times \boldsymbol{G_a}(k_0)$.

8 Final comments

Other Galois theories. In [37], Pillay proposes a Galois theory that extends Kolchin's Galois theory of strongly normal extensions. We will explain the connections to our results.

Let k be a differential field and K a Picard–Vessiot extension of k. K has the following property: for any differential extension E of K and any differential k-isomorphism ϕ of K into E, we have that $\phi(K) \cdot C = K \cdot C$, where C is the field of constants of E. Kolchin has shown ([21], Chapter VI) this is the key property for developing a Galois theory. In particular, he defines a finitely generated differential field extension K of k to be *strongly normal* if for any differential extension E of K

and any differential k-isomorphism of K into E we have that

(1) $\phi(K)\langle C\rangle = K\langle C\rangle$, where C are the constants of E and

(2) ϕ leaves each of the constants of K fixed.

For such fields, Kolchin shows that the differential Galois group of K over k has the structure of an algebraic group and that the usual Galois correspondence holds.

In [31], [35], [37], [38] Pillay considers *ordinary* differential fields and generalizes this theory. The key observation is that the condition (1) can be restated as

$$\phi(K)\langle X(E)\rangle = K\langle X(E)\rangle, \tag{1'}$$

where X is the differential algebraic variety defined by the equation $\partial Y = 0$ and $X(E)$ are the E-points of X. For X, any differential algebraic variety defined over k (or more generally, any Kolchin-constructible set), Pillay defines a differential extension K to be an X*-strongly normal* extension of k if for any differential extension E of K and any differential k-isomorphism of K into E we have that equation (1$'$) holds and that (2) is replaced by technical (but important) other conditions. Pillay then uses model theoretic tools to show that for these extensions, the Galois group is a finite dimensional differential algebraic group (note that in the PPV-theory, infinite dimensional differential algebraic groups can occur, *e.g.*, $\boldsymbol{G_a}$). The finite dimensionality results from the fact that the underlying differential fields are ordinary differential fields and that finite sets of elements in the differential closure of an ordinary differential field generate fields of finite transcendence degree (a fact that is no longer true for partial differential fields). Because of this, Pillay was able to recast his theory in [38] in the language of subvarieties of certain jet spaces. If one generalizes Pillay's definition of strongly normal to allow *partial* differential fields with derivations Δ and takes for X the differential algebraic variety defined by $\{\partial_Y = 0 \mid \partial \in \Pi\}$ where $\Pi \subset \Delta$, then this definition would include PPV-extensions. Presumably the techniques of [37] can be used to prove many of these results as well. Nonetheless, we feel that a description of the complete situation for PPV-fields is sufficiently self contained as to warrant an independent exposition.

Landesman [24] has been generalizing Kolchin's Galois theory of strongly normal extensions to differential fields having a designated subset of derivations acting as parametric derivations. When this is complete, many of our results should follow as a special case of his results.

Umemura [49]–[54] has proposed a Galois theory for general nonlinear differential equations. Instead of Galois groups, he uses Lie algebras to measure the symmetries of differential fields. Malgrange [28], [29] has proposed a Galois theory of differential equations where the role of the Galois group is taken by certain groupoids. Both Umemura and Malgrange have indicated to us that their theories can analyze parameterized differential equations as well.

Future directions. There are many questions suggested by the results presented here and we indicate a few of these.

(1) Deligne [15], [16] (see also [40]) has shown that the usual Picard–Vessiot theory can be presented in the language of Tannakian categories. Can one characterize in a similar way the category of representations of linear differential algebraic groups and use this to develop the parameterized Picard–Vessiot theory?[5]

(2) How does the parameterized monodromy sit inside the parameterized Picard–Vessiot groups? To what extent can one extend Ramis' characterization of the local Galois groups to the parameterized case?

(3) Can one develop algorithms to determine the Galois groups of parameterized linear differential equations? Sit [48] has classified the differential algebraic subgroups of SL_2. Can this classification be used to calculate Galois groups of second order parameterized differential equations in analogy to Kovacic's algorithm for second order linear differential equations?

(4) Characterize those linear differential algebraic groups that appear as Galois groups of $k_0(x)$ where k_0 is as in Example 7.1.

9 Appendix

In this Appendix, we present proofs of results that imply Theorem 3.5 and Theorem 3.12. In Section 3, Theorem 3.5 is stated for a parameterized system of *ordinary* linear differential equations but it is no harder to prove an analogous result for parameterized integrable systems of linear partial differential equations and we do this in this appendix. The first section contains a discussion of *constrained extensions*, a concept needed in the proof of the existence of PPV-extensions. In the next three sections, we prove results that simultaneously imply Theorem 2.1 and Theorem 3.5. The proofs are almost, word-for-word, the same as the proofs of the corresponding result for PV-extensions ([39], Ch. 1) once one has taken into account the need for subfields of constants to be differentially closed. Nonetheless we include the proofs for the convenience of the reader. The final section contains a proof of Theorem 3.12.

9.1 Constrained extensions

Before turning to the proof of Theorem 3.5, we shall need some more facts concerning differentially closed fields (see Definition 3.2). If $k \subset K$ are Δ-fields and $\eta = (\eta_1, \dots, \eta_r) \in K^r$, we denote by $k\{\eta\}_\Delta$ (resp. $k\langle\eta\rangle_\Delta$) the Δ-ring (resp. Δ-field) generated by k and $\eta_1, \dots, \eta_r$, that is, the ring (resp. field) generated by k and all the derivatives of the η_i. We shall denote by $k\{y_1, \dots, y_n\}_\Delta$ the ring of differential polynomials in n variables over k (*cf.,* Section 3). A k-Δ-isomorphism of $k\{\eta\}_\Delta$ is a k-isomorphism σ such that $\sigma\partial = \partial\sigma$ for all $\partial \in \Delta$.

[5] Added in proof: Alexey Ovchinnikov has done this and the details will appear in his forthcoming Ph.D. thesis at NC State University.

Definition 9.1. ([21], Ch. III.10; [20]) Let $k \subset K$ be Δ-fields.

(1) We say that a finite family of elements $\eta = (\eta_1, \dots, \eta_r) \subset K^r$ is *constrained* over k if there exist differential polynomials $P_1, \dots, P_s, Q \in k\{y_1, \dots, y_r\}_\Delta$ such that

  (a) $P_1(\eta_1, \dots, \eta_r) = \cdots = P_s(\eta_1, \dots, \eta_r) = 0$ and $Q(\eta_1, \dots, \eta_r) \neq 0$, and

  (b) for any Δ-field E, $k \subset E$, if $(\zeta_1, \dots, \zeta_r) \in E^r$ and $P_1(\zeta_1, \dots, \zeta_r) = \cdots = P_1(\zeta_1, \dots, \zeta_r) = 0$ and $Q(\zeta_1, \dots, \zeta_r) \neq 0$, then the map $\eta_i \mapsto \zeta_i$ induces a k-Δ-isomorphism of $k\{\eta_1, \dots, \eta_r\}_\Delta$ with $k\{\zeta_1, \dots, \zeta_r\}_\Delta$.

  We say that Q is the *constraint* of η over k.

(2) We say K is a *constrained* extension of k if every finite family of elements of K is constrained over k.

(3) We say k is *constrainedly closed* if k has no proper constrained extensions.

The following Proposition contains the facts that we will use:

Proposition 9.2. *Let $k \subset K$ be Δ-fields and $\eta \in K^r$*

(1) *η is* constrained *over k with constraint Q if and only if $k\{\eta, 1/Q(\eta)\}_\Delta$ is a simple Δ-ring,* i.e. *a Δ-ring with no proper nontrivial Δ-ideals.*

(2) *If η is constrained over k and $K = k\langle\eta\rangle_\Delta$, then any finite set of elements of K is constrained over k, that is, K is a constrained extension of k.*

(3) *K is differentially closed if and only if it is constrainedly closed.*

(4) *Every differential field has a constrainedly closed extension.*

One can find the proofs of these in [20], where Kolchin uses the term constrainedly closed instead of differentially closed. Proofs also can be found in [32] where the author uses a model theoretic approach. Item (1) follows from the fact that any maximal Δ-ideal in a ring containing $\mathbb{Q}$ is prime ([21], Ch. I.2, Exercise 3 or [40], Lemma 1.17.1) and that for any radical differential ideal I in $k\{y_1, \dots, y_r\}_\Delta$ there exist differential polynomials $P_1, \dots, P_s$ such that I is the smallest radical differential ideal containing $P_1, \dots, P_s$ (the Ritt–Raudenbusch Theorem [21], Ch. III.4). Item (2) is fairly deep and is essentially equivalent to the fact that the projection of a Kolchin-constructible set (an element in the boolean algebra generated by Kolchin-closed sets) is Kolchin-constructible. Items (3) and (4) require some effort but are not too difficult to prove. Generalizations to fields with noncommuting derivations can be found in [56] and [33].

In the usual Picard–Vessiot theory, one needs the following key fact: Let k be a differential field with algebraically closed subfield of constants C. If R is a simple differential ring, finitely generated over k, then any constant of R is in C (Lemma 1.17, [40]). The following result generalizes this fact and plays a similar role in the parameterized Picard–Vessiot theory. Recall that if k is a $\Delta = \{\partial_0, \dots, \partial_m\}$-field and $\Lambda \subset \Delta$, we denote by C_k^Λ the set $\{c \in k \mid \partial c = 0 \text{ for all } \partial \in \Lambda\}$. One sees that C_k^Λ is a $\Pi = \Delta \backslash \Lambda$-field.

Lemma 9.3. *Let $k \subset K$ be Δ-fields, $\Lambda \subset \Delta$, and $\Pi = \Delta \backslash \Lambda$. Assume that C_k^Λ is Π-differentially closed. If K is a Δ-constrained extension of k, then $C_K^\Lambda = C_k^\Lambda$.*

Proof. Let $\eta \in C_K^\Lambda$. Since K is a Δ-constrained extension of k, there exist $P_1, \dots, P_s$, $Q \in k\{y\}_\Delta$ satisfying the conditions of Definition 9.1 with respect to η and k. We will first show that there exist $P_1, \dots, P_s, Q \in C_k^\Lambda\{y\}_\Delta$ satisfying the conditions of Definition 9.1 with respect to η and k.

Let $\{\beta_i\}_{i \in I}$ be a C_k^Λ-basis of k. Let $R \in k\{y\}_\Delta$ and write $R = \sum R_i \beta_i$ where each $R_i \in C_k^\Lambda\{y\}_\Delta$. Since linear independence over constants is preserved when one goes to extension fields ([21], Ch. II.1), for any differential Δ-extension E of k and $\zeta \in C_E^\Lambda$, we have that $R(\zeta) = 0$ if and only if all $R_i(\zeta) = 0$ for all i. If we write $P_j = \sum P_{i,j} \beta_i$, $Q = \sum Q_i \beta_i$ then there is some i_0 such that η satisfies $\{P_{i,j} = 0\}$, $Q_{i_0} \neq 0$ and that for any $\zeta \in C_E^\Lambda$ that satisfies this system, the map $\eta \mapsto \zeta$ induces a Δ isomorphism of $k\langle\eta\rangle_\Delta$ and $k\langle\zeta\rangle_\Delta$.

We therefore may assume that there exist $P_1, \dots, P_s, Q \in C_k^\Lambda\{y\}_\Delta$ satisfying the conditions of Definition 9.1 with respect to η and k. We now show that there exist $\tilde{P}_1, \dots, \tilde{P}_s, \tilde{Q}$ in the smaller differential polynomial ring $C_k^\Lambda\{y\}_\Pi$ satisfying: If E is a Δ-extension of k and $\zeta \in C_E^\Lambda$ satisfies $\tilde{P}_1(\zeta) = \cdots = \tilde{P}_s(\zeta) = 0$, $\tilde{Q}(\zeta) \neq 0$ then there is a k-Δ-isomorphism of $k\langle\eta\rangle_\Delta$ and $k\langle\zeta\rangle_\Delta$ mapping $\eta \mapsto \zeta$. To do this, note that any $P \in C_k^\Lambda\{y\}_\Delta$ is a C_k^Λ-linear combination of monomials that are products of terms of the form $\partial_0^{i_0} \dots \partial_m^{i_m} y$. We denote by $\tilde{P}$ the differential polynomial resulting from P be deleting any monomial that contains a term $\partial_0^{i_0} \dots \partial_m^{i_m} y_j$ *with* $i_t > 0$ *for some* $\partial_{i_t} \in \Lambda$. Note that for any Δ-extension E of k and $\zeta \in C_E^\Lambda$ we have $P(\zeta) = 0$ if and only if $\tilde{P}(\zeta) = 0$. Therefore, for any $\zeta \in C_E^\Lambda$, if $\tilde{P}_1(\zeta) = \cdots = \tilde{P}_1(\zeta) = 0$ and $\tilde{Q}(\zeta) \neq 0$, then the map $\eta \mapsto \zeta$ induces a Δ-k-isomorphism of $k\{\eta\}_\Delta$ with $k\{\zeta\}_\Delta$.

We now use the fact that C_k^Λ is a Π-differentially closed field to show that any $\eta \in C_K^\Lambda$ must already be in C_k^Λ. Let $\tilde{P}_1, \dots, \tilde{P}_s, \tilde{Q} \in C_k^\Lambda\{y\}_\Pi$ be as above. Since C_k^Λ is a Π-differentially closed field and $\tilde{P}_1 = \cdots = \tilde{P}_s = 0$, $\tilde{Q} \neq 0$ has a solution in *some* Π-extension of C_k^Λ (*e.g.*, $\eta \in C_K^\Lambda$), this system has a solution $\zeta \in C_k^\Lambda \subset k$. We therefore can conclude that the map $\eta \mapsto \zeta$ induces a Π-k-isomorphism from $k\langle\eta\rangle$ to $k\langle\zeta\rangle$. Since $\zeta \in k$, we have that $\eta \in k$ and so $\eta \in C_k^\Lambda$. □

We note that if Π is empty, then Π-differentially closed is the same as algebraically closed. In this case the above result yields the important fact crucial to the Picard–Vessiot theory mentioned before the lemma.

9.2 PPV-extensions

In the next three sections, we will develop the theory of PPV-extensions for parameterized integrable systems of linear differential equations. This section is devoted to showing the existence and uniqueness of these extensions. In Section 9.3 we show that the Galois group has a natural structure as a linear differential algebraic group and in Section 9.4 we show that a PPV-extension can be associated with a torsor for the Galois group. As in the usual Picard–Vessiot theory, these results will allow us to give a complete Galois theory (see Theorem 9.5).

In this and the next three sections, we will make the following conventions. We let k be a Δ-differential field. We designate a nonempty subset $\Lambda = \{\partial_0, \ldots, \partial_r\} \subset \Delta$ and consider a system of linear differential equations

$$\begin{aligned} \partial_0 Y &= A_0 Y \\ \partial_1 Y &= A_1 Y \\ &\vdots \\ \partial_r Y &= A_r Y \end{aligned} \tag{9.1}$$

where the $A_i \in \mathrm{gl}_n(k)$, the set of $n \times n$ matrices with entries in k, such that

$$\partial_i A_j - \partial_j A_i = [A_i, A_j] \tag{9.2}$$

We denote by Π the set $\Delta \backslash \Lambda$. One sees that the derivations of Π leave the field C_k^Λ invariant and we shall think of this latter field as a Π-field. Throughout the next sections, we shall assume that $C = C_k^\Lambda$ *is a* Π*-differentially closed differential field.* The set Λ corresponds to derivations used in the linear differential equations and Π corresponds to the parametric derivations.Throughout the first part of this paper Δ was $\{\partial_0, \ldots, \partial_m\}$, $\Lambda = \{\partial_0\}$, and $\Pi = \{\partial_1, \ldots, \partial_m\}$. We now turn to a definition.

Definition 9.4. (1) A *parameterized Picard–Vessiot ring* (PPV-ring) over k for the equations (9.1) is a Δ-ring R containing k satisfying:

(a) R is a Δ-simple Δ-ring.

(b) There exists a matrix $Z \in \mathrm{GL}_n(R)$ such that $\partial_i Z = A_i Z$ for all $\partial_i \in \Lambda$.

(c) R is generated, as a Δ-ring over k, by the entries of Z and $1/\det(Z)$, *i.e.*, $R = k\{Z, 1/\det(Z)\}_\Delta$.

(2) A *parameterized Picard–Vessiot extension of* k (PPV-extension of k) for the equations (9.1) is a Δ-field K satisfying

(a) $k \subset K$.

(b) There exists a matrix $Z \in \mathrm{GL}_n(K)$ such that $\partial_i Z = A_i Z$ for all $\partial_i \in \Lambda$ and K is generated as a Δ-field over k by the entries of Z.

(c) $C_K^\Lambda = C_k^\Lambda$, *i.e.,* the Λ-constants of K coincide with the Λ-constants of k.

(3) The group $\mathrm{Gal}_\Delta(K/k) = \{\sigma : K \to K \mid \sigma$ is a k-automorphism such that $\sigma\partial = \partial\sigma$ for all $\partial \in \Delta\}$ is called the *parameterized Picard–Vessiot group (PPV-group)* associated with the PPV-extension K of k.

Note that when $\Delta = \Lambda$, $\Pi = \emptyset$ these definitions give us the corresponding definitions in the usual Picard–Vessiot theory.

Our goal in the next three sections is to prove results that will yield the following generalization of both Theorem 2.1 (when $\Delta = \Lambda$) and Theorem 3.5 (when $\Delta = \{\partial_0, \partial_1, \ldots, \partial_m\}$ and $\Lambda = \{\partial_0\}$).

Theorem 9.5. (1) *There exists a PPV-extension K of k associated with* (9.1) *and this is unique up to Δ-k-isomorphism.*

(2) *The PPV-group* $\mathrm{Gal}_\Delta(K/k)$ *equals $G(C_k^\Lambda)$, where G is a linear Π-differential algebraic group defined over C_k^Λ.*

(3) *The map that sends any Δ-subfield F, $k \subset F \subset K$, to the group* $\mathrm{Gal}_\Delta(K/F)$ *is a bijection between Δ-subfields of K containing k and Π-Kolchin closed subgroups of* $\mathrm{Gal}_\Delta(K/k)$. *Its inverse is given by the map that sends a Π-Kolchin closed group H to the field $\{z \in K \mid \sigma(z) = z$ for all $\sigma \in H\}$.*

(4) *A Π-Kolchin closed subgroup H of* $\mathrm{Gal}_\Delta(K/k)$ *is a normal subgroup of* $\mathrm{Gal}_\Delta(K/k)$ *if and only if the field K^H is left set-wise invariant by* $\mathrm{Gal}_\Delta(K/k)$. *If this is the case, the map* $\mathrm{Gal}_\Delta(K/k) \to \mathrm{Gal}_\Delta(K^H/k)$ *is surjective with kernel H and K^H is a PPV-extension of k with PPV-group isomorphic to* $\mathrm{Gal}_\Delta(K/k)/H$. *Conversely, if F is a differential subfield of K containing k and F is a PPV-extension of k, then* $\mathrm{Gal}_\Delta(K/F)$ *is a normal Π-Kolchin closed subgroup of* $\mathrm{Gal}_\Delta(K/k)$.

We shall show in this section that PPV-rings for (9.1) exist and are unique up to Δ-k-isomorphism and that every PPV-extension K of k is the quotient field of a PPV-ring (and therefore is also unique up to Δ-k-isomorphism.) We begin with

Proposition 9.6. (1) *There exists a PPV-ring R for* (9.1) *and it is an integral domain.*

(2) *The field of Λ-constants C_K^Λ of the quotient field K of a PPV-ring over k is C_k^Λ.*

(3) *Any two PPV-rings for this system are k-isomorphic as Δ-rings.*

Proof. (1) Let $(Y_{i,j})$ denote an $n \times n$ matrix of Π-indeterminates and let "det" denote the determinant of $(Y_{i,j})$. We denote by $k\{Y_{1,1}, \ldots, Y_{n.n}, 1/\det\}_\Pi$ the Π-differential polynomial ring in the variables $\{Y_{i,j}\}$ localized at det. We can make this ring into a Δ-ring by setting $(\partial_k Y_{i,j}) = A_k(Y_{i,j})$ for all $\partial_k \in \Lambda$ and using the fact that $\partial_k\partial_l = \partial_l\partial_k$ for all $\partial_k, \partial_l \in \Delta$. Let p be a maximal Δ-ideal in R. One then sees that R/p is a PPV-ring for the equation. Since maximal differential ideals are prime, R is an integral domain.

(2) Let $R = k\{Z, 1/\det(Z)\}_\Delta$. Since this is a simple differential ring, Proposition 9.2 (1) implies that Z is constrained over k with constraint det. Statement (2) of Proposition 9.2 implies that the quotient field of R is a Δ-constrained extension of k. Lemma 9.3 implies that $C_K^\Lambda = C_k^\Lambda$.

(3) Let R_1, R_2 denote two PPV-rings for the system. Let Z_1, Z_2 be the two fundamental matrices. Consider the Δ-ring $R_1 \otimes_k R_2$ with derivations $\partial_i(r_1 \otimes r_2) = \partial_i r_1 \otimes r_2 + r_1 \otimes \partial_i r_2$. Let p be a maximal Δ-ideal in $R_1 \otimes_k R_2$ and let $R_3 = R_1 \otimes_k R_2/p$. The obvious maps $\phi_i : R_i \to R_1 \otimes_k R_2$ are Δ-homomorphisms and, since the R_i are

simple, the homomorphisms ϕ_i are injective. The image of each ϕ_i is differentially generated by the entries of $\phi_i(Z_i)$ and $\det(\phi(Z_i^{-1}))$. The matrices $\phi_1(Z_1)$ and $\phi_2(Z_2)$ are fundamental matrices in R_3 of the differential equation. Since R_3 is simple, the previous result implies that C_k^Λ is the ring of Λ-constants of R_3. Therefore $\phi_1(Z_1) = \phi_2(Z_2)D$ for some $D \in \mathrm{GL}_n(C_k^\Lambda)$. Therefore $\phi_1(R_1) = \phi_2(R_2)$ and so R_1 and R_2 are isomorphic. □

Conclusion (2) of the above proposition shows that the field of fractions of a PPV-ring is a PPV-field. We now show that a PPV-field for an equation is the field of fractions of a PPV-ring for the equation.

Proposition 9.7. *Let K be a PPV-extension field of k for the system* (9.1)*, let $Z \in \mathrm{GL}_n(K)$ satisfy $\partial_i(Z) = A_i Z$ for all $\partial_i \in \Lambda$ and let* $\det = \det(Z)$.

(1) *The Δ-ring $k\{Z, 1/\det\}_\Delta$ is a PPV-extension ring of k for this system.*

(2) *If K' is another PPV-extension of k for this system then there is a k-Δ-isomorphism of K and K'.*

To simplify notation we shall use $\frac{1}{\det}$ to denote the inverse of the determinant of a matrix given by the context. For example, $k\{Y_{i,j}, \frac{1}{\det}\}_\Delta = k\{Y_{i,j}, \frac{1}{\det(Y_{i,j})}\}_\Delta$ and $k\{X_{i,j}, \frac{1}{\det}\}_\Pi = k\{X_{i,j}, \frac{1}{\det(X_{i,j})}\}_\Pi$.

As in [40], p. 16, we need a preliminary lemma to prove this proposition. Let $(Y_{i,j})$ be an $n \times n$ matrix of Π-differential indeterminates and let det denote the determinant of this matrix. For any Π-field k, we denote by $k\{Y_{i,j}, 1/\det\}_\Pi$ the Π-ring of differential polynomials in the $Y_{i,j}$ localized with respect to det. If k is, in addition, a Δ-field, the derivations $\partial \in \Lambda$ can be extended to $k\{Y_{i,j}, 1/\det\}_\Pi$ by setting $\partial(Y_{i,j}) = 0$ for all $\partial \in \Lambda$ and i, j with $1 \le i, j \le n$. In this way $k\{Y_{i,j}, 1/\det\}_\Pi$ may be considered as a Δ-ring. We consider $C_k^\Lambda\{Y_{i,j}, 1/\det\}_\Pi$ as a Π-subring of $k\{Y_{i,j}, 1/\det\}_\Pi$. For any set $I \subset k\{Y_{i,j}, 1/\det\}_\Pi$, we denote by $(I)_\Delta$ the Δ-differential ideal in $k\{Y_{i,j}, 1/\det\}_\Pi$ generated by I.

Lemma 9.8. *Using the above notation, the map $I \mapsto (I)_\Delta$ from the set of Π-ideals of $C_k^\Lambda\{Y_{i,j}, 1/\det\}_\Pi$ to the set of Δ- ideals of $k\{Y_{i,j}, 1/\det\}_\Pi$ is a bijection. The inverse map is given by $J \mapsto J \cap C_k^\Lambda\{Y_{i,j}, 1/\det\}_\Pi$.*

Proof. If $\mathcal{S} = \{s_\alpha\}_{\alpha \in \mathrm{A}}$ is a basis of k over C_k^Λ, then $\mathcal{S}$ is a module basis of $k\{Y_{i,j}, \frac{1}{\det}\}_\Pi$ over $C_k^\Lambda\{Y_{i,j}, \frac{1}{\det}\}_\Pi$. Therefore, for any ideal I of $C_k^\Lambda\{Y_{i,j}, \frac{1}{\det}\}_\Pi$, one has that $(I)_\Delta \cap C_k^\Lambda\{Y_{i,j}, \frac{1}{\det}\}_\Pi = I$.

We now prove that any Δ-differential ideal J of $k\{Y_{i,j}, \frac{1}{\det}\}_\Pi$ is generated by $I := J \cap C_k^\Lambda\{Y_{i,j}, \frac{1}{\det}\}_\Pi$. Let $\{e_\beta\}_{\beta \in \mathcal{B}}$ be a basis of $C_k^\Lambda\{Y_{i,j}, \frac{1}{\det}\}_\Pi$ over C_k^Λ. Any element $f \in J$ can be uniquely written as a finite sum $\sum_{\beta \in \mathcal{B}} m_\beta e_\beta$ with the $m_\beta \in k$. By induction on the length, $l(f)$, of f we will show that $f \in (I)_\Delta$. When $l(f) = 0, 1$, the result is clear. Assume $l(f) > 1$. We may suppose that $m_{\beta_1} = 1$ for some

$\beta_1 \in \mathcal{B}$ and $m_{\beta_2} \in k\backslash C_k^\Lambda$ for some $\beta_2 \in \mathcal{B}$. One then has that, for any $\partial \in \Lambda$, $\partial f = \sum_\beta \partial m_\beta e_\beta$ has a length smaller than $l(f)$ and so belongs to $(I)_\Delta$. Similarly $\partial(m_{\beta_2}^{-1} f) \in (I)_\Delta$. Therefore $\partial(m_{\beta_2}^{-1})f = \partial(m_{\beta_2}^{-1} f) - m_{\beta_2}^{-1}\partial f \in (I)_\Delta$. Since C_k^Λ is the field of Λ-constants of k, one has $\partial_i(m_{\beta_2}^{-1}) \neq 0$ for some $\partial_i \in \Lambda$ and so $f \in (I)_\Delta$. $\square$

Proof of Proposition 9.7. (1) Let $R_0 = k\left\{X_{i,j}, \frac{1}{\det}\right\}_\Pi$ be the ring of Π-differential polynomials over k and define a Δ-structure on this ring by setting $(\partial_i X_{i,j}) = A_i(X_{i,j})$ for all $\partial_i \in \Lambda$. Consider the Δ-rings $R_0 \subset K \otimes_k R_0 = K\left\{X_{i,j}, \frac{1}{\det}\right\}_\Pi$. Define a set of n^2 new variables $Y_{i,j}$ by $(X_{i,j}) = Z \cdot (Y_{i,j})$. Then $K \otimes_k R_0 = K\left\{Y_{i,j}, \frac{1}{\det}\right\}_\Pi$ and $\partial Y_{i,j} = 0$ for all $\partial \in \Lambda$ and all i, j. We can identify $K \otimes_k R_0$ with $K \otimes_{C_k^\Lambda} R_1$ where $R_1 := C_k^\Lambda\left\{Y_{i,j}, \frac{1}{\det}\right\}_\Pi$. Let P be a maximal Δ-ideal of R_0. P generates an ideal in $K \otimes_k R_0$ which is denoted by (P). Since $K \otimes R_0/(P) \cong K \otimes (R_0/P) \neq 0$, the ideal (P) is a proper differential ideal. Define the ideal $\tilde{P} \subset R_1$ by $\tilde{P} = (P) \cap R_1$. By Lemma 9.8 the ideal (P) is generated by $\tilde{P}$. If M is a maximal Π-ideal of R_1 containing $\tilde{P}$ then R_1/M is a simple, finitely generated Π-extension of C_k^Λ and so is a constrained extension of C_k^Λ. Since C_k^Λ is differentially closed, Proposition 9.2 (3) implies that $R_1/M = C_k^\Lambda$. The corresponding homomorphism of C_k^Λ-algebras $R_1 \to C_k^\Lambda$ extends to a differential homomorphism of K-algebras $K \otimes_{C_k^\Lambda} R_1 \to K$. Its kernel contains $(P) \subset K \otimes_k R_0 = K \otimes_{C_k^\Lambda} R_1$. Thus we have found a k-linear differential homomorphism $\psi: R_0 \to K$ with $P \subset \ker(\psi)$. The kernel of ψ is a differential ideal and so $P = \ker(\psi)$. The subring $\psi(R_0) \subset K$ is isomorphic to R_0/P and is therefore a PPV-ring. The matrix $(\psi(X_{i,j}))$ is a fundamental matrix in $\mathrm{GL}_n(K)$ and must have the form $Z \cdot (c_{i,j})$ with $(c_{i,j}) \in \mathrm{GL}_n(C_k^\Lambda)$, because the field of Λ-constants of K is C_k^Λ. Therefore, $k\{Z, 1/\det\}_\Delta$ is a PPV-extension of k.

(2) Let K' be a PPV-extension of k for $\partial_0 Y = AY$. Part (1) of this proposition implies that both K' and K are quotient fields of PPV-rings for this equation. Proposition 9.6 implies that these PPV-rings are k-Δ-isomorphic and the conclusion follows. $\square$

The following result was used in Proposition 3.9.

Lemma 9.9. *Let* $\Delta = \{\partial_0, \partial_1, \ldots, \partial_m\}$ *and* $\Lambda = \{\partial_0\}$. *Let*

$$\begin{aligned} \partial_0 Y &= AY \\ \partial_1 Y &= A_1 Y \\ &\vdots \\ \partial_m Y &= A_m Y \end{aligned} \tag{9.3}$$

be an integrable system with $A_i \in \mathrm{gl}_n(k)$. *If* K *is a PV-extension of* k *for* (9.3), *then* K *is a PPV-extension of* k *for* $\partial_0 Y = AY$

Proof. We first note that C_k^Δ is a subfield of C_k^Λ. Since this latter field is differentially closed, it is algebraically closed. Therefore, C_k^Δ is also algebraically closed. The usual Picard–Vessiot theory[6] implies that K is the quotient field of the Picard–Vessiot ring $R = k\{Z, 1/\det Z\}_\Delta$ where Z satisfies the system (9.3). Since R is a simple Δ-ring, we have that Z is constrained over k, Proposition 9.2 (2) implies that K is a Δ-constrained extension of k. Since C_k^Λ is differentially closed, Lemma 9.3 implies that $C_K^{\partial_0} = C_k^{\partial_0}$ so K is a PPV-extension of k. □

9.3 Galois groups

In this section we shall show that the PPV-group $\mathrm{Gal}_\Delta(K/k)$ of a PPV-extension K of k is a linear differential algebraic group and also show the correspondence between Kolchin-closed subgroups of $\mathrm{Gal}_\Delta(K/k)$ and Δ-subfields of K containing k. This is done in the next Proposition and conclusions (2) and (3) of Theorem 3.5 are immediate consequences.

To make things a little more precise, we will use a little of the language of affine differential algebraic geometry (see [9] or [22] for more details). We begin with some definitions that are the obvious differential counterparts of the usual definitions in affine algebraic geometry. Let k be a Δ-field. An *affine differential variety* V defined over k is given by a radical differential ideal $I \subset k\{Y_1, \ldots, Y_n\}_\Delta$. In this case, we shall say V is a differential subvariety of affine n-space and write $V \subset \mathbb{A}^n$. We will identify V with its *coordinate ring* $k\{V\} = k\{Y_1, \ldots, Y_n\}_\Delta/I$. Conversely, given a reduced Δ-ring R that is finitely generated (in the differential sense) as a k-algebra, we may associate with it the differential variety V defined by the radical differential ideal I where $R = k\{Y_1, \ldots, Y_n\}_\Delta/I$. Given any Δ-field $K \supset k$, the set of K-points of V, denoted by $V(K)$, is the set of points of K^n that are zeroes of the defining ideal of V, and may be identified with the set of k-Δ-homomorphisms of $k\{V\}$ to K. If $V \subset \mathbb{A}^n$ and $W \subset \mathbb{A}^p$ are affine differential varieties defined over k, a *differential polynomial map* $f: V \to W$ is given by a p-tuple $(f_1, \ldots, f_p) \in (k\{Y_1, \ldots, Y_n\}_\Delta)^p$ such that the map that sends an $F \in k\{Y_1, \ldots, Y_p\}_\Delta$ to $F(f_1, \ldots, f_p) \in k\{Y_1, \ldots, Y_n\}_\Delta$ induces a k-Δ-homomorphism f^* of $k\{W\}$ to $k\{V\}$. A useful criterion for showing that a p-tuple $(f_1, \ldots, f_p) \in (k\{Y_1, \ldots, Y_n\}_\Delta)^p$ defines a differential polynomial map from V to W is the following: $(f_1, \ldots, f_p)$ *defines a differential polynomial map from V to W if and only if for any Δ-field $K \supset k$ and any $v \in V(K)$, we have* $(f_1(v), \ldots, f_p(v)) \in W(K)$. This is an easy consequence of the *theorem of zeros* ([21], Ch. IV.2) which in turn is an easy consequence of the fact that a radical differential ideal is the intersection of prime differential ideals.

Given affine differential varieties V and W defined over k, we define the *product* $V \times_k W$ *of V and W* to be the differential affine variety associated with $k\{V\} \otimes_k k\{W\}$. Note that since our fields have characteristic zero, this latter ring is reduced.

[6] Proposition 1.22 of [40] proves this only for the ordinary case. Proposition 9.7 above yields this result if we let $\Lambda = \Delta$.

In this setting, a linear differential algebraic group G (defined over k) is the affine differential algebraic variety associated with a radical differential ideal $I \subset k\{Y_{1,1,}, \ldots, Y_{n,n}, Z\}_\Delta$ such that

(1) $1 - Z \cdot \det((Y_{i,j})) \in I$,

(2) $(\mathrm{id}, 1) \in G(k)$ where id is the $n \times n$ identity matrix.

(3) the map given by matrix multiplication

$$(g, (\det g)^{-1})(h, (\det h)^{-1}) \mapsto (gh, (\det(gh))^{-1})$$

(which is obviously a differential polynomial map) is a map from $G \times G$ to G and the inverse map $(g, (\det g)^{-1}) \mapsto (g^{-1}, \det g)$ (also a differential polynomial map) is a map from G to G.

Since we assume that $1 - Z \cdot \det((Y_{i,j})) \in I$, we may assume that G is defined by a radical differential ideal in the ring $k\{Y_{1,1,}, \ldots, Y_{n,n}, 1/\det(Y_{i,j})\}_\Delta$, which we abbreviate as $k\{Y, 1/\det Y\}_\Delta$. In this way, for any $K \supset k$ we may identify $G(K)$ with elements of $\mathrm{GL}_n(K)$ and the multiplication and inversion is given by the usual operations on matrices. We also note that the usual Hopf algebra definition of a linear algebraic group carries over to this setting as well. See [10] for a discussion of k-differential Hopf algebras, and criteria for an affine differential algebraic group to be linear.

Proposition 9.10. *Let $K \supset k$ be a PPV-field with differential Galois group* $\mathrm{Gal}_\Delta(K/k)$. *Then*

(1) $\mathrm{Gal}_\Delta(K/k)$ *is the group of C_k^Λ-points $G(C_k^\Lambda) \subset \mathrm{GL}_n(C_k^\Lambda)$ of a linear Π-differential algebraic group G over C_k^Λ.*

(2) *Let H be a subgroup of* $\mathrm{Gal}_\Delta(K/k)$ *satisfying $K^H = k$. Then the Kolchin closure $\bar{H}$ of H is* $\mathrm{Gal}_\Delta(K/k)$.

(3) *The field $K^{\mathrm{Gal}_\Delta(K/k)}$ of* $\mathrm{Gal}_\Delta(K/k)$*-invariant elements of the Picard–Vessiot field K is equal to k.*

Proof. (1) We shall show that there is a radical Π-ideal $I \subset S = C_k^\Lambda\{Y_{i,j}, \frac{1}{\det}\}_\Pi$ such that S/I is the coordinate ring of a linear Π-differential algebraic group G and $\mathrm{Gal}_\Delta(K/k)$ corresponds to $G(C_k^\Lambda)$.

Let K be the PPV-extension for the integrable system (9.1). Once again we denote by $k\{X_{i,j}, \frac{1}{\det}\}_\Pi$ the Π-differential polynomial ring with the added Δ-structure defined by $(\partial_r X_{i,j}) = A_r(X_{i,j})$ for $\partial_r \in \Lambda$. K is the field of fractions of $R := k\{X_{i,j}, \frac{1}{\det}\}_\Pi/q$, where q is a maximal Δ-ideal. Let $r_{i,j}$ be the image of $X_{i,j}$ in R so $(r_{i,j})$ is a fundamental matrix for the equations $\partial_i Y = A_i Y$, $\partial_i \in \Lambda$. Consider the following rings:

$$k\{X_{i,j}, \tfrac{1}{\det}\}_\Pi \subset K\{X_{i,j}, \tfrac{1}{\det}\}_\Pi = K\{Y_{i,j}, \tfrac{1}{\det}\}_\Pi \supset C_k^\Lambda\{Y_{i,j}, \tfrac{1}{\det}\}_\Pi$$

where the indeterminates $Y_{i,j}$ are defined by $(X_{i,j}) = (r_{i,j})(Y_{i,j})$. Note that $\partial Y_{i,j} = 0$ for all $\partial \in \Pi$. Since all fields are of characteristic zero, the ideal $qK\{Y_{i,j}, \frac{1}{\det}\}_\Pi \subset$

$K\{X_{i,j}, \frac{1}{\det}\}_\Pi = K\{Y_{i,j}, \frac{1}{\det}\}_\Pi$ is a radical Δ-ideal (*cf.,* [40], Corollary A.16). It follows from Lemma 9.8 that $qL[Y_{i,j}, \frac{1}{\det}]$ is generated by $I = qK\{Y_{i,j}, \frac{1}{\det}\}_\Pi \cap C_k^\Lambda\{Y_{i,j}, \frac{1}{\det}\}_\Pi$. Clearly I is a radical Δ-ideal of $S = C_k^\Lambda\{Y_{i,j}, \frac{1}{\det}\}_\Pi$. We shall show that S/I is the Π-coordinate ring of a linear differential algebraic group G, inheriting its group structure from GL_n. In particular, we shall show that $G(C_k^\Lambda)$ is a subgroup of $\mathrm{GL}_n(C_k^\Lambda)$ and that there is an isomorphism of $\mathrm{Gal}_\Delta(K/k)$ onto $G(C_k^\Lambda)$.

$\mathrm{Gal}_\Delta(K/k)$ can be identified with the set of $(c_{i,j}) \in \mathrm{GL}_n(C_k^\Lambda)$ such that the map $(X_{i,j}) \mapsto (X_{i,j})(c_{i,j})$ leaves the ideal q invariant. One can easily show that the following statements are equivalent:

(i) $(c_{i,j}) \in \mathrm{Gal}_\Delta(K/k)$,

(ii) The map $k\{X_{i,j}, \frac{1}{\det}\}_\Pi \to K$ defined by $(X_{i,j}) \mapsto (r_{i,j})(c_{i,j})$ maps all elements of q to zero.

(iii) The map $K\{X_{i,j}, \frac{1}{\det}\}_\Pi \to K$ defined by $(X_{i,j}) \mapsto (r_{i,j})(c_{i,j})$ maps all elements of $qK\{X_{i,j}, \frac{1}{\det}\}_\Pi = qK\{Y_{i,j}, \frac{1}{\det}\}_\Pi$ to zero.

(iv) Considering $qK\{Y_{i,j}, \frac{1}{\det}\}_\Pi$ as an ideal of $K\{X_{i,j}, \frac{1}{\det}\}_\Pi$, the map

$$K\{Y_{i,j}, \tfrac{1}{\det}\}_\Pi \to K, \quad (Y_{i,j}) \mapsto (c_{i,j}),$$

sends all elements of $qK\{Y_{i,j}, \frac{1}{\det}\}_\Pi$ to zero.

Since the ideal $qK\{Y_{i,j}, \frac{1}{\det}\}_\Pi$ is generated by I, the last statement above is equivalent to $(c_{i,j})$ being a zero of the ideal I, i.e., $(c_{i,j}) \in G(C_k^\Lambda)$. Since $\mathrm{Gal}_\Delta(K/k)$ is a group, the set $G(C_k^\Lambda)$ is a subgroup of $\mathrm{GL}_n(C_k^\Lambda)$. Therefore G is a linear differential algebraic group.

(2) Assuming that $\bar{H} \neq \mathrm{Gal}_\Delta$, we shall derive a contradiction. We shall use the notation of part (1) above. If $\bar{H} \neq \mathrm{Gal}_\Delta$, then there exists an element $P \in C_k^\Lambda\{Y_{i,j}, \frac{1}{\det}\}_\Pi$ such that $P \notin I$ and $P(h) = 0$ for all $h \in H$. Lemma 9.8 implies that $P \notin (I) = qk\{Y_{i,j}, \frac{1}{\det}\}_\Pi$. Let $T = \{Q \in K\{X_{i,j}, \frac{1}{\det}\}_\Pi \mid Q \notin (I) \text{ and } Q((r_{i,j})(h_{i,j})) = 0 \text{ for all } h = (h_{i,j}) \in H\}$. Since $K\{X_{i,j}, \frac{1}{\det}\}_\Pi = K\{Y_{i,j}, \frac{1}{\det}\}_\Pi \supset C_k^\Lambda\{Y_{i,j}, \frac{1}{\det}\}_\Pi$ we have that $T \neq \{0\}$. Any element of $K\{X_{i,j}, \frac{1}{\det}\}_\Pi$ may be written as $\sum_\alpha f_\alpha Q_\alpha$ where $f_\alpha \in K$ and $Q_\alpha \in k\{X_{i,j}, \frac{1}{\det}\}_\Pi$. Select $Q = f_{\alpha_1}Q_{\alpha_1} + \cdots + f_{\alpha_m}Q_{\alpha_m} \in T$ with the f_{α_i} all nonzero and m minimal. We may assume that $f_{\alpha_1} = 1$. For each $h \in H$, let $Q^h = f_{\alpha_1}^h Q_{\alpha_1} + \cdots + f_{\alpha_m}^h Q_{\alpha_m}$. One sees that $Q^h \in T$. Since $Q - Q^h$ is shorter than Q and satisfies $(Q - Q^h)((r_{i,j})(h_{i,j})) = 0$ for all $h = (h_{i,j}) \in H$ we must have that $Q - Q^h \in (I)$. If $Q - Q^h \neq 0$ then there exists an $l \in K$ such that $Q - l(Q - Q^h)$ is shorter than Q. One sees that $Q - l(Q - Q^h) \in T$ yielding a contradiction unless $Q - Q^h = 0$. Therefore $Q = Q^h$ for all $h \in H$ and so the $f_{\alpha_i} \in k$. We conclude that $Q \in k\{X_{i,j}, \frac{1}{\det}\}_\Pi$. Since $Q(r_{i,j}) = 0$ we have that $Q \in q$, a contradiction.

(3) Let $a = \frac{b}{c} \in K \backslash k$ with $b, c \in R$ and $d = b \otimes c - c \otimes b \in R \otimes_k R$. Elementary properties of tensor products imply that $d \neq 0$ since b and c are linearly independent over C_k^Λ. The ring $R \otimes_k R$ has no nilpotent elements since the characteristic of k is zero

(*cf.,* [40], Lemma A.16). We define a Δ-ring structure on $R \otimes_k R$ by letting $\partial(r_1 \otimes r_2) = \partial(r_1) \otimes r_2 + r_1 \otimes \partial(r_2)$ for all $\partial \in \Delta$. Let J be a maximal differential ideal in the differential ring $(R \otimes_k R)\left[\frac{1}{d}\right]$. Consider the two obvious morphisms $\phi_i \colon R \to N := (R \otimes_k R)\left[\frac{1}{d}\right]/J$. The images of the ϕ_i are generated (over k) by fundamental matrices of the same matrix differential equation. Therefore both images are equal to a certain subring $S \subset N$ and the maps $\phi_i \colon R \to S$ are isomorphisms. This induces an element $\sigma \in G$ with $\phi_1 = \phi_2 \sigma$. The image of d in N is equal to $\phi_1(b)\phi_2(c) - \phi_1(c)\phi_2(b)$. Since the image of d in N is nonzero, one finds $\phi_1(b)\phi_2(c) \neq \phi_1(c)\phi_2(b)$. Therefore $\phi_2((\sigma b)c) \neq \phi_2((\sigma c)b)$ and so $(\sigma b)c \neq (\sigma c)b$. This implies $\sigma\left(\frac{b}{c}\right) \neq \frac{b}{c}$. □

We have therefore completed proof of parts (2) and (3) of Theorem 9.5.

9.4 PPV-rings and torsors

In this section we will prove conclusion (4) of Theorem 9.5. As in the usual Picard–Vessiot theory, this depends on identifying the PPV-extension ring as the coordinate ring of a torsor of the PPV-group.

Definition 9.11. Let k be a Π-field and G a linear differential algebraic group defined over k. A *G-torsor (defined over k)* is an affine differential algebraic variety V defined over k together with a differential polynomial map $f \colon V \times_k G \to V \times_k V$ (denoted by $f \colon (v, g) \mapsto (vg, v)$) such that

(1) for any Π-field $K \supset k$, $v \in V(K)$, $g, g_1, g_2 \in G(K)$, $v1_G = v$, $v(g_1 g_2) = (vg_1)g_2$ and

(2) the associated homomorphism $k\{V\} \otimes_k k\{V\} \to k\{V\} \otimes_k k\{G\}$ is an isomorphism (or equivalently, for any $K \supset k$, the map $V(K) \times G(K) \to V(K) \times V(K)$ is a bijection.

We note that $V = G$ is a torsor for G over k with the action given by multiplication. This torsor is called the *trivial torsor over k*. We shall use the following notation. If V is a differential affine variety defined over k with coordinate ring $R = k\{V\}$ and $K \supset k$ we denote by V_K the differential algebraic variety (over K) whose coordinate ring is $R \otimes_k K = K\{V\}$.

We again consider the integrable system (9.1) over the Δ-field k. The PPV-ring for this equation has the form $R = k\left\{X_{i,j}, \frac{1}{\det}\right\}_\Pi / q$, where q is a maximal Δ-ideal. In the following, we shall think of q as only a Π-differential ideal. We recall that $k\left\{X_{i,j}, \frac{1}{\det}\right\}_\Pi$ is the coordinate ring of the linear Π-differential algebraic group GL_n over k. Let V be the affine differential algebraic variety associated with the ring $k\left\{X_{i,j}, \frac{1}{\det}\right\}_\Pi / q$. This is an irreducible and reduced Π-Kolchin-closed subset of GL_n. Let K denote the field of fractions of $k\left\{X_{i,j}, \frac{1}{\det}\right\}_\Pi / q$. We have shown in the previous section that the PPV-group $\mathrm{Gal}_\Delta(K/k)$ of this equation may be identified with $G(C_k^\Lambda)$, that is the C_k^Λ-points of a Π-linear differential algebraic group G over C_k^Λ. We recall how G was defined. Consider the following rings

$$k\left\{X_{i,j}, \tfrac{1}{\det}\right\}_\Pi \subset K\left\{X_{i,j}, \tfrac{1}{\det}\right\}_\Pi = K\left\{Y_{i,j}, \tfrac{1}{\det}\right\}_\Pi \supset C_k^\Lambda\left\{Y_{i,j}, \tfrac{1}{\det}\right\}_\Pi,$$

where the relation between the variables $X_{i,j}$ and the variables $Y_{i,j}$ is given by $(X_{i,j}) = (r_{i,j})(Y_{i,j})$. The $r_{a,b} \in K$ are the images of $X_{a,b}$ in $k\left\{X_{i,j}, \tfrac{1}{\det}\right\}_\Pi/q \subset K$. In Proposition 9.10 we showed that the ideal $I = qK\left\{X_{i,j}, \tfrac{1}{\det}\right\}_\Pi \cap C_k^\Lambda\left\{Y_{i,j}, \tfrac{1}{\det}\right\}_\Pi$ defines G. This observation is the key to showing the following.

Proposition 9.12. *V is a G-torsor over k.*

Proof. Let E be a Δ-field containing k. The group $G(C_k^\Lambda) \subset \mathrm{GL}_n(C_k^\Lambda)$ is precisely the set of matrices $(c_{i,j})$ such that the map $(X_{i,j}) \mapsto (X_{i,j})(c_{i,j})$ leaves the ideal q stable. In particular, for $(c_{i,j}) \in G(C_k^\Lambda)$, $(\bar{z}_{i,j}) \in V(E)$ we have that $(\bar{z}_{i,j})(c_{i,j}) \in V(E)$. We will first show that this map defines a morphism from $V \times G_k \to V$. The map is clearly defined over k so we need only show that for any $(\bar{c}_{i,j}) \in G(E)$, $(\bar{z}_{i,j}) \in V(E)$ we have that $(\bar{z}_{i,j})(\bar{c}_{i,j}) \in V(E)$. Assume that this is not true and let $(\bar{c}_{i,j}) \in G(E)$, $(\bar{z}_{i,j}) \in V(E)$ be such that $(\bar{z}_{i,j})(\bar{c}_{i,j}) \notin V(E)$. Let f be an element of q such that $f((\bar{z}_{i,j})(\bar{c}_{i,j})) \neq 0$. Let $\{\alpha_s\}$ be a basis of E considered as a vector space over C_k^Λ and let $f((\bar{z}_{i,j})(C_{i,j})) = \sum_{\alpha_s} \alpha_s f_{\alpha_s}((C_{i,j}))$ where the $C_{i,j}$ are indeterminates and the $f_{\alpha_s}((C_{i,j})) \in C_k^\Lambda\{C_{1,1}, \ldots, C_{n,n}\}_\Lambda$. By assumption (and the fact that linear independence over constants is preserved when one goes to extension fields), we have that there is an α_s such that $f_{\alpha_s}((\bar{c}_{i,j})) \neq 0$. Since C_k^Λ is a Π-differentially closed field, there must exist $(c_{i,j}) \in G(C_k^\Lambda)$ such that $f_{\alpha_s}(c_{i,j}) \neq 0$. This contradicts the fact that $f((\bar{z}_{i,j})(c_{i,j})) = 0$.

Therefore the map $(V \times_k G_k)(E) \to V(E)$ defined by $(z, g) \mapsto zg$ defines a morphism $V \times_k G_k \to V$. At the ring level, this morphism corresponds to a homomorphism of rings

$$\begin{aligned} k\left\{X_{i,j}, \tfrac{1}{\det}\right\}_\Pi/q &\to k\left\{X_{i,j}, \tfrac{1}{\det}\right\}_\Pi/q \otimes_{C_k^\Lambda} C_k^\Lambda\left[Y_{i,j}, \tfrac{1}{\det}\right]/I \\ &\simeq k\left\{X_{i,j}, \tfrac{1}{\det}\right\}_\Pi/q \otimes_k \left(k \otimes_{C_k^\Lambda} C_k^\Lambda\left\{Y_{i,j}, \tfrac{1}{\det}\right\}_\Pi/I\right) \end{aligned}$$

where the map is induced by $(X_{i,j}) \mapsto (r_{i,j})(Y_{i,j})$. We have to show that the morphism $f\colon V \times_k G_k \to V \times_k V$, given by $(z, g) \mapsto (zg, z)$ is an isomorphism of differential algebraic varieties over k. In terms of rings, we have to show that the k-algebra homomorphism $f^*\colon k\{V\} \bigotimes_k k\{V\} \to k\{V\} \otimes_{C_k^\Lambda} k\{G\}$ is an isomorphism. To do this it suffices to find a Π-field extension k' of k such that $1_{k'} \otimes_k f^*$ is an isomorphism. For this it suffices to find Π-field extension k' of k such that $V_{k'}$ is isomorphic to $G_{k'}$ as a $G_{k'}$-torsor over k' that is, for some field extension $k' \supset k$, the induced morphism of varieties over k', namely $V_{k'} \times_{k'} G'_k \to V_{k'}$, makes $V_{k'}$ into a trivial G-torsor over k'.

Let $k' = K$, the PPV-extension of k for the differential equation. We have already shown that $I = qK\{X_{i,j}, \frac{1}{\det}\}_\Pi \cap k_0\{Y_{i,j}, \frac{1}{\det}\}_\Pi$ and this fact implies that

$$\begin{aligned} K\{V\} &= K \otimes_k (k\{X_{i,j}, \tfrac{1}{\det}\}_\Pi/q) \\ &\cong K \otimes_{C_k^\Lambda} \left(C_k^\Lambda\{Y_{i,j}, \tfrac{1}{\det}\}_\Pi/I\right) = K \otimes_{C_k^\Lambda} C_k^\Lambda\{G\} = K\{G\} \end{aligned} \tag{9.4}$$

In other words, we found an isomorphism $h\colon V_K \cong G_K$. We still have to verify that V_K as a G torsor over K is, via h, isomorphic to the trivial torsor $G \times_{C_k^\Lambda} G_K \to G_K$. To do this it is enough to verify that the following diagram is commutative and we leave this to the reader. The coordinate ring $C_k^\Lambda\{G\}$ of the group appears in several places. To keep track of the variables, we will write $C_k^\Lambda\{G\}$ as $C_k^\Lambda\{T_{i,j}, \frac{1}{\det}\}_\Pi/\tilde{I}$ where $\tilde{I}$ is the ideal I with the variables $Y_{i,j}$ replaced by $T_{i,j}$.

$$\begin{tikzcd} K \otimes_k k\{X_{i,j}, \frac{1}{\det}\}_\Pi/q \arrow[r] \arrow[d, "(X_{i,j})\mapsto(r_{i,j})(Y_{i,j})"'] & K\{X_{i,j}, \frac{1}{\det}\}_{\Pi'}/qK\{X_{i,j}, \frac{1}{\det}\}_\Pi \otimes_{C_k^\Lambda} C_k^\Lambda\{G\} \arrow[d, "(X_{i,j})\mapsto(r_{i,j})(Y_{i,j})"] \\ K \otimes_{C_k^\Lambda} C_k^\Lambda\{Y_{i,j}, \frac{1}{\det}\}_\Pi/I \arrow[r] & K\{Y_{i,j}, \frac{1}{\det}\}_\Pi/(I)_\Pi \otimes_{C_k^\Lambda} C_k^\Lambda\{G\} \end{tikzcd}$$

□

In the above diagram, the top arrow represents the map $(X_{i,j}) \mapsto (X_{i,j})(T_{i,j})$ and the bottom arrow represents the map $(Y_{i,j}) \mapsto (Y_{i,j})(T_{i,j})$. Using this result (and its proof), we can now finish the proof of Theorem 9.5 by proving conclusion (4) of this theorem. As in the usual Picard–Vessiot theory, the proof depends on the following group theoretic facts. Let G be a linear differential algebraic group defined over a Π-differentially closed field C_k^Λ. For any $g \in G$ the map $\rho_g\colon G \to G$ given by $\rho_g(h) = hg$ is a differential polynomial isomorphism of G onto G and therefore corresponds to an isomorphism $\rho_g^*\colon C_k^\Lambda\{G\} \to C_k^\Lambda\{G\}$. In this way G acts on the ring $k\{G\}$. Let H be a normal linear differential algebraic subgroup of G. The following facts follow from results of [9] and [22]:

(1) The G-orbit $\{\rho_g^*(f) \mid g \in G(C_k^\Lambda)\}$ of any $f \in C_k^\Lambda\{G\}$ spans a finite dimensional C_k^Λ-vector space.

(2) The group G/H has the structure of a linear differential algebraic group (over C_k^Λ) and its coordinate ring $C_k^\Lambda\{G/H\}$ is isomorphic to the ring of H-invariants $C_k^\Lambda\{G\}^H$.

(3) The two rings $Qt(C_k^\Lambda\{G\})^H$ and $Qt(C_k^\Lambda\{G\}^H)$ are naturally Π-isomorphic, where $Qt(\,\cdot\,)$ denotes the total quotient ring.

We now can prove

Proposition 9.13. *Let K be a PPV-extension of k with Galois group G and let H be a normal Kolchin-closed subgroup. Then K^H is a PPV-extension of k.*

Proof. Let K be the quotient field of the PPV-ring $R = k\{Z, \frac{1}{\det}\}$. As we have already noted (*cf.*, (9.4)), we have

$$K \otimes_k R \cong K \otimes_{C_k^\Lambda} C_k^\Lambda\{G\}$$

that is, the torsor corresponding to R becomes trivial over K. The group G acts on $K \otimes_{C_k^\Lambda} C_k^\Lambda\{G\}$ by acting trivially on the left factor and via ρ^* on the right factor, or trivially on the left factor and with the Galois action on the right factor. In this way we have that $K \otimes_k R^H \cong K \otimes_{C_k^\Lambda} C_k^\Lambda\{G\}^H = K \otimes_{C_k^\Lambda} C_k^\Lambda\{G/H\}$ and that $K \otimes_k K^H \cong K \otimes_{C_k^\Lambda} Qt(k\{G\}^H)$ by the items enumerated above.

We now claim that R^H is finitely generated as a Π-ring over k, hence as a Δ-ring over k. Since $C_k^\Lambda\{G/H\}$ is a finitely generated Π-C_k^Λ-algebra, we have that there exist $f_1, \dots, f_s \in R^H$ that generate $K \otimes_k R^H$ as a Π-K-algebra. We claim that $f_1, \dots, f_s$ generate R^H as a Π-k-algebra. Let $\mathcal{M}$ be a k-basis of $k\{f_1, \dots, f_s\}_\Pi$. By assumption, any element of $f \in K \otimes_k R^H = K \otimes_k k\{f_1, \dots, f_s\}_\Pi$ can be written uniquely as $f = \sum_{u \in \mathcal{M}} a_u \otimes u$ where $a_u \in K$. The Galois group $G(C_k^\Lambda)$ of K over k also acts on $K \otimes_k R^H$ by acting as differential automorphisms of the left factor and trivially on the right factor. Write $1 \otimes f \in 1 \otimes R^H \subset K \otimes_k R^H$ as $1 \otimes f = \sum_{u \in \mathcal{M}} a_u \otimes u$ where $a_u \in K$. Applying $\sigma \in G(C_k^\Lambda)$ to $1 \otimes f$ we have $1 \otimes f = \sum_{u \in \mathcal{M}} \sigma(a_u) \otimes u$. Therefore $\sigma(a_u) = a_u$ for all $\sigma \in G(C_k^\Lambda)$. The parameterized Galois theory implies that $a_u \in k$ for all u. Therefore $f \in k\{f_1, \dots, f_s\}_\Pi$ and so $R^H = k\{f_1, \dots, f_s\}_\Pi$.

Using item (1) in the above list, we may assume that $f_1, \dots, f_s$ form a basis of a $G/H(C_k^\Lambda)$ invariant C_k^Λ-vector space. Let Θ be the free commutative semigroup generated by the elements of Λ. By Theorem 1, Chapter II of [21] (or Lemma D.11 of [40]), there exist $\theta_1 = 1, \dots, \theta_s \in \Theta$ such that

$$W = (\theta_i(f_j))_{1 \le i \le s, 1 \le j \le s}$$

is invertible. For each $\partial_i \in \Lambda$, we have that $A_i = (\partial_i W)W^{-1}$ is left invariant by the action of $G/H(k_0)$. Therefore each $A_i \in \mathrm{gl}_n(k)$. Furthermore, the A_i satisfy the integrability conditions. We have that K^H is generated as a Δ-field over k by the entries of W. Since the constants of K^H are C_k^Λ, we have that K^H is a PPV-field for the system $\partial_i Y = A_i Y$, $\partial_i \in \Lambda$. $\square$

We can now complete the proof of conclusion (4) of Theorem 9.5. If $F = K^H$ is left invariant by $\mathrm{Gal}_\Delta(K/k)$ then restriction to F gives a homomorphism of $\mathrm{Gal}_\Delta(K/k)$ to $\mathrm{Gal}_\Delta(F/k)$. By the previous results, the kernel of this map is H so H is normal in $\mathrm{Gal}_\Delta(K/k)$. To show surjectivity we need to show that any $\phi \in \mathrm{Gal}_\Delta(F/k)$ extends to a $\tilde{\phi} \in \mathrm{Gal}_\Delta(K/k)$. This follows from the fact of the unicity of PPV-extensions.

Now assume that H is normal in $\mathrm{Gal}_\Delta(K/k)$ and that there exists an element $\tau \in \mathrm{Gal}_\Delta(K/k)$ such that $\tau(F) \neq F$. The Galois group of K over $\tau(F)$ is $\tau H \tau^{-1}$. Since $F \neq \tau(F)$ we have $H \neq \tau H \tau^{-1}$, a contradiction.

The last sentence of conclusion (4) follows from the above proposition.

9.5 Parameterized liouvillian extensions

In this section we will prove Theorem 3.12. One may recast this latter result in the more general setting of the last three sections but for simplicity we will stay with the original formulation. Let K and k be as in the hypotheses of this theorem. Let $K_A^{\mathrm{PV}} \subset K$ be the associated PV-extension as in Proposition 3.6.

(1) $\Rightarrow$ (2): Assume that the Galois group $\mathrm{Gal}_\Delta(K/k)$ contains a solvable subgroup of finite index. We may assume this subgroup is Kolchin closed. Since $\mathrm{Gal}_\Delta(K/k)$ is Zariski-dense in $\mathrm{Gal}_{\{\partial_0\}}(K_A^{\mathrm{PV}}/k)$, we have that this latter group also contains a solvable subgroup of finite index. Theorem 1.43 of [40] implies that K_A^{PV} is a liouvillian extension of k, that is, there is a tower of ∂_0-fields $k = K_0 \subset K_1 \subset \cdots \subset K_r = K_A^{\mathrm{PV}}$ such that $K_i = K_{i-1}(t_i)$ for $i = 1, \ldots, r$ where either $\partial_0 t_i \in K_{i-1}$, or $t_i \neq 0$ and $\partial_0 t_i/t_i \in K_{i-1}$ or t_i is algebraic over K_{i-1}. We can therefore form a tower of Δ-fields $k = \tilde{K}_0 \subset \tilde{K}_1 \subset \cdots \subset \tilde{K}_r$ by inductively defining $\tilde{K}_i = \tilde{K}_{i-1}\langle t_i\rangle_\Delta$. Since $K_A^{\mathrm{PV}} = K_r$, we have $K = \tilde{K}_r$ and so K is a parameterized liouvillian extension.

(3) $\Rightarrow$ (1): Assume that K is contained in a parameterized liouvillian extension of k. We wish to show that K_A^{PV} is contained in a liouvillian extension of k. For this we need the following lemma.

Lemma 9.14. *If L is a parameterized liouvillian extension of k then $L = \bigcup_{i\in\mathbb{N}} L_i$ where $L_{i+1} = L_i(\{t_{i,j}\}_{j\in\mathbb{N}})$ and $\{t_{i,j}\}$ is a set of elements such that for each j either $\partial_0 t_{i,j} \in L_i$ or $t_{i,j} \neq 0$ and $\partial_0 t_{i,j}/t_{i,j} \in L_i$ or $t_{i,j}$ is algebraic over L_i.*

Proof. In this proof we shall refer to a tower of fields $\{L_i\}$as above, as a ∂_0-tower for L. By induction on the length of the tower of Δ-fields defining L as a parameterized liouvillian extension of k, it is enough to show the following: *Let $\{L_i\}$ be a ∂_0-tower for the Δ-field L and let $L\langle t\rangle_\Delta$ be an extension of L such that $\partial_0 t \in L$, $\partial_0 t/t \in L$ or t is algebraic of L. Then there exists a ∂_0-liouvillian tower for $L\langle t\rangle_\Delta$.* We shall deal with three cases.

If t is algebraic over L, then it is algebraic over some L_{j-1}. We then inductively define $\tilde{L}_i = L_i$ if $i < j$, $\tilde{L}_j = L_j(t)$ and $\tilde{L}_i = L_i(\tilde{L}_j)$ if $i > j$. The fields $\{\tilde{L}_i\}$ are then a ∂_0-tower for $L\langle t\rangle_\Delta$.

Now, assume that $\partial_0 t = a \in L$. Let $\Theta = \{\partial_0^{n_0}\partial_1^{n_1}\ldots\partial_m^{n_m}\}$ be the commutative semigroup generated by the derivations of Δ. Note that $L\langle t\rangle_\Delta = L(\{\theta t\}_{\theta\in\Theta})$. For any $\theta \in \Theta$ we have $\partial_0(\theta t) = \theta(\partial_0 t) = \theta(a) \in L$. We define $\tilde{L}_i = L_i(\{\theta t \mid (\theta a) \in L_{i-1}\})$. Each $\tilde{L}_i$ contains $\tilde{L}_{i-1}$ and is an extension of $\tilde{L}$ of the correct type. Since $a \in L$, we have that for any $\theta \in \Theta$ there exists an i such that $\theta(a) \in L_{i-1}$, so $\theta(t) \in \tilde{L}_i$. Therefore, $\bigcup_{i\in\mathbb{N}} \tilde{L}_i = L\langle t\rangle_\Delta$ so $\{\tilde{L}\}$ is a ∂_0-tower for $L\langle t\rangle_\Delta$.

Finally assume that $\partial_0 t/t = a \in L_j \subset L$. For $\theta = \partial_0^{n_0}\partial_1^{n_1}\ldots\partial_m^{n_m} \in \Theta$, we define $\mathrm{ord}\theta = n_0 + n_1 + \cdots + n_m$. For any $\theta \in \Theta$, the Leibnitz rule implies that $\theta(at) = p_\theta + a\theta t$ where

$$p_\theta \in \mathbb{Q}\big[\{\theta' a\}_{\mathrm{ord}(\theta')\leq\mathrm{ord}(\theta)}, \{\theta'' t\}_{\mathrm{ord}(\theta'')<\mathrm{ord}(\theta)}\big].$$

Note the strict inequality in the second subscript. Let $S_\theta = \{\theta' a\}_{\mathrm{ord}(\theta')\leq \mathrm{ord}(\theta)} \cup \{\theta'' t\}_{\mathrm{ord}(\theta'')<\mathrm{ord}(\theta)}$. We define a new tower inductively:

$$\tilde{L}_1 = L_1(t), \quad \tilde{L}_i = \text{the compositum of } L_i \text{ and } \tilde{L}_{i-1}(\{\theta t \mid S_\theta \subset \tilde{L}_{i-1}\})$$

We now show that this is a ∂_0-tower for $L\langle t\rangle_\Delta$. We first claim that $\tilde{L}_i$ is an $\{\partial_0\}$-extension of $\tilde{L}_{i-1}$ generated by ∂_0-integrals or ∂_0-exponentials of integrals or elements algebraic over $\tilde{L}_{i-1}$. For $i = 1$, we have that $\partial_0 t/t \in L_0$ and L_1 is generated by such elements. For $i > 1$, assume $\theta \in \Theta$ and $S_\theta \subset \tilde{L}_{i-1}$. We then have that

$$\partial_0\left(\frac{\theta t}{t}\right) = \frac{p_\theta}{t} \in \tilde{L}_{t-i}$$

since $t, p_\theta \in \tilde{L}_{i-1}$. Therefore $\tilde{L}_{i-1}$ is generated by the correct type of elements.

We now show that for any $\theta \in \Theta$ there is some j such that $\theta(t) \in \tilde{L}_j$. We proceed by induction on $i = \mathrm{ord}(\theta)$. For $i = 0$ this is true by construction. Assume the statement is true for $\mathrm{ord}(\theta') < i$. Since there are only a finite number of such θ, there exists an $r \in \mathbb{N}$ such that $\{\theta'' t\}_{\mathrm{ord}(\theta'')<\mathrm{ord}(\theta)} \subset \tilde{L}_r$. Since $\{\theta' a\}_{\mathrm{ord}(\theta')\leq\mathrm{ord}(\theta)}$ is a finite subset of L, there is an $s \in \mathbb{N}$ such that $\{\theta' a\}_{\mathrm{ord}(\theta')\leq\mathrm{ord}(\theta)} \subset L_s$. Therefore for $j > \max(r, s), \theta t \in \tilde{L}_j$. Thus, $\bigcup_{i\in\mathbb{N}} \tilde{L}_i = L\langle t\rangle_\Delta$ so $\{\tilde{L}\}$ is a ∂_0-tower for $L\langle t\rangle_\Delta$. □

Let L be a parameterized liouvillian extension of k containing the field K. Lemma 9.14 implies that K_A^{PV} lies in a ∂_0-tower. Since K_A^{PV} is finitely generated, one sees that this implies that K_A^{PV} lies in a liouvillian extension of k. Therefore the PV-group $\mathrm{Gal}_\Delta(K_A^{\mathrm{PV}}/k)$ has a solvable subgroup H of finite index. Since we can identify $\mathrm{Gal}_{\{\partial_0\}}(K/k)$ with a subgroup of $\mathrm{Gal}_\Delta(K_A^{\mathrm{PV}}/k)$, we have that $\mathrm{Gal}_\Delta(K/k) \cap H$ is a solvable subgroup of finite index in $\mathrm{Gal}_\Delta(K/k)$. □

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