ON DECIDING TRANSCENDENCE OF POWER SERIES

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Abstract. It is well known that algebraic power series are differentially finite (D-finite): they satisfy linear differential equations with polynomial coefficients. The converse problem, whether a given D-finite power series is algebraic or transcendental, is notoriously difficult. We prove that this problem is decidable: we give two theoretical algorithms and a transcendence test that is efficient in practice.

1. Introduction

1.1. **Stanley's problem and contribution.** A power series $f \in \mathbb{Q}[[z]]$ is called *algebraic* if it is a root of some polynomial equation $P(z, f(z)) = 0$, where $P \in \mathbb{Q}[x, y] \setminus \{0\}$; otherwise, f is called *transcendental*. A classical result, known to Abel (1827) and possibly much older, states that any algebraic power series $f \in \mathbb{Q}[[z]]$ is *D-finite* (or, *differentially finite*), that is it satisfies a linear differential equation with coefficients in $\mathbb{Q}[z]$. Moreover, the minimal-order nontrivial homogeneous linear differential equation satisfied by f has order at most equal to $\deg_y(P)$ and coefficients in $\mathbb{Q}[z]$ of degree at most $4 \deg_x(P) \deg_y(P)^2$ [11].

Conversely, not every D-finite power series is algebraic; for instance, $\exp(z)$ and $\log(1 - z)$ are both D-finite and transcendental. Several methods are available to prove transcendence of $\exp(z)$ and $\log(1-z)$ (see e.g. [10]), but in general it is notoriously difficult to decide if a given D-finite function is algebraic or transcendental. This is the topic of our article.

In his seminal article on D-finite functions [93, §4, (g), page 186], Richard Stanley asked the following question:

> *"Given the differential equation*
>
> $$L(f(z)) = a_r(z)f^{(r)}(z) + \cdots + a_0(z)f(z) = 0 \qquad (1)$$
>
> *with polynomial coefficients $a_i \in \mathbb{Q}[z]$, together with suitable (finitely many) initial conditions, satisfied by a D-finite power series f, give an algorithm suitable for computer implementation for deciding whether f is algebraic."*

For instance, $f(z) \coloneqq \sum_{n\geq 1} z^n/n$ is a D-finite power series represented by the second-order differential equation $(z - 1)f''(z) + f'(z) = 0$ and initial conditions $f(0) = 0, f'(0) = 1$, and such an algorithm should recognize the transcendence of f starting from only this data.

Deciding transcendence of formal power series has many motivations and applications. For instance, in number theory a first step towards proving the transcendence of a complex number is proving the transcendence of some suitable power

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series attached to it. Other examples come from combinatorics, where the nature of generating functions may reveal strong underlying structures [13], and from computer science, where a natural question is whether algebraic power series are easier to manipulate than transcendental ones [8].

Although Stanley insisted on the practical aspect of the targeted algorithm for deciding algebraicity of D-finite functions, not even a theoretical one has appeared so far. In this work, we make progress on Stanley's question. We give two algorithms that prove the decidability of the algebraic or transcendental nature of D-finite power series in $\mathbb{Q}[[z]]$ and we give an (incomplete) transcendence test whose implementation is efficient in practice.

The first theoretical algorithm (Algorithm 1) relies on minimization of linear differential equations [16]; the second one (Algorithm 4) uses factorization. Both rely on Singer's algorithm [86]. The practical transcendence test (Algorithm 2) combines the same minimization as the first algorithm and concludes by a local analysis; it is incomplete in the sense that it is guaranteed to be correct only when it returns 'transcendental'. If the input differential equation possesses additional properties (e.g., if it cancels the diagonal of a multivariate rational function), then a variant of it (Algorithm 3) is also guaranteed to be correct when it returns 'algebraic', modulo a conjecture in the theory of arithmetic differential equations (Conjecture 8). Designing complete and efficient algebraicity or transcendence tests is still open.

1.2. **Difficulties and analogies.** There are several *a priori* reasons why Stanley's problem is difficult. First, the minimal polynomial of an algebraic power series $f(z)$ may have arbitrarily large size (degrees) w.r.t. the size (order/degree) of its input differential equation. For instance, $f(z) = \sqrt[N]{1-z}$ has algebraicity degree N and satisfies the first-order differential equation $N(z-1)f'(z) - f(z) = 0$. Second, no "characterization" is available for coefficient sequences of algebraic power series; this is in contrast with the smaller class of rational functions (whose coefficient sequences are C-recursive, i.e. satisfy linear recursions with constant coefficients) and with the larger class of D-finite functions (whose coefficient sequences are P-recursive, i.e. satisfy linear recursions with polynomial coefficients)[1].

There are several analogies between transcendence in $\mathbb{Q}[[z]]$ and irreducibility in $\mathbb{Q}[z]$. One is that "generic" power series are transcendental, much like "generic" polynomials are irreducible. Another one is that while many sufficient criteria exist (e.g., Eisenstein's criteria for polynomial irreducibility [29], see also [84, 83, 27], respectively for power series transcendence, see §1.4.1), none of them is also necessary. This is closely related to the fact that there are no known characterizations on the level of the coefficients sequence for recognizing either irreducibility in $\mathbb{Q}[z]$, or transcendence in $\mathbb{Q}[[z]]$. However, since polynomial irreducibility is known to be decidable [61, 64], it is legitimate to hope, by analogy, that the same holds for transcendence of power series.

1.3. **Related problems.** Although Stanley's question is quite recent, it is related to many classical questions and results, formulated starting with the beginning of the 19th century.

[1] Note that for the class of *diagonals of rational functions* [23], intermediate between algebraic and D-finite power series, coefficient sequences correspond (conjecturally) to the class of P-recursive sequences with (almost) integer coefficients and geometric growth (Christol's conjecture, see Part (1) of Conjecture 8).

Given a linear differential operator L with coefficients in $\mathbb{Q}(z)$, one may ask several questions concerning the algebraic nature of its solutions:

(F) Decide if *all* solutions of L are algebraic (Fuchs' problem).
(L) Decide if L admits *at least one* nontrivial algebraic solution (Liouville's problem).
(S) Decide if *a given* solution f of L is algebraic (Stanley's problem).

Note that if L is irreducible, then the three problems **(F)**, **(L)** and **(S)** are equivalent. This is because an irreducible operator either admits a basis of algebraic solutions, or no nontrivial algebraic solution, see Proposition 2.5 in [86].

Above, the word "decide" may have several meanings. One may for instance ask for a criterion, that is for a mathematical characterization based on the inspection of the coefficients of L, or of some invariants of L (such as its singularities, exponents at singularities, etc.) The classical meaning in computer science —the one that we adopt in this article— is that one asks for an algorithmic decision procedure, which, on every specific instance of L, is able to answer the given problem using a finite number of field operations in $\mathbb{Q}$.

In addition to such a decision procedure, one may also want to exhibit an annihilating polynomial (usually, the minimal polynomial) of the algebraic solution(s). Computing such a polynomial yields the most obvious "algebraicity witness", on which the decision procedure itself may rely (see §4.2). However, these witnesses may have prohibitively large sizes; deciding algebraicity of solutions does not necessarily need them.

In this article, we add one more problem to the previous list:

(P) Compute the (monic) right-factor operator[2] L^{alg} of L whose solution set is spanned by the algebraic solutions of L.

Again, if L is irreducible, then **(P)** is equivalent to the problems **(F)**, **(L)** and **(S)**.

Note that a solution to **(P)** implies solutions to each of the problems **(F)**, **(L)** and **(S)**. Indeed, solving **(F)**, resp. **(L)**, or **(S)**, amounts to checking $L = L^{\text{alg}}$, resp. $L^{\text{alg}} \neq 1$, or $L^{\text{alg}}(f) = 0$ (the last equality can be checked as explained in [16, Lemma 2.1]).

1.4. **Sufficient transcendence criteria.** There exist several criteria that can be used to prove that a given power series $f \in \mathbb{Q}[[z]]$ is transcendental. They are simply built on properties of algebraic power series. We recall below two important ones, Eisenstein's arithmetic criterion and Flajolet's analytic criterion. Although very useful in practice, none of these sufficient criteria is also necessary, and there are concrete examples that escape them (see Example 5 for such an example). This observation brings additional motivation to Stanley's problem **(S)**.

1.4.1. *Eisenstein's criterion.*

Definition 1. *A power series $f = \sum_{n \geq 0} a_n z^n$ in $\mathbb{Q}[[z]]$ is called* globally bounded *if it has a non-zero radius of convergence and there exists $C \in \mathbb{N}^*$ such that $a_n C^n \in \mathbb{Z}$ for $n \geq 1$.*

A famous criterion, stated (and sketchily proved) in 1852 by Gotthold Eisenstein [30] and fully proved one year later by Eduard Heine [46], asserts that if a power series $f \in \mathbb{Q}[[z]]$ is algebraic, then only a finite number of prime numbers

[2]This is an operator with coefficients in $\mathbb{Q}(z)$, see Proposition 2.

can divide the denominators of the coefficients sequence $(a_n)_{n\geq 0}$. This is also a consequence of a theorem[3] by Heine [47] stating that any algebraic power series $f \in \mathbb{Q}[[z]]$ is globally bounded. This arithmetic criterion immediately implies that the power series $\log(1-z) = -\sum_{n\geq 1} z^n/n$ (more generally, the polylogarithm $\mathrm{Li}_s(z) = \sum_{n\geq 1} z^n/n^s$), and $\exp(z) = \sum_{n\geq 0} z^n/n!$ are transcendental. However, when $f(z)$ is not given in closed form, but rather like in Stanley's problem **(S)**, by a linear differential equation with sufficiently many initial conditions, then Eisenstein's criterion is difficult to apply. The reason for this is a quite fundamental one: it is currently not known (and it is considered to be a difficult open problem) how to recognize that a P-recursive sequence $(a_n)_n$ given by a linear recurrence and initial conditions has integer terms (or, that it has *almost integer* coefficients, in the sense that $a_n C^n$ are integers for all $n \geq 1$, for some constant $C \in \mathbb{N}^*$). The same difficulty arises in a different context for the mere problem of representing E-functions, see [16].

1.4.2. *Flajolet's criterion.* In many situations, the given D-finite power series $f(z)$ already has integer coefficients, therefore Eisenstein's criterion is useless. This is systematically the case in combinatorics when $f(z)$ is the generating function of a sequence whose n-th term counts the number of objects of size n is some combinatorial class. Much more useful is *Flajolet's criterion* [33, Criterion D] (see also [34, VII.7.2]) based on the Newton-Puiseux theorem and on Darboux's transfer results from the local behavior of $f(z)$ around its singularities to the asymptotic behavior of its coefficient sequence $(a_n)_{n\geq 0}$.

One form of the criterion asserts that *if $a_n \sim \gamma\, \beta^n\, n^r$ as $n \to \infty$, with either $r \notin \mathbb{Q} \setminus \mathbb{Z}_{<0}$, or $\beta \notin \overline{\mathbb{Q}}$, or $\gamma \cdot \Gamma(r+1) \notin \overline{\mathbb{Q}}$, then $f(z)$ is transcendental.*

For more general and more refined versions on the asymptotic behavior of the coefficients of algebraic power series, we refer to [34, p. 501], [74] and [80, Theorem 3.1].

This criterion is the most commonly used for transcendence questions in combinatorics. It allows for instance to show that the hypergeometric power series $\sum_{n\geq 0} \binom{2n}{n}^k z^n$ (mentioned at the end of problem (g) in [93, §4]) is algebraic if and only if $k = 1$ (see §1.6 for a different proof), and also that the non-hypergeometric power series $\sum_{n\geq 0} \sum_{k=0}^{n} \binom{n}{k}^2 \binom{n+k}{k}^2 z^n$ (occurring in Apéry's celebrated proof of the irrationality of $\zeta(3)$, see Example 4) is transcendental. A more general example that can be handled by Flajolet's criterion is

$$\sum_{n\geq 0} \left(\sum_{k=0}^{n} \binom{n}{k}^{p_0} \binom{n+k}{k}^{p_1} \binom{n+2k}{k}^{p_2} \cdots \binom{n+mk}{k}^{p_m} \right) z^n,$$

that is discussed in Appendix A.

However, there are cases of D-finite power series $f(z)$ for which both Eisenstein's criterion and Flajolet's criterion fail to detect the transcendence of $f(z)$, see Example 5.

[3] Heine's 1854 result is what is commonly called "Eisenstein's criterion" in the literature. Hence, strictly speaking there are two distinct criteria, one due to Eisenstein and the other one due to Heine. It might well be that they coincide for D-finite functions; this is related to Siegel's conjecture that the classes of "broad G-functions" and "strict G-functions" coincide, see e.g. [2, Conj. 2.3.1].

1.5. **A bit of history.** Starting with the beginning of the 19th century, many works have been dedicated to solving problems **(F)** and **(L)**, using various tools, from geometry (Schwarz, Klein), to invariants (Fuchs, Gordan) and from group theory (Jordan) to differential Galois theory (Kolchin, Singer).

Already in 1833, Liouville [66] proposed an algorithm for computing a basis of *rational* solutions of linear ODEs. This algorithm (with some enhancements and improvements) is the one currently implemented in most computer algebra systems. Liouville's algorithm clearly solves the variants $\mathbf{(F)}_{\mathrm{rat}}$ and $\mathbf{(L)}_{\mathrm{rat}}$ of **(F)** and **(L)** in which the word "algebraic" is replaced by "rational"; it can also be used to solve the variant $\mathbf{(S)}_{\mathrm{rat}}$ of problem **(S)**: from a basis of solutions $r_1(z), \dots, r_s(z)$ in $\mathbb{Q}(z)$ of L, one first computes the right factor L^{rat} of L whose solution space is spanned by the rational solutions of L, as the LCLM[4] of $\partial_z - r_j'(z)/r_j(z)$, and then one checks whether $L^{\mathrm{rat}}(f)$ is zero or not (using [16, Lemma 2.1]).

In 1839 Liouville addressed in [67] what we call problem **(L)**. He partially solved it for second order ODEs. For instance, he showed that the equation $y''+r(z)y = 0$ does not admit any nontrivial algebraic solution when $r(z) \in \mathbb{Q}[z]$. (In particular, this holds for Airy's equation $y'' = zy$.) One year later he proved the same for $r(z) = -(1 + z^2)^2/(2z - 2z^3)^2$, from which he deduced that the complete elliptic integral $f(z) = \int_0^1 dx/\sqrt{(1-x^2)(1-z^2x^2)}$ is not algebraic (and not even Liouvillian, that is, solvable in terms of integrals, exponentials and algebraic functions). In 1841, he applied his method to the Bessel equation $z^2f''(z) + zf'(z) + (z^2 - \mu^2)f(z) = 0$. However, he could not solve the problem when $r(z)$ is an arbitrary rational function; he reduced it to upper bounding the possible algebraicity degree of a solution. With such a bound B at hand, Liouville reduced problem **(L)** to finding rational solutions of the symmetric powers $L^{\circledS m}$ (that is the monic minimal order operator whose solution space contains the powers f^m for all solutions f of L) for $1 \le m \le B$.

Liouville's work was taken up by Pépin (1863, 1878, 1881), who focused on problem **(F)** and managed to remove the restriction on the algebraicity degree [76, 77]. His 1863 paper contained a few errors corrected in 1878 (after remarks by Fuchs). Pépin completed his study in his long memoir published in 1881 [75]; there, he proved that if the equation $y'' + r(z)y = 0$ has only algebraic solutions, then it admits a basis of solutions $\{y_1, y_2\}$ such that either (i) $y_1 = \sqrt[m]{a}$ and $y_2 = b/y_1$, for a, b in $\mathbb{Q}(z)$ and $m \in \mathbb{N}$; or (ii) y_i^m are both roots of a quadratic equation over $\mathbb{Q}(z)$ for $m \in \mathbb{N}$; or (iii) y_i^m are both roots of an equation of degree μ over $\mathbb{Q}(z)$, where $(m, \mu) \in \{(4, 6), (6, 8), (12, 10)\}$. In all cases, y'/y is algebraic of degree 1, 2, 4, 6 or 12.

Meanwhile, in 1873, Schwarz [85] famously solved problem **(F)** for second order operators with three singular points (the Gauss hypergeometric equation), see §1.6. Fuchs showed in 1876 that if $y''+r(z)y = 0$ has a nontrivial algebraic solution, then there exists a binary homogeneous form $F(y_1, y_2)$ of degree $d \le 12$ of a basis of solutions $\{y_1, y_2\}$ which is the k-th root of a rational function for some explicit number k depending only on d. Inspired by Fuchs's work, Klein showed in 1877 that any second-order linear differential equation with only algebraic solutions comes from some hypergeometric equation from Schwarz' list via a rational change of variables. In [4], Dwork and Baldassari discuss Klein's and Fuchs's articles and make Klein's

[4]We denote by $\mathrm{LCLM}(L_1, \dots, L_k)$ the least common left multiple of the operators $L_1, \dots, L_k$, that is the monic minimal order operator whose solution space contains the solution spaces of all L_i's.

approach algorithmic. Sanabria Malagón generalizes this approach in [82], where he also gives a history of the evolution of this approach through his work and that of others. Another approach was given by Jordan (1878) who showed that if all solutions of (1) are algebraic, then there exists a solution whose logarithmic derivative is algebraic of degree at most $J(r)$, an explicit number depending only on r. Jordan's article, as well as works by Painlevé (1887) and his student Boulanger (1898) are the starting points of modern algorithms by Singer [86, 87], who completely solved problem **(F)**, see §1.7. References to improvements on this algorithm are given in [90].

For more historical aspects related to these rich problems, we refer the interested reader to the following sources: [17, p. 1–13], [97, p. 160–165], [38, p. 5–10], [68, p. 407–413], [88, p. 25], [90, p. 533–536], [96, p. 124] and [39, p. 48–50, 273–274, and Chap. III].

1.6. **The hypergeometric case.** A very special, yet quite important special case of D-finite power series $f \in \mathbb{Q}[[z]]$ is the *hypergeometric* class. This means that the coefficient sequence of $f(z)$ satisfies a first-order homogeneous linear recurrence with polynomial coefficients. Typical examples are $\log(1-z)$, $\arcsin(\sqrt{z})/\sqrt{z}$, $(1-z)^\alpha$ for $\alpha \in \mathbb{Q}$, and more generally the Gaussian hypergeometric function with parameters $a, b, c \in \mathbb{Q}$ $(-c \notin \mathbb{N})$,

$${}_2F_1\left[\begin{matrix} a\ b \\ c \end{matrix}; z\right] \coloneqq \sum_{n=0}^{\infty} \frac{(a)_n (b)_n}{(c)_n} \frac{z^n}{n!}, \quad \text{where} \quad (d)_n \coloneqq d(d+1)\cdots(d+n-1),$$

solution of the differential equation $z(1-z)f''(z)+(c-(a+b+1)z)f'(z)-abf(z)=0$.

Deciding the algebraicity of ${}_2F_1$ functions is an old problem, first solved by Schwarz [85] using geometric and complex analytic tools, and later by Landau [62, 63] and Errera [31] using number theoretic tools. Both approaches are algorithmic: Schwarz's criterion reduces the problem to a table look-up after some preprocessing on the parameters a, b, c; the Landau-Errera criterion amounts to checking a finite number of inequalities. The reader is referred to [10, §2.1.3] and to Chap. I, II and IV of Matsuda's book [69] for more details.

The Landau-Errera criterion was extended by Beukers and Heckman [7]. To state it, we need the following definitions: for $x \in \mathbb{R}$ we denote by $\langle x \rangle$ its fractional part $x - \lfloor x \rfloor$ if $x \notin \mathbb{Z}$, and 1 if $x \in \mathbb{Z}$; we say that two equinumerous disjoint multisets of real numbers $\{u_1, \ldots, u_k\}$ and $\{v_1, \ldots, v_k\}$ with $\langle u_1 \rangle \leq \cdots \leq \langle u_k \rangle$ and $\langle v_1 \rangle \leq \cdots \leq \langle v_k \rangle$ *interlace* if $\langle u_1 \rangle < \langle v_1 \rangle < \cdots < \langle u_k \rangle < \langle v_k \rangle$. Now, let $\mathbf{a} = \{a_1, \ldots, a_k\} \subset \mathbb{Q}$ and $\mathbf{b} = \{b_1, \ldots, b_{k-1}, b_k = 1\} \subset \mathbb{Q} \setminus (-\mathbb{N})$ be two multisets of rational parameters assumed to be disjoint modulo $\mathbb{Z}$. This assumption is equivalent to the irreducibility of the generalized hypergeometric operator $H_{\mathbf{b}}^{\mathbf{a}} \coloneqq (z\partial_z + b_1 - 1)\cdots(z\partial_z + b_{k-1} - 1)z\partial_z - z(z\partial_z + a_1)\cdots(z\partial_z + a_k)$. Let D be the common denominator of $a_1, \ldots, a_k, b_1, \ldots, b_k$. Then, the generalized hypergeometric function[5]

$${}_kF_{k-1}\left[\begin{matrix} a_1 & a_2 & \cdots & a_k \\ & b_1 & \cdots & b_{k-1} \end{matrix}; z\right] \coloneqq \sum_{n=0}^{\infty} \frac{(a_1)_n \cdots (a_k)_n}{(b_1)_n \cdots (b_{k-1})_n} \frac{z^n}{n!}$$

[5]Note that when $\ell \neq k-1$, a ${}_kF_\ell$ hypergeometric function cannot be algebraic, by Stirling's formula combined with Flajolet's criterion; alternatively, because for $k > \ell + 1$ the power series ${}_kF_\ell$ has radius of convergence 0, while ${}_kF_\ell$ is an entire function for $k \leq \ell$.

is a solution of $H_{\mathbf{b}}^{\mathbf{a}}$ and the *interlacing criterion* says that it is algebraic if and only if for all $1 \le \ell < D$ with $\gcd(\ell, D) = 1$ the multisets $\ell\mathbf{a}$ and $\ell\mathbf{b}$ interlace (pictorially, this means that the points $\{e^{2\pi i\ell a_j}, j \le k\}$ and $\{e^{2\pi i\ell b_j}, j \le k\}$ interlace on the unit circle).

For instance, the Beukers-Heckman criterion immediately implies that the power series

$$\sum_{n=0}^{\infty} \binom{2n}{n}^k z^n = {}_kF_{k-1}\left[\begin{matrix}\frac{1}{2} & \frac{1}{2} & \cdots & \frac{1}{2} \\ & 1 & \cdots & 1\end{matrix}; 4^k\, z\right]$$

is transcendental for all $k \ge 2$, since the interlacing condition is violated. This example is mentioned by Stanley in [93, §4, (g)] as one of the motivations of his general question **(S)**.

The Beukers-Heckman interlacing criterion does not cover all hypergeometric cases; it cannot be used if some parameters are irrational nor if some differences between top and bottom parameters are integer numbers. For this reason, the following examples escape it

$$(2) \qquad {}_2F_1\left[\begin{matrix}1 & 1 \\ & 2\end{matrix}; z\right], \quad {}_2F_1\left[\begin{matrix}2 & 2 \\ & 1\end{matrix}; z\right] \quad \text{and} \quad {}_3F_2\left[\begin{matrix}1/2 & 1+\sqrt{2} & 1-\sqrt{2} \\ & \sqrt{2} & -\sqrt{2}\end{matrix}; z\right].$$

Fürnsinn and Yurkevich [36] provided a complete classification of the algebraic generalized hypergeometric functions with no restriction on the set of their parameters, thus answering completely Stanley's question **(S)** for power series whose coefficient sequences satisfy a recurrence of order 1. Their result is algorithmic and allows to prove that, among the three examples in Eq. (2), the first power series is transcendental, while the last two are algebraic. Their algorithm relies on an elementary reduction [36, Fig. 1] of the general case to the interlacing criterion of Beukers and Heckman.

1.7. **Singer's algorithm for problem (F).** In the 1980s, Michael Singer [86, 87] designed algorithms that solve Fuchs' problem **(F)**. In fact, the problem has (almost) been solved in the second half of the 19th century, for order 2 equations by Schwarz, Klein, Fuchs, and for order 3 equations by Painlevé and Boulanger. For an irreducible operator L of order r, Singer's algorithm in [86] relies on differential Galois theory and proceeds in two main steps:

(i) (Jordan's bound) decide if the (nonlinear) Riccati differential equation[6] $R_L(u) = 0$ attached to L admits an algebraic solution u of degree at most $(49r)^{r^2}$;

(ii) (Abel's problem) given an algebraic u, decide whether $y'/y = u$ admits an algebraic solution y.

Step (ii) is solved by Risch's algorithm [79] (and independently by Baldassarri and Dwork [4, §6]); it was the missing ingredient in the 19th century procedures. Singer solves Step (i) by reducing it to an algebraic elimination problem. Part of the proof of the algorithm is that if step (ii) finds an algebraic y such that $y'/y = u$ with u as in step (i), then y is a nontrivial algebraic solution of L, and since L is irreducible, all solutions of L are algebraic.

In the reducible case, Singer's algorithm proceeds first to a factorization step that writes L as a product of irreducible operators. The corresponding detailed results

[6] By definition, R_L has order $r-1$ and has the property that f is a nonzero solution of L if and only if f'/f is a solution of R_L.

and procedures can be found in (the proofs of) [86, Theorem 1] in the irreducible case, and [86, Theorem 3] in the general case. A slightly different route is proposed by Singer in [87, Corollary 4.3], where first a basis of all Liouvillian solutions is computed, and then an algorithm by Rothstein and Caviness [81, §4] is used to decide if all elements in this basis are algebraic.

1.8. **Other approaches to problem (F).** Another way of seeing that Fuchs' problem **(F)** is decidable, in a spirit similar to Singer's algorithm but of even higher algorithmic complexity, is based on the equivalence for a linear differential operator L between having a full basis of algebraic solutions and having a finite differential Galois group. Indeed, since Hrushovski's seminal article [52], it is known that one can fully determine algorithmically the Galois group of L. (Prior to [52], it was only known how to compute the Galois group of *completely reducible operators*, that is for operators L that can be written as LCLM of irreducible operators [25], a condition that can itself be algorithmically tested [89].) The complexity of Hrushovski's algorithm is not yet fully understood, and its simplification is the object of ongoing works, e.g. [32, 94, 1]. An approach to understanding the Galois group of L by calculating the Lie algebra of its identity component is given in [5, 6] and [28]. In a different direction, van der Hoeven proposed in [49] a symbolic-numeric approach for computing differential Galois groups, but for the moment its potential has not been exploited further.

To our knowledge, none of the algorithms mentioned in Sections 1.7 and 1.8 has been implemented yet, nor are they expected to provide good practical behavior (except perhaps for very moderate orders, as in e.g., [91, 48]).

Let us finally mention a very recent sufficient (algorithmic) criterion for problem (**F**). Assume that the linear differential equation $L(y) = 0$ admits a basis of Puiseux series solutions at each of its singularities, and moreover that there exists another differential operator P such that $\partial_z P$ is not divisible by L and such that at all singularities of L and for any element f in a basis of Puiseux solutions of L, the Puiseux series $P(f)$ has only nonnegative exponents. Then L admits a transcendental solution, in other words Fuchs' problem (**F**) is solved by the negative. This is the content of Theorem 5 in [56], where such an operator P is called a *pseudo-constant*. Moreover, Algorithm 10 in [56] is able to compute a pseudo-constant for L, or to certify that none exists. More generally, if any symmetric power of L admits a pseudoconstant, then L admits a transcendental solution (Theorem 18 in [56]). Conversely, it is an open question whether the fact that L admits transcendental solutions implies the existence of pseudoconstants for some symmetric power of L (Question 20 in [56]). If the answer to this open question was positive, then this would provide another way to show that Fuchs' problem is decidable.

2. Solving problem (S)

For a D-finite power series $f \in \mathbb{Q}[[z]]$, we write $L_f^{\min}$ for the linear differential operator in $\mathbb{Q}[z]\langle\partial_z\rangle$ of minimal order that cancels f, and whose coefficients in $\mathbb{Q}[z]$ have a trivial gcd. $L_f^{\min}$ can be computed efficiently [16] starting from any linear differential equation satisfied by f together with sufficiently many initial terms of f.

Our solution to Stanley's problem **(S)** is based on the properties of $L_f^{\min}$ that we review now.

2.1. **Properties of $L_f^{\min}$.** In simple terms, the minimal differential operator $L_f^{\min}$ is a differential analogue for D-finite functions of the more classical notion of minimal polynomial for algebraic power series. Indeed, any $L \in \mathbb{Q}[z]\langle\partial_z\rangle$ such that $L(f(z)) = 0$ is a left multiple of $L_f^{\min}$ in the ring $\mathbb{Q}(z)\langle\partial_z\rangle$, and conversely any left multiple of $L_f^{\min}$ is an annihilator for f.

But there are two main differences between these similar notions. One is that, for such an L, although its order upper bounds the order of $L_f^{\min}$, it is in general not the case that the maximal degree of the coefficients of L is an upper bound on the maximal degree of the coefficients of $L_f^{\min}$; this is actually the main difficulty in the algorithmic computation of $L_f^{\min}$, see [16] for more details. Another difference important to note is that $L_f^{\min}$ does not need to be irreducible in $\mathbb{Q}(z)\langle\partial_z\rangle$. This is clear if f is transcendental, as illustrated by $L^{\min}_{\log(1-z)} = ((1-z)\partial_z - 1)\partial_z$. The same also holds when f is algebraic. For instance, for the algebraic power series of degree 2

$$f = \sqrt{1-4z} + z = 1 - z - 2z^2 - 4z^3 - 10z^4 - \cdots,$$

we have that

$$\begin{aligned} L_f^{\min} &= (1-2z)\,(1-4z)\,\partial_z^2 - 4z\partial_z + 4 \\ &= \left((1-2z)\,(1-4z)\,\partial_z + 4z - 6 + \frac{1}{z}\right)\cdot\left(\partial_z - \frac{1}{z}\right). \end{aligned}$$

Moreover, this is actually the generic behavior for an algebraic power series $f \in \mathbb{Q}[[z]]$: if $P = y^d + c_{d-1}(z)y^{d-1} + \cdots + c_0(z) = \prod_{\ell=1}^d (y - f_\ell(z))$ is the minimal polynomial of $f = f_1$ in $\mathbb{Q}(z)[y]$, then $L_f^{\min}$ cancels all the conjugated roots $f_2, \ldots, f_d$ of $f = f_1$ (see Proposition 2). Therefore, $L_f^{\min}$ admits $\sum_{\ell=1}^d f_\ell(z) = -c_{d-1}(z)$ as rational solution, hence if $c_{d-1} \neq 0$ then $L_f^{\min}$ is right-divisible by $\partial_z - c'_{d-1}/c_{d-1}$. This was remarked by Tannery, [95, p. 132], see also [26, Prop. 4.2].

In Proposition 2 below we give properties of $L_f^{\min}$ and related operators. The proofs rely on some basic facts from the Galois theory of linear differential equations (see [88, 96]) so we begin in this general setting. Let k be a differential field of characteristic zero with derivation ∂ and with algebraically closed subfield of constants $C = \{c \in k \mid \partial c = 0\}$. Let $L \in k\langle\partial\rangle$ be an operator of order n and let K be the associated Picard-Vessiot extension[7]. Let G be the differential Galois group of K over k and let $f \in K$ be a solution of $L(y) = 0$. We define $L_f^{\min}$ to be the monic operator in $k\langle\partial\rangle$ of smallest order vanishing on f.

The C-space of algebraic solutions of $L(y) = 0$ is left invariant by the action of G so it is the solution space of a monic operator in $k\langle\partial\rangle$ denoted by L^{alg} ([89, Lemma 2.2] or [96, Lemma 2.17 p. 48]). Let $P(Y)$ be an irreducible polynomial in $k[Y]$ and assume that there is a $z \in K$ such that $P(z) = 0$. The differential Galois theory implies that the splitting field of P over k lies in K [96, Prop 1.34.3]. Since G permutes the roots of $P(Y)$, we have that the C-span of these roots is left invariant under this action. As before this implies that this vector space is the solution space of a monic operator in $k\langle\partial\rangle$ denoted by L_P. Note that since Picard-Vessiot extensions for L are unique up to k-differential isomorphisms, the

[7] This is a field generated over k by a fundamental set of solutions and all their $n-1^{st}$ derivatives and having the same constants C as k; one sees that it is again a differential field. Since we assume that C is algebraically closed, such a field exists and any two are differentially isomorphic over k.

operators $L^{\mathrm{alg}}, L_f^{\min}$ and L_P will be independent of such an extension since any such isomorphism will preserve all the properties of these operators.

We shall apply the above constructions to a linear differential operator $L \in \overline{\mathbb{Q}}(z)\langle \frac{d}{dz}\rangle$ where $\overline{\mathbb{Q}}$ is the algebraic closure of $\mathbb{Q}$. To be precise, by an algebraic solution we mean an element $y \in \mathcal{P} = \cup_{n\in\mathbb{N}}\overline{\mathbb{Q}}((z^{\frac{1}{n}}))$, the differential field of formal Puiseux series where $\frac{d}{dz}(z^{\frac{1}{n}}) = \frac{1}{n}z^{\frac{1}{n}-1}$ such that y is algebraic over $\overline{\mathbb{Q}}(z)$ and $L(y) = 0$. For a given operator L, the set $\mathcal{A}^L$ of algebraic solutions forms a vector space over $\overline{\mathbb{Q}}$ of dimension at most equal to the order of L. Furthermore, the derivation $\frac{d}{dz}$ extends uniquely to the field $E = \overline{\mathbb{Q}}(z, \mathcal{A}^L)$. We now construct a Picard-Vessiot extension K of $k = \overline{\mathbb{Q}}(z)$ that contains $\mathcal{A}^L$. Let F be the Picard-Vessiot extension of E corresponding to L. Since $\mathcal{A}^L$ lies in the solution space of L in E, there exists a basis $\{y_1, \dots, y_n\}$ of this space that contains a basis of $\mathcal{A}^L$. The field $K = \overline{\mathbb{Q}}(z)(y_1, \dots, y_n, \dots, y_1^{(n-1)}, \dots, y_n^{(n-1)}) \subset E$ has no new constants and so is the required Picard-Vessiot extension.

The following proposition states that if L has coefficients in $\mathbb{Q}(z)$, then the associated operators $L^{\mathrm{alg}}, L_f^{\min}$ and L_P also have coefficients in $\mathbb{Q}(z)$ and gives further information concerning $L_f^{\min}$.

Proposition 2. *Let $L \in \mathbb{Q}(z)\langle\partial\rangle$ and $f \in \mathbb{Q}((z))$ be an algebraic solution of $L(y) = 0$ with minimal polynomial $P(Y) \in \mathbb{Q}(z)[Y]$. If L^{alg}, L_P, and $L_f^{\min}$ are the operators defined above over $\overline{\mathbb{Q}}(z)$, then these operators have coefficients in $\mathbb{Q}(z)$. Furthermore, $L_f^{\min} = L_P$ so the solution space of $L_f^{\min}$ is spanned by the roots of $P(Y)$. In particular, the local solutions of $L_f^{\min}$ at any point of its singularities do not contain logarithms.*

Proof. Once again let $\mathcal{P}$ be the differential field of Puiseux series and let $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ be the Galois group of $\overline{\mathbb{Q}}$ over $\mathbb{Q}$. For $\sigma \in \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$, we define the action of σ on $\mathcal{P}$ as $\sigma(\sum_i a_i z^{\frac{i}{n}}) = \sum_i \sigma(a_i) z^{\frac{i}{n}}$. In this way $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ acts on $\mathcal{P}$ as a group of differential automorphism with fixed field $\cup_{n\in\mathbb{N}}\mathbb{Q}((z^{\frac{1}{n}}))$.

$\underline{L^{\mathrm{alg}}}$: Since L has coefficients in $\mathbb{Q}(z)$, $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ leaves the $\overline{\mathbb{Q}}$-space of algebraic solutions invariant. Denoting by σL^{alg} the operator obtained from L^{alg} by applying σ to the coefficients of L, we have that σL^{alg} and L have the same solution spaces and so must be equal. Therefore L^{alg} has coefficients in $\mathbb{Q}(z)$.

$\underline{L_P}$: Since $P(Y)$ has coefficients in $\mathbb{Q}(z)$, $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ permutes its roots and therefore the $\overline{\mathbb{Q}}$-space spanned by these roots is left invariant by this action. Arguing as above we have that L_P has coefficients in $\mathbb{Q}(z)$.

$\underline{L_f^{\min}}$: Any $\sigma \in \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ leaves f fixed and sends $L_f^{\min}$ to $\sigma L_f^{\min}$. By uniqueness of $L_f^{\min}$, we have $L_f^{\min} = \sigma L_f^{\min}$ and this implies, as above, that the coefficients of $L_f^{\min}$ lie in $\mathbb{Q}(z)$.

$\underline{L_f^{\min} = L_P}$: Let F be the splitting field of $P(Y)$ over $\mathbb{Q}(z)$ and let H be the (usual) Galois group of this extension. The derivation on $\mathbb{Q}(z)$ extends uniquely to a derivation on F and the elements of H commute with this derivation. Since $P(Y)$ is irreducible, H acts transitively on the roots of $P(Y)$. Since $L_f^{\min}$ has coefficients in $\mathbb{Q}(z)$ and vanishes on one root of $P(Y)$, $L_f^{\min}$ vanishes on all the roots and therefore on the $\overline{\mathbb{Q}}$-space spanned by the roots of $P(Y)$. This implies that L_P is a right factor of $L_f^{\min}$. Minimality of $L_f^{\min}$ and the fact that $L_P(f) = 0$ implies that $L_f^{\min}$ is a right factor of L_P. Therefore these two operators are equal. $\square$

Proposition 2 implies the following classical result (for definitions of the terms see [53]):

Corollary 3. *The differential operator $L_f^{\min}$ of an algebraic power series $f \in \mathbb{Q}[[z]]$ enjoys the following properties:*

(1) *it is Fuchsian;*
(2) *it admits only rational exponents at all its singular points (including infinity);*
(3) *the indicial polynomials at all of its singular points (including infinity) split in $\mathbb{Q}[z]$ into distinct linear factors.*

Corollary 3 was essentially proved by Fuchs in [35, §6]. Its conclusions (1) and (2) also hold for the larger class of *diagonals of rational functions*, and even for the much larger class of *G-functions*, by results of Katz (1970), Chudnovsky and Chudnovsky (1985) and André (1989), see e.g. [2, p. 719] or [65, Thm. 1] and the references therein. Conclusion (3) is a consequence of the fact that a multiple root of the indicial equation at $z = a$ always introduces logarithmic terms in the basis of local solutions at $z = a$ (see [53, p. 405]). Contrary to (1) and (2), it is false for the larger class of diagonals, as the simple example $f = \sum_{n\geq 0} \binom{2n}{n}^2 z^n$ demonstrates, with indicial polynomial z^2 at $z = 0$.

2.2. **Examples.**

Example 4. *Here is a proof of the transcendence of Apéry's power series*

$$f(z) = \sum_{n\geq 0} A_n z^n, \quad \text{where } A_n = \sum_{k=0}^{n} \binom{n}{k}^2 \binom{n+k}{k}^2.$$

First, creative telescoping [100] produces a linear recurrence with polynomial coefficients satisfied by the sequence $(A_n)_n$,

$$(n+1)^3 A_{n+1} + n^3 A_{n-1} = (2\,n+1)\,(17\,n^2 + 17\,n + 5) A_n, \quad A_0 = 1,\ A_1 = 5.$$

This recurrence is then converted into a linear differential equation $L(f) = 0$ satisfied by $f(z)$, where L is the differential operator

$$L = (z^4 - 34z^3 + z^2)\partial_z^3 + (6z^3 - 153z^2 + 3z)\partial_z^2 + (7z^2 - 112z + 1)\partial_z + z - 5.$$

In the third step one certifies that $L_f^{\min} = L$, meaning that L is already the minimal-order differential equation satisfied by $f(z)$. This can be done using the minimization procedure in [16, Algorithm 1][8].

Finally, the indicial polynomial of $L_f^{\min}$ at $z = 0$ is z^3, and the transcendence of $f(z)$ follows from part (3) of Corollary 3.

The approach used in Example 4 is the basis of Algorithms 2 and 3 below.

Note that there are alternative proofs for the transcendence of the Apéry series $f(z)$. One of them relies on the combination of three (non-trivial) ingredients: (i) the asymptotics $a_n \sim \frac{(1+\sqrt{2})^{4n+2}}{2^{9/4}\pi^{3/2}n^{3/2}}$ [71]; (ii) Flajolet's criterion (§1.4.2); (iii) the fact that π is transcendental. Our proof based on differential operators appears to be more "natural", since proving the transcendence of a function should

[8]Note that in this very particular case, one could alternatively prove that L is actually irreducible (for instance, by showing that neither L, nor its adjoint, admit any non-trivial hyperexponential solutions).

Algorithm 1 Deciding transcendence of D-finite functions

Input: $L = a_r(z)\partial_z^r + \cdots + a_0(z)$ with $a_i(z) \in \mathbb{Q}(z)$;
ini: f_0 a truncated power series at precision $p_0 \geq r$
specifying a unique solution $f \in \mathbb{Q}[[z]]$ of $L(f) = 0$.
Output: Either T if f is transcendental, or A if f is algebraic.

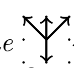
$L_f^{\min}$:=MinimalRightFactor(L, ini) // *Bostan-Rivoal-Salvy algorithm* [16]
if $L_f^{\min}$ has a basis of algebraic solutions **then** $\mathsf{B} := \mathsf{A}$ // *Singer's algorithm* [86]
else $\mathsf{B} := \mathsf{T}$
return B

be easier than proving the transcendence of a number. Another alternative proof uses [20, Thm. 5.1] (see also [102, §6]); it relies on the fact that $f(z)$ admits modular parametrizations, implying that, up to algebraic pullbacks, $f(z)$ coincides with the hypergeometric series ${}_3F_2\left[\begin{matrix}1/2 & 1/2 & 1/2 \\ & 1 & 1\end{matrix}; z\right]$, which is transcendental (§1.6). This proof does not extend to more general Apéry-like series, as those in Appendix A.

The next example is of a more combinatorial flavor. It exhibits a case of a nontrivial D-finite power series $f(z)$ in $\mathbb{Q}[[z]]$ whose minimal-order operator $L_f^{\min}$ is not only reducible, but (almost) split.

Example 5. *The generating function of the so-called "trident walks in the quarter plane" is the power series*

$$f(z) = \sum_n a_n z^n = 1 + 2\,z + 7\,z^2 + 23\,z^3 + 84\,z^4 + 301\,z^5 + 1127\,z^6 + \cdots .$$

Its n-th coefficient a_n counts all the  *-walks of length n in $\mathbb{N}^2$ starting at $(0,0)$. It was proved in* [13] *that $f(z)$ satisfies $L(f) = 0$, where L is a linear differential operator of order 5 and polynomial degree at most 15. In* [13, §4.2] *it was shown that $f(z)$ is transcendental by exploiting the full factorization of L. An alternative proof of the transcendence of f (and actually of all $4 \times 19 - 4 = 72$ similar power series in* [13, Thm. 2]*, see §5) uses minimization to first prove that the differential equation $L(f) = 0$ has minimal order, that is $L_f^{\min} = L$, and concludes using the presence of logarithmic terms in the local basis of L at $z = 0$. One advantage of this proof is that factorization of differential operators is avoided.*

Note that in Example 5, one cannot conclude transcendence of f as in Example 4. Indeed, the asymptotic behaviour[9] $a_n \sim \gamma\beta^n n^r$ with $\gamma = 4/(3\sqrt{\pi})$, $\beta = 4$, $r = -1/2$, is compatible with algebraicity, since $r \in \mathbb{Q} \setminus \mathbb{Z}_{<0}$, $\beta \in \overline{\mathbb{Q}}$ and $\gamma\Gamma(r+1) = 4/3 \in \overline{\mathbb{Q}}$.

Another combinatorial example, also arising from the world of lattice path enumeration, was considered in [9, Proposition 8.4]. In that case, all known criteria for transcendence fail to apply, as well as all previously known algorithms. Moreover, the defining differential operator (of order 11 and polynomial degree 73) is too big to

[9] This asymptotic estimate was conjectured in [13, Conjecture 1], shown to be equivalent to an integral evaluation in [13, Theorem 8], and proved using analytic combinatorics in several variables in [73, §6].

Algorithm 2 Transcendence test for D-finite functions

Input: $L = a_r(z)\partial_z^r + \cdots + a_0(z)$ with $a_i(z) \in \mathbb{Q}(z)$;
ini: f_0 a truncated power series at precision $p_0 \geq r$
specifying a unique solution $f \in \mathbb{Q}[[z]]$ of $L(f) = 0$.
Output: Either T if f is transcendental, or FAIL.

```
L_f^min := MinimalRightFactor(L, ini)          // Bostan-Rivoal-Salvy algorithm [16]
for s ∈ Singularities(L_f^min) do
    P := IndicialPolynomial(L_f^min, z = s)
    if deg P < ord L_f^min then return T                              // not Fuchsian
    Z := DistinctRationalZeros(P)
    if card Z < deg P then return T        // either an irrational exponent or a logarithm
    D := {|z_i − z_j|, z_i ∈ Z, z_j ∈ Z} ∩ ℕ_{>0}
    if D ≠ ∅ then
        B := FormalSolutions(L_f^min, z = s, order = max(D))
        if B has logarithms then return T
return FAIL
```

be factored with the current computer algebra algorithms. The proof in [9, Proposition 8.4] relies on the computation of the corresponding minimal operator $L_f^{\min}$. It is the starting point for Algorithms 1 to 3.

2.3. **An algorithm for Stanley's problem (S).** Based on Proposition 2, a first solution to Stanley's problem **(S)** is Algorithm 1. It solves it *in principle*, in the sense that it shows that **(S)** is decidable. However, its computational complexity is too high to be of practical use, mainly because it relies on the costly algorithm from [86].

Theorem 6. *Given a power series $f \in \mathbb{Q}[[z]]$ by a linear differential operator $L \in \mathbb{Q}[z]\langle\partial_z\rangle$ such that $L(f) = 0$, together with sufficiently many initial conditions, Algorithm 1 decides whether f is algebraic or transcendental.*

Proof. If f is algebraic then, by Proposition 2, $L_f^{\min}$ admits a full basis of algebraic solutions, and this is detected by Singer's algorithm [86]. If f is transcendental, then $L_f^{\min}$ has no nontrivial algebraic solution, and again this is detected by the algorithm in [86]. □

2.4. **An efficient transcendence test.** To go further towards a more efficient variant of Algorithm 1, we use Proposition 2 and Corollary 3.

Theorem 7. *Given a power series $f \in \mathbb{Q}[[z]]$ by a linear differential operator $L \in \mathbb{Q}[z]\langle\partial_z\rangle$ such that $L(f) = 0$, together with sufficiently many initial conditions, Algorithm 2 either proves that f is transcendental or returns* FAIL.

In terms of completeness, Algorithm 2 is not as good as Algorithm 1, since it does not detect algebraicity. As such it is not a full decision procedure for transcendence. However, in terms of efficiency, Algorithm 2 is much better than Algorithm 1, as it only relies on the minimization algorithm from [16], plus a few computationally cheap local tests.

2.5. **An efficient conditional algorithm in the globally bounded case.** To state our next results, we now recall a collection of conjectures on D-finite globally bounded power series.

Conjecture 8 (Christol, André)**.** *Let $f \in \mathbb{Q}[[z]]$ be a globally bounded and D-finite power series. Then:*

1. *f is the diagonal of a rational function;*
2. *If $z = 0$ is an ordinary point for $L_f^{\min}$, then f is algebraic;*
3. *If the monodromy of $L_f^{\min}$ at $z = 0$ is semisimple (i.e., $z = 0$ is not a logarithmic singularity of $L_f^{\min}$), then f is algebraic.*

Part (1) of Conjecture 8 is due to Gilles Christol, and formulated in the late 1980s [21], see also [22, Conj. 4] and [23, Conj. 10]. Part (2) was formulated around 1997 in private discussions between Gilles Christol and Yves André, and appeared in print in [3, Rem. 5.3.2] (there, the connection with the Grothendieck-Katz p-curvatures conjecture is also discussed). See also [70, §3.3] for a similar conjecture (called "Eisenstein's Algebraicity Criterion" by the author). Part (3) is due to Yves André (private communication).

The global boundedness assumption in Conjecture 8 is crucial for the three parts. For instance, without this assumption, Part (1) does not hold for $f(z) = \log(1-z)$, Part (2) does not hold for $f(z) = \exp(z)$ and Part (3) does not hold for the power series $f(z) = {}_2F_1\left[\begin{matrix}1/6\ 5/6\\ 7/6\end{matrix};z\right]$ considered in [57, Prop. 2.3], which actually is the particular case $(a,b) = (1/6, 5/6)$ of the next remark.

Example 9. *There exist D-finite transcendental power series whose minimal-order operator is Fuchsian and such that all singular points (including infinity) are algebraic (not logarithmic) singularities. Take a and b in $\mathbb{Q}\setminus\mathbb{Z}$ such that $b-a$ is not a positive integer. Then $f(z) = {}_2F_1\left[\begin{matrix}a\ b\\ a+1\end{matrix};z\right]$ is such that its L_f^{min} is reducible, and equals*

$$z(1-z)\partial_z^2 + (a+1-(a+b+1)z)\partial_z - ab = \Big(z(1-z)\partial_z + 1-(b+1)z\Big)\Big(\partial_z + a/z\Big).$$

Thus, $L_f^{\min}$ is Fuchsian, it admits the algebraic solution z^{-a}, and its local exponents are $\{0,-a\}$ at $z=0$, $\{0,1-b\}$ at $z=1$, $\{a,b\}$ at $z=\infty$. Since the exponent differences are not integers, the three singularities are algebraic. Moreover, $f(z)$ is transcendental, since otherwise the operator $L_f^{\min}$ would have finite cyclic monodromy and this is impossible by [98, Thm. 2.2]*. However, $f(z)$ is not globally bounded by* [21, Prop. 1]*, so this example does not contradict Part (3) of Conjecture 8.*

We design now a more efficient version of Algorithm 1; this is Algorithm 3, based on Proposition 2, Corollary 3, and Conjecture 8.

Theorem 10. *Given a linear differential operator $L \in \mathbb{Q}[z]\langle\partial_z\rangle$ and a power series $f \in \mathbb{Q}[[z]]$ such that $L(f) = 0$, Algorithm 3 is correct when it outputs* T. *Moreover, if f is globally bounded, then assuming Part (3) of Conjecture 8, Algorithm 3 is also correct when it outputs* A.

Proof. If f is algebraic, then, by Prop. 3, $L_f^{\min}$ is Fuchsian and it admits only rational exponents at all its singular points. Moreover, if $L_f^{\min}$ has a logarithmic

Algorithm 3 Transcendence of globally bounded D-finite functions

Input: $L = a_r(z)\partial_z^r + \cdots + a_0(z)$ with $a_i(z) \in \mathbb{Q}(z)$;
ini: f_0 a truncated power series at precision $p_0 \geq r$
specifying a unique solution $f \in \mathbb{Q}[[z]]$ of $L(f) = 0$.
It is assumed that f is globally bounded.
Output: Either T if f is transcendental, or A if f is algebraic.

In Algorithm 2, replace FAIL *by* A *and line 8 by*
8: **if** $s = 0$ and $D \neq \emptyset$ **then**

singularity at $z = 0$, then $L_f^{\min}$ does not have algebraic solutions, hence f is transcendental by Prop. 2. This proves the first part of the result. The last part follows from Conjecture 8. ☐

Remark 11. *Example 9 shows that the assumption "$f(z)$ is globally bounded" is necessary in Algorithm 3, even if we replaced "$L_f^{\min}$ has a logarithmic singularity at $z = 0$" by "$L_f^{\min}$ has a logarithmic singularity at one of its singular points".*

Remark 12. *As already mentioned in §1.4, it is currently not known how to decide if a given D-finite power series $f(z)$ is globally bounded. However, this property is known to be satisfied by large classes of power series, such as generating functions of multiple binomial sums or, equivalently, by diagonals of rational functions, see* [15, Theorem 3.5]*. In this case, the operator L annihilating f does not need to be part of the input: it can be computed from a diagonal representation of f using* creative telescoping*, see* [15] *and the references therein (Section 5.2 provides several non-trivial examples). Still in this case, Algorithm 3 can be enhanced in the following way: after computing $L_f^{\min}$, if it detects logarithms in the local basis of solutions of $L_f^{\min}$ at $z = 0$, then it concludes transcendence; if it detects a term $\ln^s(z)$ in that local basis (e.g., if the indicial polynomial of $L_f^{\min}$ at $z = 0$ is $z^{s+1} \cdot P(z)$ with $P(0) \neq 0$), then it can conclude that the number of variables needed to express $f(z)$ as the diagonal of a rational function is at least $s + 2$* [43, Corollary 2.6]*, thus proving a strong form of transcendence of $f(z)$ if $s > 1$. For instance, in Example 4, the presence of terms $\ln^2(z)$ in the local basis of $L_A^{\min}$ at $z = 0$ implies that this number is at least 4 (and in fact exactly 4, e.g. using the diagonal representations in Section A.2), see also* [43, Example 3.9].

3. Solving problem (P)

Given an operator $L \in \mathbb{Q}(x)\langle\partial\rangle$ the aim in this section is to calculate another operator $L^{\text{alg}} \in \mathbb{Q}(x)\langle\partial\rangle$ whose solution space is spanned by the algebraic solutions of $Ly = 0$. This is done by extending some of Singer's results [86, 89].

Such an algorithm for computing L^{alg} can be used to solve Stanley's problem **(S)** in full generality, providing a second proof that Stanley's problem **(S)** is decidable. Starting with the D-finite power series f given as a solution of the input operator L together with finitely many initial conditions, one first computes L^{alg}, and then checks if f is a solution of L^{alg} (using [16, Lemma 2.1]). If it is, then f is algebraic, if it is not, then f is transcendental. Note however that the corresponding algorithm relies on several highly non-trivial algorithmic bricks, such as ODE factorization

Algorithm 4 Algebraic Solutions

Input: $L \in \mathbb{C}(x)\langle\partial\rangle$.
Output: $L^{\mathrm{alg}} \in \mathbb{C}(x)\langle\partial\rangle$ with solution space spanned by the algebraic solutions of $Ly = 0$.

1: factor L over $\mathbb{C}(x)$ as a product $L_1 L_2 \cdots L_t$ where each L_i is irreducible;
2: select from the factors L_i those factors $L_{i_1}, \dots, L_{i_s}$ that admit a full basis of algebraic solutions; // *Singer's algorithm* [86]
3: for $j = 1, \dots s$, construct an operator $\widehat{L}_{i_j}$ having the following property:// *See §3.1*

$$\widehat{L}_{i_j} = \mathrm{LCLM}\{R \in \mathbb{C}(x)\langle\partial\rangle \mid R \text{ is equivalent to } L_{i_j} \text{ and divides } L \text{ on the right}\}.$$

4: Compute $L^{\mathrm{alg}} = \mathrm{LCLM}(\widehat{L}_{i_1}, \dots, \widehat{L}_{i_s})$.
5: **return** L^{alg}

and Risch's algorithm for Abel's problem. Moreover, Jordan's bound $(49r)^{r^2}$ is so large that it prevents one from obtaining a practical algorithm (except for small orders r, where one can use more refined bounds [60, 48, 92]).

Still, the computation of L^{alg} is a question of independent interest. We first recall the following definition:

Definition 13. *Let k be a differential field and $L_1, L_2 \in k\langle\partial\rangle$ be irreducible operators of order n. We say L_1 is* equivalent *to L_2 if there exist $a_0, a_1, \dots, a_{n-1} \in k$, not all zero, such that L_1 divides $L_2 \circ (a_{n-1}\partial^{n-1} + \cdots + a_0)$ on the right.*

This is a special case of a more general definition but we will only need this version and the fact that this is an equivalence relation [89, Cor. 2.6]. If K is a Picard-Vessiot extension of k containing a full set of solutions of $L_1 y = 0$ and of $L_2 y = 0$ then the map $\phi : y \mapsto a_{n-1}\partial^{n-1} y + \cdots + a_0 y$ maps the solution space of $L_1 y = 0$ to the solution space of $L_2 y = 0$. Since L_1 and L_2 are irreducible, this map is a bijection. If all solutions of $L_1 y = 0$ are algebraic then so are their images by ϕ and therefore the same is true for $L_2 y = 0$.

The algorithm computing L^{alg} is given in Algorithm 4. Step 1 is performed using any factorization algorithm, e.g. [40], or [51]; step 2 is solved by Singer's algorithms [86] or [87]; for the computation of the LCLM in step 4, see e.g. [12], or [50].

We will now justify the equation in the last step and show how one can construct the operator in step 3.

3.1. **Proof of the algorithm.** This proof uses concepts from the differential Galois theory of linear differential equations. These can be found in [96]. Let k be a differential field, $L \in k\langle\partial\rangle$, K the Picard-Vessiot extension of k corresponding to $Ly = 0$ and G the Galois group of K over k.

The vector space of solutions of $Ly = 0$ algebraic over k is invariant under G and so span the solution space V^{alg} of a linear differential equation $L^{\mathrm{alg}} y = 0$ with $L^{\mathrm{alg}} \in k\langle\partial\rangle$, [89, Lemma 2.2]. Note that L^{alg} divides L on the right [89, Lemma 2.1]. The equation $L^{\mathrm{alg}} y = 0$ obviously has a full set of solutions in K and they generate a Picard-Vessiot extension E over k with $k \subset E \subset K$. Furthermore the Galois group H of E over k is finite. A finite group acting on a vector space allows one to decompose the vector space as a direct sum of irreducible subspaces. Therefore $V^{\mathrm{alg}} = \sum_i V_i$, each V_i an H-irreducible space. Furthermore, each V_i is the solution space of an irreducible $L_i \in k\langle\partial\rangle$. Factorizations of linear operators are unique up to equivalence [89, Proposition 2.11]. It follows that L_i must be equivalent to some L_{i_j} and so divides some $\widehat{L}_{i_j}$. Therefore the solution space of $\mathrm{LCLM}\{\widehat{L}_{i_1}, \dots, \widehat{L}_{i_s}\}y = 0$ contains the solution space of $L^{\mathrm{alg}}y = 0$. Conversely all solutions of $\mathrm{LCLM}\{\widehat{L}_{i_1}, \dots, \widehat{L}_{i_s}\}y = 0$ are algebraic so its solution space is contained in the solution space of $L^{\mathrm{alg}}y = 0$. Therefore these monic operators are equal.

3.2. **Calculating the $\widehat{L}_{i_j}$.** This will follow from

Proposition 14. *Let k be a differential field satisfying the following property:*

> *If $L \in k\langle\partial\rangle$ then one can effectively find a basis of the space $\{u \in k \mid Lu = 0\}$.*

Let $L_1, L \in k\langle\partial\rangle$ and assume that L_1 is irreducible. One can effectively find

$$\widehat{L}_1 = \mathrm{LCLM}\{R \in k\langle\partial\rangle \mid R \textit{ is equivalent to } L_1 \textit{ and divides } L \textit{ on the right.}\}$$

Proof. Let L_1 have order n. The set

$$W = \{\mathbf{a} = (a_{n-1}, \dots, a_0) \in k^n \mid L_1 \text{ divides } L \circ (a_{n-1}\partial^{n-1} + \ldots + a_0) \text{ on the right}\}$$

is a vector space over the constants. It is not hard to see (see the discussion following Example 2.8 in [89]) that there is an $n \times n$ matrix $\mathcal{A}$ with entries in $k\langle\partial\rangle$ such that $\mathbf{a} \in W$ if and only if $\mathcal{A}\mathbf{a} = 0$. Finding solutions of $\mathcal{A}Y = 0$ in k can be reduced to finding solutions of scalar linear differential equations in k and so one can find a basis of W.

For $\mathbf{a} = (a_{n-1}, \dots, a_0) \in W$, we construct an operator $L_{\mathbf{a}} \in k\langle\partial\rangle$ of order n so that the map $y \mapsto a_{n-1}\partial^{n-1}y + \cdots + a_0 y$ maps the solution space of $L_1 y = 0$ to the solutions space of $L_{\mathbf{a}}y = 0$. This is done by successively differentiating $a_{n-1}\partial^{n-1}y + \ldots + a_0 y$ n times and using the equation $L_1 y = 0$ to express each derivative in terms of $y, \dots, \partial^{n-1}y$. The resulting $n+1$ expressions must be linearly dependent over k and this yields the equation $L_{\mathbf{a}}$. Furthermore $L_{\mathbf{a}}$ has the following properties:

- $L_{\mathbf{a}}$ is equivalent to L_1.
- By definition, the map $y \mapsto a_{n-1}\partial^{n-1}y + \ldots + a_0 y$ also maps the solution space of $L_1 y = 0$ into the solution space of $Ly = 0$, so $L_{\mathbf{a}}$ will divide L on the right.

Let $\{\mathbf{a}_1, \dots, \mathbf{a}_r\}$ be a basis of W and let $\widetilde{L}_1 = \mathrm{LCLM}\{L_{\mathbf{a}_1}, \dots, L_{\mathbf{a}_r}\}$. We will show that $\widetilde{L}_1 = \widehat{L}_1$. Since each $L_{\mathbf{a}_i}$ is equivalent to L_i and divides L on the right, we have that $\mathrm{LCLM}\{L_{\mathbf{a}_1}, \dots, L_{\mathbf{a}_r}\}$ divides $\widehat{L}_1$ on the right.

Now assume that an operator R is equivalent to L_1 and divides L on the right. Let $a_{n-1}\partial^{n-1} + \ldots + a_0$ be such that L_1 divides $R \circ (a_{n-1}\partial^{n-1} + \ldots + a_0)$. Since R divides L on the right, we have that L_1 divides $L \circ (a_{n-1}\partial^{n-1} + \ldots + a_0)$ on the right. Therefore $\mathbf{a} = (a_{n-1}, \dots, a_0) \in W$ and so there exist $c_1, \dots, c_s \in C$ such that $\mathbf{a} = \sum c_i \mathbf{a}_i$. This implies that $y \mapsto a_{n-1}\partial^{n-1}y + \ldots + a_0 y$ takes the solution space of R into the sum of the solution spaces of the $L_{\mathbf{a}_i}y = 0$ which is the solution space of $\mathrm{LCLM}\{L_{\mathbf{a}_1}, \dots, L_{\mathbf{a}_r}\}y = 0$. Therefore R divides $\mathrm{LCLM}\{L_{\mathbf{a}_1}, \dots, L_{\mathbf{a}_r}\}$ on the right. $\widehat{L}_1$ is the LCLM of all such R so we have that $\widehat{L}_1$ divides $\mathrm{LCLM}\{L_{\mathbf{a}_1}, \dots, L_{\mathbf{a}_r}\}$ on the right. Since both of these operators are monic, we have shown they are equal. □

Remark 15. *Proposition 14 can be used to construct the Loewy decomposition of an operator into the product of completely reducible operators.*

4. Other approaches

4.1. **Using the Grothendieck-Katz p-curvatures conjecture.** In the early 1970s, Alexander Grothendieck proposed a conjectural "arithmetic" characterization for linear differential operators with coefficients in $\mathbb{Q}(z)$: such an operator admits a basis of algebraic solutions if and only if the same holds for its reductions modulo p for almost all primes p. Grothendieck's conjecture can be seen as a differential generalization of a particular case, due to Kronecker, of Chebotarev's theorem: the roots of a polynomial in $\mathbb{Q}(z)$ are all rational if and only if the roots of its reductions modulo p are in $\mathbb{F}_p$ for almost all primes p. This conjecture was studied in depth and popularized by Nicholas Katz [55], who proved it for the Gauss hypergeometric equations, and more generally for the so-called Picard-Fuchs operators. These are precisely operators of the form $L_f^{\min}$, where $f(z)$ is an r-multiple integral of a rational (or of an algebraic) function taken over a cycle in $\mathbb{C}^r$ with a parameter z. It particular, the Grothendieck-Katz conjecture holds for $L_f^{\min}$, where $f(z)$ is the diagonal of a rational function, or equivalently where $f(z)$ is the generating function of a multiple binomial sum.

Now, for any fixed prime p, the linear differential operator $L_f^{\min} \bmod p$, with coefficients in $\mathbb{F}_p(z)$, admits a basis of algebraic solutions if and only if it admits a basis of rational solutions [19], and this is verified by a very simple algorithm, based on checking the nullity of the p-curvature of $L_f^{\min} \bmod p$.

In practice, computing p-curvatures for a bunch of primes p allows to conclude, at least heuristically: if $L_f^{\min} \bmod p$ has some nonzero p-curvature, then $f(z)$ is transcendental; otherwise $f(z)$ is algebraic. To turn this heuristic into a decision algorithm, one would need an upper bound on the number of primes whose p-curvature is to be tested. In other words, one needs an effective Grothendieck-Katz theorem, similar to effective versions of Chebotarev's theorem. The existence of such results, at least under some additional assumptions, is alluded to by André [3, §16.3.1].

4.2. **Guess-and-prove.** To prove algebraicity of a given power series $f(z)$, a very popular method in computer algebra is to first guess a polynomial $P(z,u) \in \mathbb{Q}[z,u]$ such that $P(z, f(z)) = O(z^\sigma)$ for some value σ, and then to certify that $P(z, f(z)) = 0$ by manipulating the roots of P via the differential operator L_P whose solution space is generated by these roots. This strategy is explained for instance in the proof of [14, Corollary 2]. The guessing part is based on (structured) linear algebra (over $\mathbb{Q}$) in size σ and the most efficient algorithms use Hermite-Padé approximants (which reduce the problem to linear algebra over $\mathbb{Q}(z)$ in a size much smaller than σ). The proving part is based on the computation of L_P, which ultimately also relies on linear algebra [11]. A delicate issue is the choice of σ; usually one keeps doubling its value until a candidate P is found. If $f(z)$ is algebraic, then the procedure will eventually discover it (modulo computational difficulties related to the size of σ that can be huge). The strength of the approach is that it is able to recognize and prove algebraicity of $f(z)$ even in situations where $f(z)$ is not *a priori* known to be D-finite (e.g., it is given not by an ODE, but by a different kind of functional equation), see e.g. [14]. The weak point of the approach is that it cannot prove that $f(z)$ is transcendental. This is why it is usually used in conjunction with other heuristics such as computations of p-curvatures, or numerical computations of local monodromies.

4.3. **Factoring.** If the input L is checked to be irreducible, then its solution f is algebraic if and only if all solutions of L are algebraic (by Proposition 2.5 in [86]), and this can be tested using Singer's 1979 algorithm [86]. But if L is reducible, then it is *a priori* not clear how to combine factoring algorithms [51, 40] and Singer's algorithm [86] in order to solve Stanley's problem **(S)**. One practical way is: factor the input $L = AB$ for irreducible A of order less than the order of L and check if A has a basis of algebraic solutions. If yes, then we cannot conclude. If no, then A has no nontrivial algebraic solutions, and there are two possible cases. Either $g \coloneqq B(f)$ is nonzero, in which case it is transcendental (as it is a nontrivial solution of A), hence f itself is transcendental (otherwise g would be algebraic). Or, $g = 0$, and we can repeat the whole procedure on B. This procedure is guaranteed to succeed in the particular case when L factors as a product of irreducible operators each of them having no algebraic solutions.

5. Examples

Starting from an implementation of the minimization algorithm [16], the implementation of the transcendence test of Algorithm 2 in Maple is straightforward and fits in 40 lines.[10] We list a few experiments performed with this implementation.

5.1. **Combinatorial examples.** Examples with a combinatorial origin for which our implementation gives an automatic proof of transcendence are:

- the differential equation of order 11 and degree 73 from 3-dimensional walks confined to the positive octant mentioned after Example 5, coming from [9, Proposition 8.4] (in 10 sec.)[11];
- the 72 transcendental cases from [13, Theorem 2] (in total time 25 sec.);
- the lattice Green functions of the face-centered cubic lattice in dimensions $d = 3, \ldots, 10$. These (D-finite) functions are defined by

$$G_d(z) \coloneqq \frac{1}{\pi^d} \int_0^{\pi} \cdots \int_0^{\pi} \frac{d\theta_1 \cdots d\theta_d}{1 - z\binom{d}{2}^{-1} \sum_{1 \le i < j \le d} \cos\theta_i \cos\theta_j}$$

  When $d = 2$ and $d = 3$, we have hypergeometric expressions [42, Eq. (53)], [54, Eq. (3.6)]

$$G_2(z) = {}_2F_1\left[\begin{matrix}\frac{1}{2}\ \frac{1}{2}\\ 1\end{matrix}; z^2\right], \quad G_3(z) = {}_2F_1\left[\begin{matrix}\frac{1}{6}\ \frac{1}{3}\\ 1\end{matrix}; \frac{z^2(z+3)^2}{4}\right]^2$$

  that can be seen to be transcendental by the arguments of Section 1.6. For $d \ge 4$, no such closed formulas are known, and it is not clear how to decide transcendence of $G_d(z)$ using asymptotic arguments. Linear differential equations for $G_d(z)$ were first conjectured by Guttmann in dimension 4 [41]

[10] It is implemented in the function `istranscendental` in versions larger than 4.05 of the package `gfun`, available at https://perso.ens-lyon.fr/bruno.salvy/software/the-gfun-package/, where a Maple session with these examples can also be found.

[11] Timings were obtained with Maple2024.2 on a MacBook Pro 2017 with a quad-core Intel i7.

dim	order	degree	transc. (sec.)	factor. (sec.)
3	3	5	2.2	0.9
4	4	10	0.1	0.2
5	6	17	0.6	1.3
6	8	43	4.2	18.3
7	11	68	24.0	265.
8	14	126	174.9	4706.
9	18	169	771.6	>10000
10	22	300	8817.1	

TABLE 1. Examples from lattice Green functions.

and by Broadhurst in dimension 5 [18], after intensive computations. They were then computed by Koutschan for $d = 3, 4, 5, 6$ using creative telescoping [59]. For $d = 7$, an equation was conjectured by Hassani, Maillard and Zenine [101] and then equations were conjectured up to $d = 11$ by Hassani, Koutschan, Maillard and Zenine [44]. While these differential equations have been obtained after days of computations, it is relatively simple for our code to prove that they are minimal. (Note that their minimality for $d \leq 11$, and even their irreducibility for $d \leq 7$, were claimed in [44], but the arguments there are based on heuristics.) Since all these equations are minimal and have a logarithmic singularity at 0, Proposition 2 implies that $G_d(z)$ is transcendental for $d \leq 10$ (assuming the conjectured differential equations to be correct). Alternatively, one can use factorization of linear differential operators to prove that the differential equations for $G_d(z)$ are irreducible for $d \leq 8$, and use Proposition 2.5 in [86] (rather than Proposition 2) to deduce the transcendence of $G_d(z)$. Sizes and timings are reported in Table 1. (The column 'transc' indicates the time taken by our code to prove transcendence of $G_d(z)$; the time taken by the factorization program from [24] is given in the column 'factor'. Maple's native command `DFactor` was much slower.)

In practice, for all these examples, all the time is spent in the first step where a differential equation of certified minimality is computed.

5.2. **Diagonals.** A classical result by Pólya and Furstenberg states that the diagonal of any bivariate rational function is algebraic [78, 37]. In a larger number of variables, that property does not hold in general (in characteristic zero) and our approach proves or disproves algebraicity. For instance, using Koutschan's `HolonomicFunctions` package [58], the diagonal of

$$\frac{1}{(1 - 5x - 7yz - 13z^2)(1 - x - xy)}$$

is found to satisfy a linear differential equation of order 5 with coefficients of degree up to 24. This equation has logarithmic singularities at 0 and 1/140. However, minimization shows that the diagonal actually satisfies an equation of order only 3 with coefficients of degree up to 13. From that equation, Algorithm 3 strongly suggests that the diagonal is algebraic (being a diagonal, it is globally bounded).

Actually, the more general situation of the diagonal of

$$F(x,y,z) = \frac{1}{(1-ax-byz-cz^2)(1-x-xy)}$$

for arbitrary a, b, c can be seen to be algebraic by a direct residue computation: the diagonal is the residue at $x = y = 0$ of

$$G(x,y,z,t) = \frac{1}{xy}F(x,y,t/(xy)) = \frac{xy}{((x-ax^2-bt)xy^2-ct^2)(1-x-xy)}$$

when $t \to 0$. The second factor of the denominator has a root in y that is away from 0 while the first factor has two roots in y that both tend to 0 when $t \to 0$. The sum of their residues is therefore a rational function in x, t. From there it follows that the sum of residues in x is algebraic. An explicit computation is possible: by a Rothstein-Trager resultant, one obtains a polynomial cancelling these two residues in y, from whose coefficients their sum is deduced to be

$$\frac{x-1}{a\,x^4 - 2a\,x^3 + bt\,x^2 + c\,t^2x + a\,x^2 - 2btx - x^3 + bt + 2x^2 - x}.$$

Only one of the roots of the denominator tends to 0 with t. Again, a Rothstein-Trager resultant gives a polynomial of degree 4 cancelling the diagonal.

A very similar-looking diagonal is that of

$$\frac{1}{(1-x-y-z^2)(1-x-xy)}.$$

`HolonomicFunctions` gives a linear differential operator L_7 of order 7 with coefficients of degree up to 19. This operator is not irreducible: it is the product T_4A_3 of an operator T_4 that does not have nonzero algebraic solutions with an operator A_3 all of whose solutions are algebraic. Our code proves that L_7 is minimal for the diagonal Δ which is then proved transcendental by observing that L_7 has a logarithmic singularity at 0. Another proof is by observing that A_3 does not cancel Δ, which implies that $A_3\Delta$ is a solution of T_4 and thus necessarily transcendental and therefore so is Δ. That alternative way does not work on the adjoint L_7^* of L_7. If one takes the solution $s(t)$ which is its unique power series solution with $s(0) = -1/2$, $s'(0) = 0$, $s''(0) = -21121726441/112000$, then again our program proves that it is transcendental. A factorization of L_7^* of the form $A_3^*T_4^*$ still has A_3^* whose solutions are all algebraic, but a conclusion from there does not seem direct.

Finally, the diagonal of

$$\frac{1}{(1-x-y-z^2)(1-x-xy^2)}$$

is even more challenging. It is annihilated by a linear differential operator L_9 of order 9 and degree 60, that factors as a product T_4A_5[12]. Our code detects that L_9 is minimal for the diagonal and proves that it is transcendental. The direct computations suggested above become more involved. Even proving that all solutions of A_5 are algebraic seems challenging[13].

[12]This operator and its factorization were communicated to us by Jean-Marie Maillard. Similar examples can be found in [45].

[13]Gilles Villard found a polynomial of degree 120 with coefficients of degree 460 that cancels the power series with largest valuation among the solutions of A_5, modulo a large prime (892901). Similar polynomials can be found for a basis of solutions. Proving that the solutions of such a polynomial are a basis of those of A_5 is feasible in theory, but is very demanding computationally.

5.3. **Apéry-like series.** For $p, q \in \mathbb{N}\setminus\{0\}$, we consider the following generalization of the power series in Example 4:

$$f_{p,q}(z) = \sum_{n\geq 0} \left(\sum_{k=0}^{n} \binom{n}{k}^{p} \binom{n+k}{k}^{q} \right) z^n.$$

In Appendix A we use Flajolet's criterion to prove that $f_{p,q}(z)$ is transcendental if and only if $(p, q) \neq (1, 1)$. It was conjectured by Sergey Yurkevich in [99, Conjecture 10.5] that the minimal number $\mu_{p,q}$ of variables needed to represent $f_{p,q}(z)$ as the diagonal of a rational power series is equal to $p + q$. Yurkevich showed that $\mu_{p,q} \leq p + q$, by using the identity (due to Wadim Zudilin)

$$f_{p,q} = \operatorname{Diag}\left(\frac{1}{\left(\prod_{j=1}^{q}(1-y_j) - x_1\right) \cdot \prod_{k=2}^{p}(1-x_k) - \prod_{k=1}^{p} x_k \cdot \prod_{j=1}^{q} y_j} \right). \tag{3}$$

For $p+q \leq 10$, we checked the opposite inequality $p+q \leq \mu_{p,q}$ by using [43, Corollary 2.6] and the enhanced version of our Algorithm 3, mentioned in Remark 12. Indeed, in all these 45 cases, the indicial polynomial $I_{p,q}(z)$ of $L^{\min}_{f_{p,q}}$ at $z = 0$ equals $z^{p+q-1} \cdot R_{p,q}(z)$ where $R_{p,q}(0) \neq 0$, hence Yurkevich's conjecture holds. By looking closely at the indicial polynomials $I_{p,q}(z)$, we made a few additional observations. First, the degree of $I_{p,q}(z)$, that is the order of $L^{\min}_{f_{p,q}}$, is equal to $\lfloor (p+q)^2/4 \rfloor$ except when p and q are both even and equal, in which case it equals $p^2 - 1$. In particular, the order of $L^{\min}_{f_{p,q}}$ is equal to that of $L^{\min}_{f_{q,p}}$ in the 45 cases. Second, if p is odd or if $p < q$, then $I_{p,q}(z)$ factors $(z-1)^{p+q-3} \cdots (z-r)^{(r \bmod 2)+1}$ where $r = \lfloor (p+q)/2 \rfloor - 1$; when p and q are both even and equal, then $I_{p,q}(z)$ equals $z^{2p-1} \cdot (z-1)^{2p-3} \cdots (z-(p-2))^3$. We leave all these questions open for general values of p and q. (Similar considerations for the exponential generating versions of $f_{p,q}$ appear in Question 5.2 in [16].)

Appendix A. Generalized Apéry series

A.1. **A multiple binomial sum.** A direct application of Flajolet's criterion proves the transcendence of the following family of power series, extending the one in Section 5.3.

Proposition 16. *Let* $(p_0, \dots, p_m) \in \mathbb{N}^{m+1}$ *with* $p_0 \geq 1$. *Then the power series*

$$f_{\mathbf{p}}(z) = \sum_{n\geq 0} \left(\sum_{k=0}^{n} \binom{n}{k}^{p_0} \binom{n+k}{k}^{p_1} \binom{n+2k}{k}^{p_2} \cdots \binom{n+mk}{k}^{p_m} \right) z^n$$

is transcendental if and only if $p := p_0 + \cdots + p_m > 2$.

Proof. The case $p > 2$ is covered by Flajolet's criterion. Asymptotically, the nth coefficient $S_{\mathbf{r}}(n)$ of $f_{\mathbf{p}}(z)$ behaves like

$$S_{\mathbf{p}}(n) \sim \gamma (\pi n)^{\frac{1-p}{2}} \beta^n, \quad n \to \infty \tag{4}$$

with γ and β algebraic [71]. If $p > 2$ is odd, the exponent of n is a negative integer making $f_{\mathbf{r}}(z)$ transcendental; if p is even, $\Gamma((3-p)/2)$ is a rational multiple of $\sqrt{\pi}$ and thus $\gamma\Gamma((3-p)/2)\pi^{(1-p)/2}$ is an algebraic multiple of $\pi^{1-p/2}$, which is not algebraic for $p > 2$.

For $p = 1$, the series $f_{(1,0,\dots,0)}$ is the rational $1/(1-2z)$. Finally, for $p = 2$, either $p_0 = 2$ and all the other p_i are 0, in which case $f_{\mathbf{p}} = (1-4z)^{-1/2}$, or $p_0 = 1$, $p_j = 1$ for one $j \neq 0$ and all other p_i are 0. In that case, the power series can be seen to be algebraic as the diagonal of a bivariate rational function. Here, the rational function

$$\frac{1}{1 - z(1+y)(y + (1+y)^j)} = \sum_{n \geq 0} z^n \sum_{k=0}^{n} \binom{n}{k} y^{n-k} (1+y)^{n+jk},$$

has a Taylor expansion whose coefficient of $z^n y^n$ is the desired

$$\sum_{k=0}^{n} \binom{n}{k} \binom{n+jk}{k}. \qquad \square$$

A.2. **Generic diagonals.** It is well known that the Apéry power series $f_{2,2}$ can be written as the diagonal of a rational power series in 4 variables, e.g. as the diagonal of $1/\left(1 - z\,(1+x)\,(1+y)\,(1+t)\,(txy + (1+y)\,(1+t))\right)$, or of $1/((1-x-y)(1-t-z) - xyzt)$. From such a diagonal representation, the approach of analytic combinatorics in several variables (ACSV) allows to deduce the asymptotics of the n-th Apéry number A_n as in Example 4, see e.g. [72, Example 1]. More generally, the generalized Apéry power series $f_{p,q}$ can be written as the diagonal of a rational power series in $m = p + q$ variables *with nonnegative coefficients*, e.g. via (3). Now, [72, Result 1] states that under certain assumptions that hold generically, the n-th coefficient of the diagonal of a rational power series in m variables with nonnegative coefficients grows asymptotically like $\gamma \beta^n n^r$ where $r = (1-m)/2$ and $\gamma/(2\pi)^r \in \overline{\mathbb{Q}}$. (This is coherent with the asymptotics (4) in the proof of Proposition 16.) Combined with Flajolet's criterion, the formula of [72, Result 1] implies that diagonals of "generic" rational functions with nonnegative coefficients are transcendental if $m > 2$; conversely, the bivariate case is algebraic by the result of Pólya and Furstenberg. Note that the genericity assumption is essential, as shown by the example of the diagonal of $1/(1 - x - y - yz - xyz)$, equal to $f_{1,1}(z)$ which is algebraic by Proposition 16, or by the algebraic examples of Section 5.2.

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