

ON DECIDING TRANSCENDENCE OF POWER SERIES

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ABSTRACT. It is well known that algebraic power series are differentially finite (D-finite): they satisfy linear differential equations with polynomial coefficients. The converse problem—determining whether a given D-finite power series is algebraic or transcendental—is notoriously difficult. We prove that this problem is decidable: we give two theoretical algorithms and a transcendence test that results in an efficient Maple implementation.

1. INTRODUCTION

1.1. Stanley’s problem and our contribution. A power series $f \in \mathbb{Q}[[z]]$ is called *algebraic* if it is a root of some polynomial equation $P(z, f(z)) = 0$, where $P \in \mathbb{Q}[x, y] \setminus \{0\}$; otherwise, f is called *transcendental*. A classical result, known to Abel (1827) and possibly much older, states that any algebraic power series $f \in \mathbb{Q}[[z]]$ is *D-finite* (or, *differentially finite*)—that is, it satisfies a linear differential equation with coefficients in $\mathbb{Q}[z]$. Moreover, the minimal-order nontrivial homogeneous linear differential equation satisfied by f has order at most $\deg_y(P)$ and coefficients in $\mathbb{Q}[z]$ of degree at most $4 \deg_x(P) \deg_y(P)^2$ [13].

Conversely, not every D-finite power series is algebraic; for instance, $\exp(z)$ and $\log(1 - z)$ are both D-finite and transcendental. Several methods are available to prove the transcendence of $\exp(z)$ and $\log(1 - z)$ (see e.g., [12]), but in general it is notoriously difficult to decide if a given D-finite function is algebraic or transcendental. This is the topic of our article.

In his seminal article on D-finite functions [103, §4, (g), page 186], Richard Stanley asked the following question:

“Given the differential equation

$$(1) \quad L(f(z)) = a_r(z)f^{(r)}(z) + \cdots + a_0(z)f(z) = 0$$

with polynomial coefficients $a_i \in \mathbb{Q}[z]$, together with suitable (finitely many) initial conditions, satisfied by a D-finite power series f , give an algorithm suitable for computer implementation for deciding whether f is algebraic.”

For instance, $f(z) := \sum_{n \geq 1} z^n/n$ is a D-finite power series represented by the second-order differential equation $(z - 1)f''(z) + f'(z) = 0$ and initial conditions $f(0) = 0, f'(0) = 1$, and such an algorithm would have to recognize the transcendence of f starting from this data alone.

Deciding transcendence of formal power series has many motivations and applications. For instance, in number theory a first step towards proving the transcendence of a complex number consists in proving the transcendence of some suitable power

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series attached to it. Other examples come from combinatorics, where the nature of generating functions may reveal strong underlying structures [15], and from computer science, where a natural question is whether algebraic power series are easier to manipulate than transcendental ones [9].

Although Stanley insisted on the practical aspect of the targeted algorithm for deciding algebraicity of D-finite functions, not even a theoretical one has appeared so far. In this work, we make progress on Stanley’s question: we give two algorithms that prove the decidability of the algebraic or transcendental nature of D-finite power series in $\mathbb{Q}[[z]]$ and we give an (incomplete) transcendence test whose implementation is efficient in practice.

The first theoretical algorithm (Algorithm 1) relies on minimization of linear differential equations [18]; the second one (Algorithm 5) uses factorization. Both rely on Singer’s algorithm [96]. The practical transcendence test (Algorithm 2) combines the same minimization as the first algorithm with a local analysis; it is incomplete in the sense that it is guaranteed to be correct only when it returns “transcendental”. If the input differential equation possesses additional properties (e.g., if it annihilates the diagonal of a multivariate rational function [25]), then a variant of it (Algorithm 3) is also guaranteed to be correct when it returns “algebraic”, modulo a conjecture in the theory of arithmetic differential equations (Conjecture 10). Designing complete and efficient algebraicity or transcendence tests is still open.

1.2. Difficulties and analogies. There are several *a priori* reasons why Stanley’s problem is difficult. First, the minimal polynomial of an algebraic power series $f(z)$ may have arbitrarily large size (degrees) with respect to the size (order/degree) of its defining differential equation. For instance, $f(z) = \sqrt[N]{1-z}$ has algebraicity degree N and satisfies the first-order differential equation $N(z-1)f'(z) - f(z) = 0$. Second, no characterization is available for coefficient sequences of algebraic power series; this is in contrast with the smaller class of rational functions (whose coefficient sequences are C-recursive, i.e., satisfy linear recursions with constant coefficients) and with the larger class of D-finite functions (whose coefficient sequences are P-recursive, i.e., satisfy linear recursions with polynomial coefficients)¹.

There are several analogies between the problems of transcendence in $\mathbb{Q}[[z]]$ and of irreducibility in $\mathbb{Q}[z]$. One is that “generic” power series in $\mathbb{Q}[[z]]$ are transcendental, much like “generic” polynomials in $\mathbb{Q}[z]$ are irreducible. Another one is that, while many sufficient criteria exist (e.g., Eisenstein’s criteria for polynomial irreducibility [34], see also [94, 93, 30], and for power series transcendence, see Section 1.4.1), none of them is also necessary. This is closely related to the fact that there are no known characterizations at the level of the coefficient sequence for recognizing either irreducibility in $\mathbb{Q}[z]$, or transcendence in $\mathbb{Q}[[z]]$. However, since polynomial irreducibility is known to be decidable [68, 71], it is legitimate to hope, by analogy, that the same holds for transcendence of power series.

1.3. Related problems. Although Stanley’s question is quite recent, it is related to many classical questions and results, dating back to the beginning of the 19th century.

¹Note that for the class of diagonals of rational functions, which is intermediate between algebraic and D-finite power series, coefficient sequences correspond (conjecturally) to the class of P-recursive sequences with (almost) integer coefficients and geometric growth (this is predicted by the “Christol conjecture”, see Part (1) of Conjecture 10).

Given a linear differential operator $L = \sum_{i=0}^r a_i(z) \partial_z^i$ (where ∂_z stands for $\frac{d}{dz}$) with coefficients in $\mathbb{Q}[z]$, modeling the linear differential equation (1), one may ask several questions concerning the algebraic nature over $\mathbb{Q}(z)$ of its solutions:

- (F) Decide if *all* solutions of L are algebraic (Fuchs' problem).
- (L) Decide if L admits *at least one* nontrivial algebraic solution (Liouville's problem).
- (S) Decide if a *given* nonzero solution f of L is algebraic (Stanley's problem).

Remark 1. In problems (F) and (L) an “algebraic solution” of L is an element f of the field of formal Puiseux series $\mathcal{P} = \bigcup_{n \in \mathbb{N}^*} \overline{\mathbb{Q}}((z^{\frac{1}{n}}))$, such that f is algebraic over $\mathbb{Q}(z)$ and $L(f) = 0$. (Here $\overline{\mathbb{Q}}$ is the algebraic closure of $\mathbb{Q}$.)

“All solutions of L are algebraic” means that L admits a full basis, i.e., a linearly independent family of cardinality $r = \text{ord}(L)$, of solutions in $\mathcal{P}$.

Note also that if L is irreducible², then the three problems (F), (L) and (S) are equivalent. This is because an irreducible operator either admits a full basis of algebraic solutions, or no nontrivial algebraic solution, see [96, Prop. 2.5].

Above, the word “decide” may have several meanings. One may for instance ask for a criterion, that is for a mathematical characterization based on the inspection of the coefficients of L , or of some invariants of L (such as its singularities, exponents at singularities, etc.³). The classical meaning in computer science—the one that we adopt in this article—is that one asks for an algorithmic decision procedure, which, on every specific instance of L , is able to answer the given problem using a finite number of operations $(+, \times, \div, =)$ in $\mathbb{Q}$.

In addition to such a decision procedure, one may also want to exhibit an annihilating polynomial (usually, the minimal polynomial) of the algebraic solution(s). Computing such a polynomial yields the most obvious “algebraicity witness”, on which the decision procedure itself may rely (see Section 4.2). However, these witnesses may have prohibitively large sizes; deciding algebraicity of solutions does not necessarily need them.

In this article, we add one more problem to the previous list:

- (P) Compute the (monic) right factor⁴ L^{alg} of L whose solution space is spanned by the algebraic solutions of L .

Again, if L is irreducible, then (P) is equivalent to the problems (F), (L) and (S).

Note that a solution to (P) implies solutions to each of the problems (F), (L) and (S). Indeed, solving (F), (L), and (S) amounts, respectively, to checking $\text{ord}(L) = \text{ord}(L^{\text{alg}})$, resp. $L^{\text{alg}} \neq 1$, or $L^{\text{alg}}(f) = 0$ (the last equality can be checked as explained in [18, Lemma 2.1]).

²This means that L cannot be written $L = L_1 L_2$ in the (non-commutative) ring $\mathbb{Q}(z)\langle \partial_z \rangle$ of linear differential operators, with $1 \leq \text{ord}(L_2) < r = \text{ord}(L)$. Irreducibility of linear differential operators was first studied by Frobenius (1873), and then uniqueness properties for factorization into irreducible operators were further studied by Landau (1902) and Loewy (1903). For historical accounts on the rich structural analogy between polynomials and linear differential equations, we refer to [32], [99, §2.4] and [14, p. 100], and the references therein.

³Finite singularities (or singular points) of Eq. (1) are points where one of the rational functions $a_i(z)/a_r(z)$ has a pole. Infinity is also a singular point when 0 is a singularity of the equation obtained by making the change of variables $z \mapsto 1/z$. For more information, see [59, §15.1].

⁴This is a linear differential operator with coefficients in $\mathbb{Q}(z)$, see Proposition 3.

1.4. Sufficient transcendence criteria. There exist several criteria that can be used to prove that a given power series $f \in \mathbb{Q}[[z]]$ is transcendental. They are simply built on properties of algebraic power series. We recall below two important ones, Eisenstein's arithmetic criterion and Flajolet's analytic criterion. Although very useful in practice, none of these sufficient criteria is also necessary, and there are concrete examples that escape them (see Example 6). This observation brings additional motivation to Stanley's problem (S).

1.4.1. Eisenstein's criterion.

Definition 2. A power series $f = \sum_{n \geq 0} a_n z^n$ in $\mathbb{Q}[[z]]$ is called globally bounded if it has a non-zero and finite radius of convergence, and if there exists $C \in \mathbb{N}^*$ such that $a_n C^n \in \mathbb{Z}$ for all $n \geq 1$.

A famous criterion, stated (and sketchily proved) in 1852 by Gotthold Eisenstein [35] and fully proved one year later by Eduard Heine [52], asserts that if a power series $f \in \mathbb{Q}[[z]]$ is algebraic, then only a finite number of prime numbers can divide the denominators of the coefficient sequence $(a_n)_{n \geq 0}$. This is also a consequence of a theorem⁵ by Heine [53] stating that any algebraic power series $f \in \mathbb{Q}[[z]]$ is globally bounded. This arithmetic criterion immediately implies that the power series $\log(1 - z) = -\sum_{n \geq 1} z^n/n$ (more generally, the polylogarithm $\text{Li}_s(z) = \sum_{n \geq 1} z^n/n^s$ for $s \in \mathbb{N}^*$), and $\exp(z) = \sum_{n \geq 0} z^n/n!$ are transcendental. However, when $f(z)$ is not given in closed form, but rather like in Stanley's problem (S), by a linear differential equation with sufficiently many initial conditions, then Eisenstein's criterion is difficult to apply. The main reason for this is a quite fundamental one: it is currently not known (and it is considered to be a difficult open problem) how to recognize that a P-recursive sequence $(a_n)_n$ given by a linear recurrence and initial conditions has integer terms (or, that it has *almost integer* coefficients, in the sense that $a_n C^n$ are integers for all $n \geq 1$, for some constant $C \in \mathbb{N}^*$). The same difficulty arises in a different context for the mere problem of representing E -functions, see [18].

1.4.2. Flajolet's criterion. In many situations, the given D-finite power series $f(z)$ already has integer coefficients, therefore Eisenstein's criterion is useless. This is systematically the case in combinatorics when $f(z)$ is the generating function of a sequence whose n th term counts the number of objects of size n in some combinatorial class. Much more useful is *Flajolet's criterion* [38, Criterion D] (see also [39, VII.7.2]) based on the Newton-Puiseux theorem and on Darboux's transfer results from the local behavior of $f(z)$ around its complex singularities to the asymptotic behavior of its coefficient sequence $(a_n)_{n \geq 0}$.

One form of the criterion asserts that if $a_n \sim \gamma \beta^n n^r$ as $n \rightarrow \infty$, with either r an irrational number or a negative integer, or β a transcendental number, or $\gamma \cdot \Gamma(r+1)$ a transcendental number, then $f(z)$ is transcendental.

For more general and more refined versions on the asymptotic behavior of the coefficients of algebraic power series, we refer to [39, p. 501], [82] and [90, Theorem 3.1].

⁵Heine's 1854 result is what is commonly called "Eisenstein's criterion" in the literature. Hence, strictly speaking there are two distinct criteria, one due to Eisenstein and the other one due to Heine. It might well be that they coincide for D-finite functions; this is related to Siegel's conjecture that the classes of "broad G -functions" and "strict G -functions" coincide, see e.g., [3, Conj. 2.3.1].

This criterion is the most commonly used for transcendence questions in combinatorics. It allows for instance to show that if $k \in \mathbb{N}^*$, then the hypergeometric power series $\sum_{n \geq 0} \binom{2n}{n}^k z^n$ (mentioned at the end of problem (g) in [103, §4]) is algebraic if and only if $k = 1$ (see Section 1.6 for a different proof), and also that the non-hypergeometric power series $\sum_{n \geq 0} \sum_{k=0}^n \binom{n}{k}^2 \binom{n+k}{k}^2 z^n$ (occurring in Apéry's celebrated proof of the irrationality of $\zeta(3)$, see Example 5) is transcendental. A more general example that can be handled by Flajolet's criterion is

$$\sum_{n \geq 0} \left(\sum_{k=0}^n \binom{n}{k}^{p_0} \binom{n+k}{k}^{p_1} \binom{n+2k}{k}^{p_2} \cdots \binom{n+mk}{k}^{p_m} \right) z^n,$$

with $p_i \in \mathbb{N}$, that is discussed in Appendix A.

However, there are cases of D-finite power series $f(z)$ for which both Eisenstein's criterion and Flajolet's criterion fail to detect the transcendence of $f(z)$, see Example 6.

1.5. A bit of history. Since the beginning of the 19th century, many works have been dedicated to solving problems (F) and (L), using various tools, from geometry (Schwarz, Klein), to invariants (Fuchs, Gordan) and from group theory (Jordan) to differential Galois theory (Kolchin, Singer).

Already in 1833, Liouville [73] proposed an algorithm for computing a basis of *rational* solutions (that is, solutions in $\mathbb{Q}(z)$) of linear differential equations like (1). This algorithm (with some enhancements and improvements, e.g., [1]) is the one currently implemented in most computer algebra systems. Liouville's algorithm clearly solves the variants (F)_{rat} and (L)_{rat} of problems (F) and (L), in which the word “algebraic” is replaced by “rational”; it can also be used to solve the variant (S)_{rat} of problem (S): from a basis of solutions $r_1(z), \dots, r_s(z)$ in $\mathbb{Q}(z)$ of L , one first computes the right factor L^{rat} of L whose solution space is spanned by the rational solutions of L , as the LCLM⁶ of the differential operators $\partial_z - r'_j(z)/r_j(z)$, and then one checks whether $L^{\text{rat}}(f)$ is zero or not (using [18, Lemma 2.1]).

In 1839 Liouville addressed in [74] what we call problem (L). He partially solved it for second-order differential operators. For instance, he showed that the equation $y'' + r(z)y = 0$ does not admit any nontrivial algebraic solution when $r(z) \in \mathbb{Q}[z] \setminus \{0\}$. (In particular, this holds for Airy's equation $y'' = zy$.) One year later he proved the same for $r(z) = (1 + z^2)^2 / (2z - 2z^3)^2$, from which he deduced that the complete elliptic integral $f(z) = \int_0^1 dx / \sqrt{(1-x^2)(1-z^2x^2)}$ is not algebraic (and not even Liouvillian, that is, solvable in terms of integrals, exponentials and algebraic functions). In 1841, he applied his method to the Bessel equation $z^2 f''(z) + z f'(z) + (z^2 - \mu^2) f(z) = 0$. However, he could not solve the problem when $r(z)$ is an arbitrary rational function; he reduced it to upper bounding the possible algebraicity degree of a solution. With such a bound B at hand, Liouville reduced problem (L) to finding rational solutions of the symmetric powers $L^{\odot m}$ (that is the monic minimal order operator whose solution space contains the powers f^m for all solutions f of L) for $1 \leq m \leq B$.

Liouville's work was taken up by Pépin (1863, 1878, 1881), who focused on problem (F) and managed to remove the restriction on the algebraicity degree [84,

⁶We denote by $\text{LCLM}\{L_1, \dots, L_k\}$ the *least common left multiple* of the operators $L_1, \dots, L_k$, that is the monic minimal order operator whose solution space contains the solution spaces of all L_i 's.

85]. His 1863 paper contained a few errors corrected in 1878 (after remarks by Fuchs). Pépin completed his study in his long memoir published in 1881 [83]; there, he proved that if the equation $y'' + r(z)y = 0$ has algebraic solutions only, then it admits a basis of solutions $\{y_1, y_2\}$ such that either (i) $y_1 = \sqrt[m]{a}$ and $y_2 = b/y_1$, for a, b in $\mathbb{Q}(z)$ and $m \in \mathbb{N}^*$; or (ii) y_i^m are both roots of a quadratic equation over $\mathbb{Q}(z)$ for $m \in \mathbb{N}^*$; or (iii) y_i^m are both roots of an equation of degree μ over $\mathbb{Q}(z)$, where $(m, \mu) \in \{(4, 6), (6, 8), (12, 10)\}$. In all cases, y'_i/y_i is algebraic of degree 1, 2, 4, 6 or 12.

Meanwhile, in 1873, Schwarz [95] famously solved problem (F) for second-order operators with three singular points (the Gauss hypergeometric equation), see Section 1.6. Fuchs showed in 1876 that if $y'' + r(z)y = 0$ has a nontrivial algebraic solution, then there exists a binary homogeneous form F of degree $d \leq 12$ such that, for some basis $\{y_1, y_2\}$ of solutions, $F(y_1, y_2)$ is the k -th root of a rational function for some explicit number k depending only on d . Inspired by Fuchs's work, Klein showed in 1877 that any second-order linear differential equation with only algebraic solutions comes from some hypergeometric equation from Schwarz's list via a rational change of variables. In [5], Dwork and Baldassarri discuss Klein's and Fuchs's articles and make Klein's approach algorithmic. Sanabria Malagón generalizes this approach in [92], where he also gives a history of the evolution of this approach through his work and that of others. Another approach was proposed by Jordan (1878) who showed that if all solutions of (1) are algebraic, then there exists a solution whose logarithmic derivative is algebraic of degree at most $J(r)$, an explicit number depending only on the order r . Jordan's article and the works by Painlevé (1887) and his student Boulanger (1898) are the starting points of modern algorithms by Singer [96, 97], who completely solved problem (F), see Section 1.7. References to improvements on this algorithm are given in [100].

For more historical aspects related to these rich problems, we refer the interested reader to the following sources: [19, p. 1–13], [108, p. 160–165], [44, p. 5–10], [75, p. 407–413], [98, p. 25], [100, p. 533–536], [107, p. 124] and [45, p. 48–50, 273–274, and Chap. III].

1.6. The hypergeometric case. A very particular, yet quite important special case of D-finite power series $f \in \mathbb{Q}[[z]]$ is the *hypergeometric* class. This means that the coefficient sequence of $f(z)$ satisfies a first-order homogeneous linear recurrence with polynomial coefficients. Typical examples are $\log(1-z)$, $\arcsin(\sqrt{z})/\sqrt{z}$, $(1-z)^\alpha$ for $\alpha \in \mathbb{Q}$, and more generally the Gaussian hypergeometric function with parameters $a, b, c \in \mathbb{Q}$ ($-c \notin \mathbb{N}$),

$${}_2F_1 \left[\begin{matrix} a & b \\ c \end{matrix} ; z \right] := \sum_{n=0}^{\infty} \frac{(a)_n (b)_n}{(c)_n} \frac{z^n}{n!}, \quad \text{where } (d)_n := d(d+1) \cdots (d+n-1),$$

solution of the differential equation $z(1-z)f''(z) + (c - (a+b+1)z)f'(z) - abf(z) = 0$.

Deciding the algebraicity of ${}_2F_1$ functions is an old problem, first solved by Schwarz [95] using geometric and complex analytic tools, and later by Landau [69, 70] and Errera [36] using number theoretic tools. Both approaches are algorithmic: Schwarz's criterion reduces the problem to a table look-up after some preprocessing on the parameters a, b, c ; the Landau-Errera criterion amounts to checking a finite number of inequalities. The reader is referred to [12, §2.1.3] and to Matsuda's book [76, Chap. I, II and IV] for more details.

The definition of the Gaussian hypergeometric function extends to the *generalized hypergeometric function* with complex parameters a_i and b_i with $-b_i \notin \mathbb{N}$

$${}_kF_\ell \left[\begin{matrix} a_1 & a_2 & \cdots & a_k \\ b_1 & \cdots & b_\ell \end{matrix} ; z \right] := \sum_{n=0}^{\infty} \frac{(a_1)_n \cdots (a_k)_n}{(b_1)_n \cdots (b_\ell)_n} \frac{z^n}{n!}.$$

When $\ell \neq k-1$, a ${}_kF_\ell$ non-polynomial hypergeometric function in $\mathbb{Q}[[z]]$ cannot be algebraic, by Stirling's formula combined with Flajolet's criterion; alternatively, because for $k > \ell+1$ the power series ${}_kF_\ell$ has radius of convergence 0, while ${}_kF_\ell$ is an entire function for $k \leq \ell$.

When $\ell = k-1$ and the parameters are rational numbers, the Landau-Errera criterion was extended by Christol [23, §VI] under a conjecture, and by Beukers and Heckman [8] unconditionally. To state it, we need the following definitions: for $x \in \mathbb{R}$ we denote by $\langle x \rangle$ its fractional part $x - \lfloor x \rfloor$ if $x \notin \mathbb{Z}$, and 1 if $x \in \mathbb{Z}$; we say that two equinumerous disjoint multisets of real numbers $\{u_1, \dots, u_k\}$ and $\{v_1, \dots, v_k\}$ with $\langle u_1 \rangle \leq \cdots \leq \langle u_k \rangle$ and $\langle v_1 \rangle \leq \cdots \leq \langle v_k \rangle$ *interlace* if $\langle u_1 \rangle < \langle v_1 \rangle < \cdots < \langle u_k \rangle < \langle v_k \rangle$. Now, let $\mathbf{a} = \{a_1, \dots, a_k\} \subset \mathbb{Q}$ and $\mathbf{b} = \{b_1, \dots, b_{k-1}, b_k = 1\} \subset \mathbb{Q} \setminus (-\mathbb{N})$ be two multisets of rational parameters assumed to be disjoint modulo $\mathbb{Z}$. This assumption is equivalent to the irreducibility of the generalized hypergeometric operator $H_{\mathbf{b}}^{\mathbf{a}} := (z\partial_z + b_1 - 1) \cdots (z\partial_z + b_{k-1} - 1)z\partial_z - z(z\partial_z + a_1) \cdots (z\partial_z + a_k)$. Let D be the common denominator of $a_1, \dots, a_k, b_1, \dots, b_{k-1}$. Then, the function ${}_kF_{k-1}$ is a solution of $H_{\mathbf{b}}^{\mathbf{a}}$ and the *interlacing criterion* says that it is algebraic if and only if for all $1 \leq \ell < D$ with $\gcd(\ell, D) = 1$ the multisets $\ell\mathbf{a}$ and $\ell\mathbf{b}$ interlace (pictorially, this means that the points $\{e^{2\pi i \ell a_j}, j \leq k\}$ and $\{e^{2\pi i \ell b_j}, j \leq k\}$ interlace on the unit circle).

For instance, the Beukers-Heckman criterion immediately implies that the power series

$$\sum_{n=0}^{\infty} \binom{2n}{n}^k z^n = {}_kF_{k-1} \left[\begin{matrix} \frac{1}{2} & \frac{1}{2} & \cdots & \frac{1}{2} \\ 1 & \cdots & 1 \end{matrix} ; 4^k z \right]$$

is transcendental for all $k \geq 2$, since the interlacing condition is violated. This example is mentioned by Stanley in [103, §4, (g)] as one of the motivations of his general question **(S)**.

The Beukers-Heckman interlacing criterion does not cover all cases of hypergeometric functions in $\mathbb{Q}[[z]]$; it cannot be used if some parameters are irrational nor if a difference between a top and a bottom parameter is an integer. For this reason, the following examples escape it

$$(2) \quad {}_2F_1 \left[\begin{matrix} 1 & 1 \\ 2 \end{matrix} ; z \right], \quad {}_2F_1 \left[\begin{matrix} 2 & 2 \\ 1 \end{matrix} ; z \right] \quad \text{and} \quad {}_3F_2 \left[\begin{matrix} 1/2 & 1 + \sqrt{2} & 1 - \sqrt{2} \\ \sqrt{2} & -\sqrt{2} \end{matrix} ; z \right].$$

Fürnsinn and Yurkevich [41] provided a complete classification of the algebraic generalized hypergeometric functions with no restriction on the set of their parameters, thus answering completely Stanley's question **(S)** for power series whose coefficient sequences satisfy a recurrence of order 1. Their result is algorithmic and makes it possible to prove that, among the three examples in Eq. (2), the first power series is transcendental, while the last two are algebraic. Their algorithm relies on an ingenious, albeit elementary, reduction of the general case to the interlacing criterion of Beukers and Heckman [41, Fig. 1].

1.7. Singer’s algorithm for problem (F). In the 1980s, Michael Singer [96, 97] designed algorithms that solve Fuchs’ problem (F). In fact, the problem had essentially been solved in the second half of the 19th century, for order 2 equations by Schwarz, Klein, Fuchs, and for order 3 equations by Painlevé and Boulanger. For an irreducible operator L of order r , Singer’s algorithm in [96] relies on differential Galois theory and proceeds in two main steps:

- (i) (Jordan’s bound) decide if the (nonlinear) Riccati differential equation⁷
 $R_L(u) = 0$ attached to L admits an algebraic solution u of degree at most $(49r)^{r^2}$;
- (ii) (Abel’s problem) given an algebraic u , decide whether $y'/y = u$ admits an algebraic solution y .

Step (ii) is solved by Risch’s algorithm [89] (and independently by Baldassarri and Dwork [5, §6]); it was the missing ingredient in the 19th century procedures. Singer solves Step (i) by reducing it to an algebraic elimination problem. Part of the proof of the algorithm is that if step (ii) finds an algebraic y such that $y'/y = u$ with u as in step (i), then y is a nontrivial algebraic solution of L , and since L is irreducible, all solutions of L are algebraic.

In the reducible case, Singer’s algorithm first proceeds with a factorization step that writes L as a product of irreducible operators. The corresponding detailed results and procedures can be found in (the proofs of) [96, Theorem 1] in the irreducible case, and [96, Theorem 3] in the general case. A slightly different route is proposed by Singer in [97, Corollary 4.3], where first a basis of all Liouvillian solutions is computed, and then an algorithm by Rothstein and Caviness [91, §4] is used to decide if all elements in this basis are algebraic.

1.8. Other approaches to problem (F). Another way of seeing that Fuchs’ problem (F) is decidable, in a spirit similar to Singer’s algorithm but of even higher algorithmic complexity, is based on the equivalence for a linear differential operator L between having a full basis of algebraic solutions and having a finite differential Galois group. Indeed, since Hrushovski’s seminal article [58], it is known that one can fully determine algorithmically the Galois group of L . (Prior to [58], it was only known how to compute the Galois group of *completely reducible operators*, that is for operators L that can be written as LCLM of irreducible operators [28], a condition that can itself be algorithmically tested [99].) The complexity of Hrushovski’s algorithm is not yet fully understood, and its simplification is the object of ongoing works, e.g., [37, 105, 2]. An approach to understanding the Galois group of L by computing the Lie algebra of its identity component is given in [6, 7] and [33]. In a different direction, van der Hoeven proposed in [55] a symbolic-numeric approach for computing differential Galois groups, but for the moment its potential has not been exploited further.

To our knowledge, none of the algorithms mentioned in Sections 1.7 and 1.8 has been implemented yet, nor are they expected to provide good practical behavior (except perhaps for very moderate orders, as in e.g., [101, 54]).

Let us finally mention a very recent sufficient (algorithmic) criterion for problem (F). Assume that the linear differential equation $L(y) = 0$ admits a basis of Puiseux series solutions at each of its singularities. Assume moreover that there

⁷By definition, R_L has order $r - 1$ and has the property that f is a nonzero solution of L if and only if f'/f is a solution of R_L .

exists another differential operator P such that $\partial_z P$ is not right-divisible by L and such that for all singularities σ of L and for all elements f in a basis of Puiseux solutions of L at σ , the Puiseux series $P(f)$ has only nonnegative exponents. Then L admits a transcendental solution, in other words Fuchs' problem **(F)** is answered by the negative. This is the content of [62, Thm. 5], where such an operator P is called a *pseudoconstant*. Moreover, Algorithm 10 in [62] is able to compute a pseudoconstant for L , or to certify that none exists. More generally, if any symmetric power of L admits a pseudoconstant, then L admits a transcendental solution (Theorem 18 in [62]). Conversely, it is an open question whether the fact that L admits transcendental solutions implies the existence of pseudoconstants for some symmetric power of L [62, Question 20]. If the answer to this open question were positive, then this would provide another way to show that Fuchs' problem is decidable.

2. SOLVING PROBLEM **(S)**

For a D-finite power series $f \in \mathbb{Q}[[z]]$, we write $L_f^{\min}$ for the monic linear differential operator in $\mathbb{Q}(z)\langle\partial_z\rangle$ of minimal order that annihilates f . By a slight abuse of notation, we sometimes write $L_f^{\min}$ for the operator in $\mathbb{Q}[z]\langle\partial_z\rangle$ of the same order, obtained by clearing denominators. These two operators differ by a multiplicative factor in $\mathbb{Q}(z)$. Both operators have the same main properties (solution space, local bases, singularities, ...). The *degree* of $L_f^{\min}$ is the maximal degree of its coefficients in the second representation. $L_f^{\min}$ can be computed efficiently [18] starting from any linear differential equation satisfied by f together with sufficiently many initial terms of f .

Our solution to Stanley's problem **(S)** is based on the properties of $L_f^{\min}$ that we review now.

2.1. Properties of $L_f^{\min}$. In simple terms, the minimal differential operator $L_f^{\min}$ is a differential analogue for D-finite functions of the more classical notion of minimal polynomial for algebraic power series. Indeed, any $L \in \mathbb{Q}[z]\langle\partial_z\rangle$ such that $L(f(z)) = 0$ is a left multiple of $L_f^{\min}$ in the ring $\mathbb{Q}(z)\langle\partial_z\rangle$, and conversely any left multiple of $L_f^{\min}$ is an annihilator for f .

But there are two main differences between these similar notions. One is that, for such an L , although its order upper bounds the order of $L_f^{\min}$, it is in general not the case that the maximal degree of the coefficients of L is an upper bound on the maximal degree of the coefficients of $L_f^{\min}$; this is actually the main difficulty in the algorithmic computation of $L_f^{\min}$, see [18] for more details. Another difference important to note is that $L_f^{\min}$ does not need to be irreducible in $\mathbb{Q}(z)\langle\partial_z\rangle$. Reducibility can already occur when f is transcendental, as illustrated by $L_{\log(1-z)}^{\min} = ((1-z)\partial_z - 1)\partial_z$. The same also holds when f is algebraic. For instance, for the algebraic power series of degree 2

$$f = \sqrt{1-4z} + z = 1 - z - 2z^2 - 4z^3 - 10z^4 - \dots,$$

we have that

$$\begin{aligned} L_f^{\min} &= (1-2z)(1-4z)\partial_z^2 - 4z\partial_z + 4 \\ &= \left((1-2z)(1-4z)\partial_z + 4z - 6 + \frac{1}{z}\right) \cdot \left(\partial_z - \frac{1}{z}\right). \end{aligned}$$

Moreover, this is actually the “generic behavior” for an algebraic power series $f \in \mathbb{Q}[[z]]$: if $P = y^d + c_{d-1}(z)y^{d-1} + \dots + c_0(z) = \prod_{\ell=1}^d (y - f_\ell(z))$ is the minimal polynomial of $f = f_1$ in $\mathbb{Q}(z)[y]$, then $L_f^{\min}$ annihilates all the conjugate roots $f_2, \dots, f_d$ of $f = f_1$ (see Proposition 3). Therefore, $L_f^{\min}$ admits $\sum_{\ell=1}^d f_\ell(z) = -c_{d-1}(z)$ as rational solution, hence if $c_{d-1} \neq 0$ then $L_f^{\min}$ is right-divisible by $\partial_z - c'_{d-1}/c_{d-1}$. This was remarked by Tannery [106, p. 132], see also [29, Prop. 4.2].

In Proposition 3 below we give properties of $L_f^{\min}$ and related operators. The proofs rely on some basic facts from the Galois theory of linear differential equations (see [98, 107]), so we begin in this general setting. Let k be a differential field of characteristic zero with derivation ∂ and whose field of constants $C = \{c \in k \mid \partial c = 0\}$ is algebraically closed. Let $L \in k\langle\partial\rangle$ be an operator of order r and let K be the associated Picard-Vessiot extension⁸. Let G be the differential Galois group of K over k and let $f \in K$ be a solution of $L(y) = 0$. We define $L_f^{\min}$ to be the monic operator in $k\langle\partial\rangle$ of smallest order vanishing on f ⁹.

The C -space of algebraic solutions of $L(y) = 0$ is left invariant by the action of G , so it is the solution space of a monic operator in $k\langle\partial\rangle$ denoted by L^{alg} ([99, Lemma 2.2] or [107, Lemma 2.17 p. 48])⁹. Let $P(Y)$ be an irreducible polynomial in $k[Y]$ and assume that there is a $u \in K$ such that $P(u) = 0$. The differential Galois theory implies that the splitting field of P over k lies in K [107, Prop. 1.34.3]. Since G permutes the roots of $P(Y)$, we have that the C -span of these roots is left invariant under this action. As before, this implies that this vector space is the solution space of a monic operator in $k\langle\partial\rangle$, denoted by L_P . Note that since Picard-Vessiot extensions for L are unique up to k -differential isomorphisms, the operators L^{alg} , $L_f^{\min}$ and L_P will be independent of such an extension since any such isomorphism will preserve all the properties of these operators.

We shall apply the above constructions to a linear differential operator $L \in \overline{\mathbb{Q}}(z)\langle\partial_z\rangle$. To be precise, by an algebraic solution we mean an element $y \in \mathcal{P} = \bigcup_{n \in \mathbb{N}^*} \overline{\mathbb{Q}}((z^{\frac{1}{n}}))$, the differential field of formal Puiseux series where $\partial_z(z^{\frac{1}{n}}) = \frac{1}{n}z^{\frac{1}{n}-1}$ such that y is algebraic over $\overline{\mathbb{Q}}(z)$ and $L(y) = 0$. For a given operator L , the set $\mathcal{A}^L$ of algebraic solutions forms a vector space over $\overline{\mathbb{Q}}$ of dimension at most $r = \text{ord}(L)$. Furthermore, the derivation ∂_z extends uniquely to the field $E = \overline{\mathbb{Q}}(z, \mathcal{A}^L)$. We now construct a Picard-Vessiot extension K of $k = \overline{\mathbb{Q}}(z)$ that contains $\mathcal{A}^L$. Let F be the Picard-Vessiot extension of E corresponding to L . Since $\mathcal{A}^L$ lies in the solution space of L in E , there exists a basis $\{y_1, \dots, y_r\}$ of this space that contains a basis of $\mathcal{A}^L$. The field $K = \overline{\mathbb{Q}}(z)(y_1, \dots, y_r, \dots, y_1^{(r-1)}, \dots, y_r^{(r-1)}) \subset F$ has no new constants and so is the required Picard-Vessiot extension.

If both $k = \overline{\mathbb{Q}}(z)$ and $f \in \mathbb{Q}[[z]]$, Proposition 3 below shows that $L_f^{\min}$ and L^{alg} have coefficients in $\mathbb{Q}(z)$. Therefore in this special case, the definitions of $L_f^{\min}$ and L^{alg} coincide with those given earlier.

Proposition 3. *Let $L \in \mathbb{Q}(z)\langle\partial\rangle$ and $f \in \mathbb{Q}[[z]]$ be an algebraic solution of $L(y) = 0$ with minimal polynomial $P(Y) \in \mathbb{Q}(z)[Y]$. If L^{alg} , L_P , and $L_f^{\min}$ are the operators*

⁸This is a field generated over k by a fundamental set of solutions and all their derivatives through order $r-1$ and having the same constants C as k ; one sees that it is again a differential field. Since we assume that C is algebraically closed, such a field exists and any two are differentially isomorphic over k .

⁹This definition is reconciled with the previous one before Proposition 3.

defined above over $\overline{\mathbb{Q}}(z)$, then these operators have coefficients in $\mathbb{Q}(z)$. Furthermore, $L_f^{\min} = L_P$ so the solution space of $L_f^{\min}$ is spanned by the roots of $P(Y)$. In particular, the local solutions of $L_f^{\min}$ at any of its singularities do not contain logarithms.

Proof. Once again let $\mathcal{P}$ be the differential field of Puiseux series and let $\text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ be the Galois group of $\overline{\mathbb{Q}}$ over $\mathbb{Q}$. For $\sigma \in \text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$, we define the action of σ on $\mathcal{P}$ as $\sigma(\sum_i a_i z^{\frac{i}{n}}) = \sum_i \sigma(a_i) z^{\frac{i}{n}}$. In this way $\text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ acts on $\mathcal{P}$ as a group of differential automorphisms with fixed field $\bigcup_{n \in \mathbb{N}^*} \mathbb{Q}((z^{\frac{1}{n}}))$.

L^{alg} : Since L has coefficients in $\mathbb{Q}(z)$, $\text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ leaves the $\overline{\mathbb{Q}}$ -space of algebraic solutions invariant. Denoting by σL^{alg} the operator obtained from L^{alg} by applying σ to the coefficients of L^{alg} , we have that σL^{alg} and L^{alg} have the same solution spaces and so must be equal. Therefore L^{alg} has coefficients in $\mathbb{Q}(z)$.

L_P : Since $P(Y)$ has coefficients in $\mathbb{Q}(z)$, $\text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ permutes its roots and therefore the $\overline{\mathbb{Q}}$ -space spanned by these roots is left invariant by this action. Arguing as above we have that L_P has coefficients in $\mathbb{Q}(z)$.

$L_f^{\min}$: Any $\sigma \in \text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ leaves f fixed and sends $L_f^{\min}$ to $\sigma L_f^{\min}$. By uniqueness of $L_f^{\min}$, we have $L_f^{\min} = \sigma L_f^{\min}$ and this implies, as above, that the coefficients of $L_f^{\min}$ lie in $\mathbb{Q}(z)$.

$L_f^{\min} = L_P$: Let F be the splitting field of $P(Y)$ over $\mathbb{Q}(z)$ and let H be the (usual) Galois group of this extension. The derivation on $\mathbb{Q}(z)$ extends uniquely to a derivation on F and the elements of H commute with this derivation. Since $P(Y)$ is irreducible, H acts transitively on the roots of $P(Y)$. Since $L_f^{\min}$ has coefficients in $\mathbb{Q}(z)$ and vanishes on one root of $P(Y)$, $L_f^{\min}$ vanishes on all the roots and therefore on the $\overline{\mathbb{Q}}$ -space spanned by the roots of $P(Y)$. This implies that L_P is a right factor of $L_f^{\min}$. Minimality of $L_f^{\min}$ and the fact that $L_P(f) = 0$ implies that $L_f^{\min}$ is a right factor of L_P . Therefore these two operators are equal.

Finally, the local branches of the roots of P are Puiseux series at every finite point and at infinity. Since these roots span the solution space of $L_f^{\min}$, it admits at every point a local fundamental system of Puiseux solutions and therefore no logarithmic local solutions. $\square$

Proposition 3 implies the following classical result (for definitions of the terms see [59, Chap. 15-16]):

Corollary 4. *The differential operator $L_f^{\min}$ of an algebraic power series $f \in \mathbb{Q}[[z]]$ enjoys the following properties:*

- (1) *it is Fuchsian;*
- (2) *it admits only rational exponents at all its singular points;*
- (3) *the monic indicial polynomials at all of its singular points split in $\mathbb{Q}[z]$ into distinct linear factors.*

Proof. By Proposition 3, all solutions of $L_f^{\min}$ are algebraic. Fuchs proved that in this situation, the differential operator satisfies properties (1) and (2) [40, §6.II]. Conclusion (3) is a consequence of the fact that a multiple root of the indicial equation at $z = a$ always introduces logarithmic terms in the basis of local solutions at $z = a$ [40, §6.I], see also [59, p. 405]. $\square$

Algorithm 1 Deciding transcendence of D-finite functions

Input: $L = a_r(z)\partial_z^r + \cdots + a_0(z)$ with $a_i(z) \in \mathbb{Q}[z]$;
 f_0 : a truncated power series specifying
a unique solution $f \in \mathbb{Q}[[z]]$ of $L(f) = 0$.
Output: Either **T** if f is transcendental, or **A** if f is algebraic.
1: $L_f^{\min} := \text{MinimalRightFactor}(L, f_0)$ // *Bostan-Rivoal-Salvy* [18, Algorithm 1]
2: **if** $L_f^{\min}$ has a basis of algebraic solutions **then** // *Singer's algorithm* [96]
3: **B** := **A**
4: **else** **B** := **T**
5: **return** **B**

2.2. Examples.

Example 5. Here is a proof of the transcendence of Apéry's power series

$$f(z) = \sum_{n \geq 0} A_n z^n, \quad \text{where } A_n = \sum_{k=0}^n \binom{n}{k}^2 \binom{n+k}{k}^2.$$

First, creative telescoping [111] produces a linear recurrence with polynomial coefficients satisfied by the sequence $(A_n)_n$: $A_0 = 1$, $A_1 = 5$,

$$(n+1)^3 A_{n+1} + n^3 A_{n-1} = (2n+1)(17n^2 + 17n + 5)A_n, \quad n \geq 1.$$

This recurrence is then converted into a linear differential equation $L(f) = 0$ satisfied by $f(z)$, where L is the differential operator

$$L = (z^4 - 34z^3 + z^2)\partial_z^3 + (6z^3 - 153z^2 + 3z)\partial_z^2 + (7z^2 - 112z + 1)\partial_z + z - 5.$$

In the third step one certifies that $L_f^{\min} = L$, meaning that L is already the minimal-order differential equation satisfied by $f(z)$. This can be done using the minimization procedure in [18, Algorithm 1]¹⁰.

Finally, the indicial polynomial of $L_f^{\min}$ at $z = 0$ is z^3 , and the transcendence of $f(z)$ follows from part (3) of Corollary 4.

The approach used in Example 5 is the basis of Algorithms 2 and 3 below.

Note that there are alternative proofs for the transcendence of the Apéry series $f(z)$. One of them relies on the combination of three (non-trivial) ingredients: (i) the asymptotics $A_n \sim \frac{(1+\sqrt{2})^{4n+2}}{2^{9/4}\pi^{3/2}n^{3/2}}$ [78]; (ii) Flajolet's criterion (Section 1.4.2); (iii) the fact that π is transcendental. Our proof based on differential operators appears to be more “natural”, since proving the transcendence of a function should be easier than proving the transcendence of a number. Another alternative proof uses [22, Thm. 5.1] (see also [113, §6]); it relies on the fact that $f(z)$ admits modular parametrizations, implying that, up to an algebraic transformation, $f(z)$ coincides with the hypergeometric series ${}_3F_2\left[\begin{smallmatrix} 1/2 & 1/2 & 1/2 \\ 1 & 1 \end{smallmatrix}; z\right]$, which is transcendental (Section 1.6). This proof does not extend to more general Apéry-like series, such as those in Appendix A.

¹⁰Note that in this very particular case, one could alternatively prove that L is actually irreducible (for instance, by showing that neither L , nor its adjoint, admit any non-trivial hyperexponential solutions).

The next example is of a more combinatorial flavor. It exhibits a case of a nontrivial D-finite power series $f(z)$ in $\mathbb{Q}[[z]]$ whose minimal-order operator $L_f^{\min}$ is not only reducible, but (almost) split.

Example 6. *The generating function of the “trident walks in the quarter plane” is the power series*

$$f(z) = \sum_{n \geq 0} a_n z^n = 1 + 2z + 7z^2 + 23z^3 + 84z^4 + 301z^5 + 1127z^6 + \cdots$$

Its n th coefficient a_n counts all the $\begin{smallmatrix} \nearrow & \Uparrow & \nwarrow \\ \downarrow & & \end{smallmatrix}$ -walks of length n in $\mathbb{N}^2$ starting at $(0, 0)$. It was proved in [15] that $f(z)$ satisfies $L(f) = 0$, where L is a linear differential operator of order 5 and degree at most 15. In [15, §4.2] it was shown that $f(z)$ is transcendental by exploiting the full factorization of L . An alternative proof of the transcendence of f (and actually of all $4 \times 19 - 4 = 72$ similar power series in [15, Thm. 2], see Section 5) uses minimization to first prove that the differential equation $L(f) = 0$ has minimal order, that is $L_f^{\min} = L$, and concludes using the presence of logarithmic terms in the local fundamental system of L at $z = 0$. One advantage of this proof is that factorization of linear differential operators is avoided.

Flajolet’s asymptotic criterion used in the alternative proof for the Apéry series does not prove transcendence in Example 6. Indeed, the asymptotic behavior¹¹ $a_n \sim \gamma \beta^n n^r$ with $\gamma = 4/(3\sqrt{\pi})$, $\beta = 4$, $r = -1/2$, is compatible with algebraicity, since $r \in \mathbb{Q} \setminus \mathbb{Z}_{<0}$, $\beta \in \mathbb{Q}$ and $\gamma \Gamma(r+1) = 4/3 \in \mathbb{Q}$.

Another combinatorial example, also arising from the world of lattice path enumeration, was considered in [11, Proposition 8.4]. In that case, all known criteria for transcendence fail to apply, as well as all previously known algorithms. Moreover, the defining differential operator (of order 11 and polynomial degree 73) could not be factored with the computer algebra systems of the time. The proof in [11, Proposition 8.4] relies on the computation of the corresponding minimal operator $L_f^{\min}$. This example was the starting point for Algorithms 1 to 3.

2.3. An algorithm for Stanley’s problem (S). Based on Proposition 3, a first solution to Stanley’s problem (S) is Algorithm 1. It solves it *in principle*, in the sense that it shows that (S) is decidable. However, its computational complexity is too high to be of practical use, mainly because it relies on the costly algorithm from [96].

Theorem 7. *Given a power series $f \in \mathbb{Q}[[z]]$ by a linear differential operator $L \in \mathbb{Q}[z]\langle \partial_z \rangle$ such that $L(f) = 0$, together with sufficiently many initial conditions, Algorithm 1 decides whether f is algebraic or transcendental.*

Proof. If f is algebraic then, by Proposition 3, $L_f^{\min}$ admits a full basis of algebraic solutions, and this is detected by Singer’s algorithm [96]. Conversely, if $L_f^{\min}$ admits a full basis of algebraic solutions then f is a linear combination of the elements of this basis and therefore is algebraic. $\square$

2.4. An efficient transcendence test. To go further towards a more efficient variant of Algorithm 1, we use Proposition 3 and Corollary 4.

¹¹This asymptotic estimate was conjectured in [15, Conjecture 1], shown to be equivalent to an integral evaluation in [15, Theorem 8], and proved using analytic combinatorics in several variables in [80, §6].

Algorithm 2 Transcendence test for D-finite functions

Input: $L = a_r(z)\partial_z^r + \cdots + a_0(z)$ with $a_i(z) \in \mathbb{Q}(z)$;
 f_0 : a truncated power series specifying
a unique solution $f \in \mathbb{Q}[[z]]$ of $L(f) = 0$.
Output: Either **T** if f is transcendental, or **FAIL**.

```

1:  $L_f^{\min} := \text{MinimalRightFactor}(L, f_0)$            // Bostan-Rivoal-Salvy algorithm [18]
2: for  $s \in \text{Singularities}(L_f^{\min})$  do
3:    $P := \text{IndicialPolynomial}(L_f^{\min}, z = s)$ 
4:   if  $\deg P < \text{ord}(L_f^{\min})$  then return T           // not Fuchsian
5:    $Z := \text{DistinctRationalZeros}(P)$ 
6:   if  $\text{card } Z < \deg P$  then return T // the indicial polynomial does not have
   distinct rational roots
7:    $D := \{|z_i - z_j|, z_i \in Z, z_j \in Z\} \cap \mathbb{N}_{>0}$ 
8:   if  $D \neq \emptyset$  then
9:      $B := \text{FormalSolutions}(L_f^{\min}, z = s, \text{order} = \max(D))$ 
10:    if  $B$  has logarithms then return T
11: return FAIL

```

Theorem 8. *Given a power series $f \in \mathbb{Q}[[z]]$ by a linear differential operator $L \in \mathbb{Q}[z]\langle \partial_z \rangle$ such that $L(f) = 0$, together with sufficiently many initial conditions, Algorithm 2 either proves that f is transcendental or returns **FAIL**.*

Proof. Assume that f is algebraic. By Corollary 4, $L_f^{\min}$ is Fuchsian and, at each of its singularities, its indicial polynomial has degree $\text{ord}(L_f^{\min})$ and has distinct rational roots. Therefore neither of the tests in lines 4 and 6 can return **T**.

Moreover, by Proposition 3, the local solutions of $L_f^{\min}$ contain no logarithmic terms. Thus the logarithm test cannot return **T** either. Consequently, Algorithm 2 returns **FAIL** whenever f is algebraic. Hence, whenever it returns **T**, the power series f is transcendental. $\square$

In terms of completeness, Algorithm 2 is not as good as Algorithm 1, since it does not detect algebraicity. As such, it is not a full decision procedure for transcendence. However, in terms of efficiency, Algorithm 2 is much better than Algorithm 1, as it only relies on the minimization algorithm from [18], plus a few computationally cheap local tests.

2.5. An efficient conditional algorithm in the globally bounded case. To state our next results, we now recall a collection of results and conjectures on D-finite globally bounded power series (see Definition 2). An important source of globally bounded power series is the class of *diagonals of rational functions* defined as follows.

Definition 9. *For a rational function $R \in \mathbb{Q}(x_1, \dots, x_s)$ with Taylor expansion*

$$\sum_{i_1 \geq 0, \dots, i_s \geq 0} a_{i_1, \dots, i_s} x_1^{i_1} \cdots x_s^{i_s}$$

in the neighborhood of the origin, the diagonal $\text{Diag}(R)$ is the power series

$$\text{Diag}(R) = \sum_{i \geq 0} a_{i, \dots, i} z^i \in \mathbb{Q}[[z]].$$

A power series in $\mathbb{Q}[[z]]$ is a diagonal of a rational function if it equals $\text{Diag}(R)$ for some rational function R .

Algebraic functions are diagonals [42]. Diagonals of rational functions are D-finite and globally bounded [24, Proposition 5]. Moreover, a D-finite and globally bounded power series is a G -function. A consequence is an extension of parts (1) and (2) of Corollary 4: $L_f^{\min}$ is Fuchsian with rational exponents at all its singular points. This follows from results of Katz (1970), Chudnovsky and Chudnovsky (1985) and André (1989), see e.g., [3, p. 719] or [72, Thm. 1] and the references therein¹².

The converse of some of these implications is the object of the following conjecture. Notably, part (3) extends part (2) and gives a converse to Proposition 3 that allows (conjecturally) to detect algebraic solutions of linear differential equations under the global boundedness assumption.

Conjecture 10 (Christol, André). *Let $f \in \mathbb{Q}[[z]]$ be a globally bounded and D-finite power series. Then:*

- (1) f is the diagonal of a rational function;
- (2) If $z = 0$ is an ordinary point for $L_f^{\min}$, then f is algebraic;
- (3) If $L_f^{\min}$ admits a logarithm-free fundamental system at $z = 0$, then f is algebraic.

Part (1) of Conjecture 10 is due to Gilles Christol, and was formulated in the late 1980s [23], see also [24, Conj. 4] and [25, Conj. 10]. Part (2) was formulated around 1997 in private discussions between Gilles Christol and Yves André, and appeared in print in [4, Rem. 5.3.2] (there, the connection with the Grothendieck-Katz p -curvature conjecture is also discussed). See also [77, §3.3] for a similar conjecture (called “Eisenstein’s Algebraicity Criterion” by the author). Part (3) is due to Yves André (private communication).

The global boundedness assumption in Conjecture 10 is crucial for the three parts. For instance, without this assumption, Part (1) does not hold for $f(z) = \log(1 - z)$, Part (2) does not hold for $f(z) = \exp(z)$ and Part (3) does not hold for the power series $f(z) = {}_2F_1\left[\begin{smallmatrix} 1/6 & 5/6 \\ 7/6 \end{smallmatrix}; z\right]$ considered in [63, Prop. 2.3], which actually is the particular case $(a, b) = (1/6, 5/6)$ of the following example.

Example 11. *There exist D-finite transcendental power series whose minimal-order operator is Fuchsian and such that all singular points are algebraic (not logarithmic) singularities. Take a and b in $\mathbb{Q} \setminus \mathbb{Z}$ such that $b - a$ is not an integer.*

Then $f(z) = {}_2F_1\left[\begin{smallmatrix} a & b \\ a + 1 \end{smallmatrix}; z\right]$ is such that its $L_f^{\min}$ is reducible, and equals

$$z(1 - z)\partial_z^2 + (a + 1 - (a + b + 1)z)\partial_z - ab = \left(z(1 - z)\partial_z + 1 - (b + 1)z\right)\left(\partial_z + a/z\right).$$

¹²Note that part (3) of Corollary 4 is false for the larger class of diagonals, as the simple example $f = \text{Diag}(1/(1 - (x_1 + x_2)(x_3 + 1))) = \sum_{n \geq 0} \binom{2n}{n}^2 z^n$ demonstrates, since $L_f^{\min} = (16z^2 - z)\partial_z^2 + (32z - 1)\partial_z + 4$ has indicial polynomial z^2 at $z = 0$.

Thus, $L_f^{\min}$ is Fuchsian, it admits the algebraic solution z^{-a} , and its local exponents are $\{0, -a\}$ at $z = 0$, $\{0, 1 - b\}$ at $z = 1$, $\{a, b\}$ at $z = \infty$. Since the exponent differences are not integers, the three singularities are algebraic. Moreover, $f(z)$ is transcendental, since otherwise the operator $L_f^{\min}$ would have finite cyclic monodromy and this is impossible by [109, Thm. 2.2 (2)]. However, $f(z)$ is not globally bounded by [23, Prop. 1], so this example does not contradict Part (3) of Conjecture 10.

We design now a more efficient version of Algorithm 1; this is Algorithm 3, based on Proposition 3, Corollary 4, and Conjecture 10.

Theorem 12. *Given a linear differential operator $L \in \mathbb{Q}[z]\langle \partial_z \rangle$ and sufficiently many initial conditions specifying a unique solution $f \in \mathbb{Q}[[z]]$, every output $\mathbf{T}$ of Algorithm 3 is unconditionally correct. If f is globally bounded, then, assuming Part (3) of Conjecture 10, every output $\mathbf{A}$ is also correct.*

Proof. If f is algebraic, then, by Corollary 4, $L_f^{\min}$ is Fuchsian and it admits only rational exponents at all its singular points. Moreover, if $L_f^{\min}$ has a logarithmic singularity at $z = 0$, then f is transcendental by Proposition 3. This proves the first part of the result. The last part follows from Conjecture 10. $\square$

Remark 13. *Example 11 shows that the assumption “ $f(z)$ is globally bounded” is necessary in Algorithm 3, even if we replaced “ $L_f^{\min}$ has a logarithmic singularity at $z = 0$ ” by “ $L_f^{\min}$ has a logarithmic singularity at one of its singular points”.*

Remark 14. *As already mentioned in Section 1.4, it is currently not known how to decide if a given D -finite power series $f(z)$ is globally bounded. However, this property is known to be satisfied by large classes of power series, such as generating functions of multiple binomial sums or, equivalently, by diagonals of rational functions, see [17, Theorem 3.5]. In this case, the operator L annihilating f does not need to be part of the input: it can be computed from a diagonal representation of f using creative telescoping, see [17] and the references therein (Section 5.2 provides several non-trivial examples). Still in this case, Algorithm 3 can be enhanced in the following way: after computing $L_f^{\min}$, if it detects logarithms in the local basis of solutions of $L_f^{\min}$ at $z = 0$, then it concludes transcendence; if it detects a term $\log^s(z)$ in that local basis (e.g., if the indicial polynomial of $L_f^{\min}$ at $z = 0$ is $z^{s+1} \cdot P(z)$ with $P(0) \neq 0$), then it can conclude that the number of variables needed to express $f(z)$ as the diagonal of a rational function is at least $s + 2$ [49, Corollary 2.7], thus proving a strong form of transcendence of $f(z)$ if $s \geq 1$. For instance, in Example 5, the presence of terms $\log^2(z)$ in the local basis of $L_f^{\min}$ at $z = 0$ implies that this number is at least 4 (and in fact exactly 4, e.g., using the diagonal representations in Section A.2), see also [49, Example 3.9].*

3. SOLVING PROBLEM (P)

Given an operator $L \in \mathbb{Q}(z)\langle \partial_z \rangle$ the aim of this section is to present Algorithm 4 that computes the operator $L^{\text{alg}} \in \mathbb{Q}(z)\langle \partial_z \rangle$ whose solution space is spanned by the algebraic solutions of $L(y) = 0$ (see Section 2.1). This is done by extending some of Singer’s results [96, 99]. Algorithm 4 is then used in Algorithm 5 to solve Stanley’s

Algorithm 3 Transcendence of globally bounded D-finite functions

Input: $L = a_r(z)\partial_z^r + \cdots + a_0(z)$ with $a_i(z) \in \mathbb{Q}(z)$;

f_0 : a truncated power series specifying
a unique solution $f \in \mathbb{Q}[[z]]$ of $L(f) = 0$.

It is assumed that f is globally bounded.

Output: Either T if f is transcendental, or A if f is algebraic (under Conjecture 10).

In Algorithm 2, replace FAIL by A and line 8 by

8: **if** $s = 0$ and $D \neq \emptyset$ **then**

Algorithm 4 Algebraic Right Factor

Input: $L \in \overline{\mathbb{Q}}(z)\langle\partial\rangle$.

Output: $L^{\text{alg}} \in \overline{\mathbb{Q}}(z)\langle\partial\rangle$ with solution space spanned by the algebraic solutions of $L(y) = 0$.

- 1: factor L over $\overline{\mathbb{Q}}(z)$ as a product $L_1 L_2 \cdots L_t$ where each L_i is irreducible
- 2: select from the factors L_i those factors $L_{i_1}, \dots, L_{i_s}$
that admit a full basis of algebraic solutions // *Singer's algorithm* [96]
- 3: for $j = 1, \dots, s$, construct an operator $\hat{L}_{i_j}$ such that // *See Proposition 16*
 $\hat{L}_{i_j} = \text{LCLM}\{R \in \overline{\mathbb{Q}}(z)\langle\partial\rangle \mid R \text{ is equivalent to } L_{i_j} \text{ and divides } L \text{ on the right}\}$
- 4: compute $L^{\text{alg}} = \text{LCLM}\{\hat{L}_{i_1}, \dots, \hat{L}_{i_s}\}$
- 5: **return** L^{alg}

problem (S) in full generality, providing a second proof that Stanley's problem (S) is decidable.

Both algorithms rely on Singer's algorithm [96], which suffers from a very large bound, except for very small order, where one can use more refined bounds [67, 54, 102], see Section 1.7. This prevents us from obtaining practical versions of Algorithms 4 and 5. Still, the computation of L^{alg} is a question of independent interest. We first recall the following definition:

Definition 15. Let k be a differential field of characteristic zero and $L_1, L_2 \in k\langle\partial\rangle$ be irreducible operators of order n . We say L_1 is equivalent to L_2 if there exist $a_0, a_1, \dots, a_{n-1} \in k$, not all zero, such that L_1 divides $L_2 \circ (a_{n-1}\partial^{n-1} + \cdots + a_0)$ on the right.

This is a special case of a more general definition¹³ but we will only need this version and the fact that this is an equivalence relation [99, Cor. 2.6]. If K is a Picard-Vessiot extension of k containing a full set of solutions of $L_1(y) = 0$ and of $L_2(y) = 0$ then the map $\phi : y \mapsto a_{n-1}\partial^{n-1}y + \cdots + a_0y$ maps the solution space of $L_1(y) = 0$ to the solution space of $L_2(y) = 0$. Since L_1 and L_2 are irreducible, this map is a bijection [99, Cor. 2.6]. All solutions of $L_1(y) = 0$ are algebraic then so are their images by ϕ and therefore the same is true for $L_2(y) = 0$. The converse is true, since the inverse of the bijection is of the same type.

¹³Of operators of the same type, whose definition does not require irreducibility.

Algorithm 5 Deciding transcendence of D-finite functions

Input: $L = a_r(z)\partial_z^r + \cdots + a_0(z)$ with $a_i(z) \in \mathbb{Q}[z]$;
 f_0 : a truncated power series specifying
a unique solution $f \in \mathbb{Q}[[z]]$ of $L(f) = 0$.
Output: Either **T** if f is transcendental, or **A** if f is algebraic.

```

1:  $L^{\text{alg}} := \text{AlgebraicRightFactor}(L)$  // Algorithm 4
2: if  $L^{\text{alg}}(f_0) = 0$  then  $B := A$  // [18, Lemma 2.1]
3: else  $B := T$ 
4: return  $B$ 

```

Algorithm 4 computes L^{alg} when $k = \overline{\mathbb{Q}}(z)$ and $C = \overline{\mathbb{Q}}$. Step 1 is performed using any factorization algorithm, e.g., [46], or [57]; step 2 is solved by Singer's algorithms [96] or [97]; for the computation of the LCLM in step 4, see e.g., [14], or [56]. By Proposition 3, if the input L is in $\mathbb{Q}(z)\langle\partial\rangle$, then even if the intermediate computations take place in $\overline{\mathbb{Q}}(z)\langle\partial\rangle$, the output L^{alg} does belong to $\mathbb{Q}(z)\langle\partial\rangle$.

We will now justify the equation in the last step and show how one can construct the operator in step 3.

3.1. Proof of the algorithm. This proof uses again concepts from the differential Galois theory of linear differential equations as in Section 2.1. Let k be a differential field of characteristic 0 with algebraically closed field of constants C , $L \in k\langle\partial\rangle$, K the Picard-Vessiot extension of k corresponding to $L(y) = 0$ and G the Galois group of K over k .

We recall that the vector space of solutions of $L(y) = 0$ that are algebraic over k is invariant under G and so spans the solution space V^{alg} of a linear differential equation $L^{\text{alg}}(y) = 0$ with $L^{\text{alg}} \in k\langle\partial\rangle$. Note that L^{alg} divides L on the right [99, Lemma 2.1]. The equation $L^{\text{alg}}(y) = 0$ obviously has a full set of solutions in K and they generate a Picard-Vessiot extension E over k with $k \subset E \subset K$. Furthermore the Galois group H of E over k is finite. A finite group acting on a vector space allows one to decompose the vector space as a direct sum of irreducible subspaces. Therefore $V^{\text{alg}} = \oplus_i V_i$, each V_i an H -irreducible space. Furthermore, each V_i is the solution space of an irreducible $M_i \in k\langle\partial\rangle$. Factorizations of linear operators are unique up to equivalence [99, Proposition 2.11]. It follows that M_i must be equivalent to some L_{i_j} and so must divide some $\widehat{L}_{i_j}$. Therefore the solution space of $\text{LCLM}\{\widehat{L}_{i_1}, \dots, \widehat{L}_{i_s}\}(y) = 0$ contains the solution space of $L^{\text{alg}}(y) = 0$. Conversely all solutions of $\text{LCLM}\{\widehat{L}_{i_1}, \dots, \widehat{L}_{i_s}\}(y) = 0$ are algebraic so its solution space is contained in the solution space of $L^{\text{alg}}(y) = 0$. Therefore these monic operators are equal.

3.2. Computing the $\widehat{L}_{i_j}$. This will follow from

Proposition 16. *Let k be a differential field of characteristic 0 with algebraically closed field of constants C and satisfying the following property:*

If $L \in k\langle\partial\rangle$ then one can effectively find a C -basis of the space

$$\{u \in k \mid Lu = 0\}.$$

Let $M, L \in k\langle\partial\rangle$ and assume that M is irreducible. One can effectively find¹⁴

$$\widehat{M} = \text{LCLM}\{R \in k\langle\partial\rangle \mid R \text{ is equivalent to } M \text{ and divides } L \text{ on the right}\}.$$

Note that in the application of this proposition in Algorithm 4, where the field k is $\overline{\mathbb{Q}}(z)$, the property is satisfied by Liouville's algorithm [73], see also [1].

Proof. Let M have order n . The set

$$W = \{\mathbf{a} = (a_{n-1}, \dots, a_0) \in k^n \mid M \text{ divides } L \circ (a_{n-1}\partial^{n-1} + \dots + a_0) \text{ on the right}\}$$

is a vector space over the constants. By computing the product $L \circ (a_{n-1}\partial^{n-1} + \dots + a_0)$, its right Euclidean division by M and writing that the remainder is 0, we get an $n \times n$ matrix $\mathcal{A}$ with entries in $k\langle\partial\rangle$ such that $\mathbf{a} \in W$ if and only if $\mathcal{A}\mathbf{a} = 0$ (see the discussion following Example 2.8 in [99]). Finding solutions of $\mathcal{A}Y = 0$ in k^n can be reduced to finding solutions of scalar linear differential equations in k and so one can find a C -basis $\{\mathbf{a}_1, \dots, \mathbf{a}_s\}$ of W [88, Chap. III, §11]. If $W = \{0\}$, there is no R equivalent to M that divides L on the right and the result is $\widehat{M} = 1$.

For $\mathbf{a} = (a_{n-1}, \dots, a_0) \in W \setminus \{0\}$, we construct an operator $L_{\mathbf{a}} \in k\langle\partial\rangle$ of order at most n so that the map $A_{\mathbf{a}} : y \mapsto a_{n-1}\partial^{n-1}y + \dots + a_0y$ maps the solution space of M to the solution space of $L_{\mathbf{a}}$. This is done by differentiating j times the equality $v = a_{n-1}\partial^{n-1}y + \dots + a_0y$ for $j = 0, \dots, n$ and using the equation $My = 0$ to express each derivative $v^{(j)}$ in terms of $y, \dots, \partial^{n-1}y$. The resulting $n+1$ expressions must be linearly dependent over k and this yields an equation $L_{\mathbf{a}}$. The operator $L_{\mathbf{a}}$ obtained that way has order exactly n and is irreducible: by its construction, there exists an operator Q_1 such that $L_{\mathbf{a}} \circ A_{\mathbf{a}} = Q_1 \circ M$ with $A_{\mathbf{a}}$ relatively prime to M since M is irreducible; the conclusion follows from general decomposition theorems [81, Ch. II. Thm. 1], [99, Prop. 2.11]. This shows that $L_{\mathbf{a}}$ is equivalent to M . Moreover, $L_{\mathbf{a}}$ divides L on the right since $A_{\mathbf{a}}$ maps the solution space of M bijectively into the solution space of $L_{\mathbf{a}}$, while it also maps it into the solution space of L .

Let $\widetilde{M} = \text{LCLM}\{L_{\mathbf{a}_1}, \dots, L_{\mathbf{a}_s}\}$. We will show that $\widetilde{M} = \widehat{M}$. Since each $L_{\mathbf{a}_i}$ is equivalent to M and divides L on the right, we have that $\widetilde{M}$ divides $\widehat{M}$ on the right.

Now assume that an operator R is equivalent to M and divides L on the right. Let $A_{\mathbf{a}} = a_{n-1}\partial^{n-1} + \dots + a_0$ be such that M divides $R \circ A_{\mathbf{a}}$ on the right. Since R divides L on the right, we have that M divides $L \circ A_{\mathbf{a}}$ on the right. Therefore $\mathbf{a} = (a_{n-1}, \dots, a_0) \in W$ and so there exist $c_1, \dots, c_s \in C$ such that $\mathbf{a} = \sum c_i \mathbf{a}_i$. Moreover, $A_{\mathbf{a}}$ maps the solution space of M bijectively into the solution space of R . Since $A_{\mathbf{a}} = \sum c_i A_{\mathbf{a}_i}$, the latter space is included in the sum of the solution spaces of the $L_{\mathbf{a}_i}$, which is the solution space of $\widetilde{M}$. Therefore R divides $\widetilde{M}$ on the right.

Since $\widehat{M}$ is the LCLM of all such R , we have that $\widehat{M}$ divides $\widetilde{M}$ on the right. Since both of these operators are monic, we have shown that they are equal. $\square$

Remark 17. Proposition 16 can be used as a building block to construct the Loewy decomposition of an operator into the product of completely reducible operators.

¹⁴This is a LCLM of a potentially infinite set of operators that are all right divisors of L . Thus, the earlier definition extends and the LCLM is a right divisor of L . By convention, if the set is empty, the LCLM is taken to be 1.

4. OTHER ALGORITHMIC APPROACHES TO ALGEBRAICITY AND TRANSCENDENCE

4.1. Using the Grothendieck-Katz p -curvature conjecture. In the 1970s, Alexander Grothendieck proposed a conjectural “arithmetic” characterization for linear differential operators with coefficients in $\mathbb{Q}(z)$: such an operator admits a full basis of algebraic solutions if and only if the same holds for its reductions modulo p for almost all primes p . Grothendieck’s conjecture can be seen as a differential generalization of a particular case, due to Kronecker, of Chebotarev’s theorem: the roots of a polynomial in $\mathbb{Q}[z]$ are all rational if and only if the roots of its reductions modulo p are in $\mathbb{F}_p$ for almost all primes p . (See [12, 43], and the references therein.)

Although it remains largely open in the general case, the p -curvature conjecture has been proven in several important cases. It was first studied in depth and popularized by Nicholas Katz [61], who proved it for the Gauss hypergeometric equations, a result later extended to the generalized hypergeometric equations by Beukers and Heckman [8]. More generally, it was proved in [61] and [4, §III] for Picard-Fuchs differential operators (satisfied by the periods of families of smooth algebraic varieties) and for certain of their direct factors and subquotients (sometimes called “differential equations coming from geometry”). In a different direction, the p -curvature conjecture was proved by the Chudnovsky brothers [26] (see also [4, §6]) for linear differential operators of first order with algebraic function coefficients in $\overline{\mathbb{Q}(z)}$. (This last arithmetic approach was used to prove the algebraicity of some power series e.g., in [64, 31, 10].)

Consider now problem **(F)** for any linear differential operator L for which the Grothendieck-Katz conjecture is known to hold. For any fixed prime p , the linear differential operator $L \bmod p$, with coefficients in $\mathbb{F}_p(z)$, admits a basis of algebraic solutions if and only if it admits a basis of rational solutions [21], and this is verified by a very simple algorithm, based on checking the nullity of the p -curvature of L .

In practice, computing p -curvatures for a few primes p provides a very efficient heuristic: if L has several zero p -curvatures, then it very likely admits a full basis of algebraic solutions. In particular, for problem **(S)**, if the Grothendieck-Katz conjecture is known to hold for $L_f^{\min}$ and if $L_f^{\min}$ has several zero p -curvatures, then $f(z)$ is very likely algebraic, otherwise it is very likely transcendental. To turn this heuristic into a decision algorithm, one would need an upper bound on the number of primes p for which the p -curvature is to be tested. In other words, one needs an *effective* Grothendieck-Katz theorem, similar to effective versions of Chebotarev’s theorem (e.g., [86] and the references therein). For Picard-Fuchs operators, the existence of such results is alluded to by André [4, §16.3.1]. A first complete effective result was recently obtained, for first-order linear operators with rational function coefficients, by Fürnsinn and Pannier [43]. Much remains to be done in this direction, for more general classes of linear differential operators.

4.2. Guess-and-prove. To prove algebraicity of a given power series $f(z)$, a very popular method in computer algebra is to first guess a polynomial $P(z, u) \in \mathbb{Q}[z, u]$ such that $P(z, f(z)) = O(z^\sigma)$ for some value σ , and then to certify that $P(z, f(z)) = 0$ by manipulating the roots of P via the differential operator L_P whose solution space is generated by these roots. This strategy is explained for instance in the proof of [16, Corollary 2]. The guessing part is based on (structured) linear algebra

(over $\mathbb{Q}$) in size σ and the most efficient algorithms use Hermite-Padé approximants (which reduce the problem to linear algebra over $\mathbb{Q}(z)$ in a size much smaller than σ). The proving part is based on the computation of L_P , which ultimately also relies on linear algebra [13], and sufficiently many initial conditions. A delicate issue is the choice of σ ; usually one keeps doubling its value and degree bounds on P until a candidate P is found. If $f(z)$ is algebraic, then the procedure will eventually discover it (modulo computational difficulties related to the size of σ that can be huge).

This approach applies to problem (S) when f is D-finite. Its strength is that it is able to recognize and prove algebraicity of $f(z)$ even in situations where $f(z)$ is not *a priori* known to be D-finite (e.g., it is given not by an ODE, but by a different kind of functional equation), see e.g., [16]. The weak point of the approach is that it cannot prove that $f(z)$ is transcendental. This is why it is usually used in conjunction with other heuristics such as computations of p -curvatures, or numerical computations of local monodromies.

4.3. Factoring. If the input L is checked to be irreducible, then its solution f is algebraic if and only if all solutions of L are algebraic (by Proposition 2.5 in [96]), and this can be tested using Singer's 1979 algorithm [96]. But if L is reducible, then it is *a priori* not clear how to combine factoring algorithms [57, 46] and Singer's algorithm [96] in order to solve Stanley's problem (S). One practical way is: factor the input $L = AB$ for irreducible A of order less than the order of L and check if A has a basis of algebraic solutions. If yes, then we cannot conclude. (See the examples at the end of Section 5.2.) If no, then A has no nontrivial algebraic solutions, and there are two possible cases. Either $g := B(f)$ is nonzero, in which case it is transcendental (as it is a nontrivial solution of A), hence f itself is transcendental (otherwise g would be algebraic). Or, $g = 0$, and we can repeat the whole procedure on B . This procedure is guaranteed to succeed in the particular case when L factors as a product of irreducible operators each of them having no algebraic solutions.

5. EXAMPLES

Starting from an implementation of the minimization algorithm [18], the implementation of the transcendence test of Algorithm 2 in Maple is straightforward and fits in 40 lines.¹⁵ We list a few experiments performed with this implementation.

5.1. Combinatorial examples. Examples with a combinatorial origin for which our implementation gives an automatic proof of transcendence are:

- the differential equation of order 11 and degree 73 from 3-dimensional walks confined to the positive octant mentioned after Example 6, coming from [11, Proposition 8.4] (in 10 seconds)¹⁶;
- the 72 transcendental cases from [15, Theorem 2] (in a total of 25 seconds);

¹⁵It is implemented in the function `istranscendental` in versions later than 4.05 of the package `gfun`, available at <https://perso.ens-lyon.fr/bruno.salvy/software/the-gfun-package/>, where a Maple session with these examples can also be found.

¹⁶Timings were obtained with Maple2024.2 on a MacBook Pro 2017 with a quad-core Intel i7.

dim	order	degree	transc. (sec.)	factor. (sec.)
3	3	5	2.2	0.9
4	4	10	0.1	0.2
5	6	17	0.6	1.3
6	8	43	4.2	18.3
7	11	68	24.0	265.
8	14	126	174.9	4706.
9	18	169	771.6	>10000
10	22	300	8817.1	—
11	27	409	46743.	—

TABLE 1. Transcendence of lattice Green functions.

- the lattice Green functions of the face-centered cubic lattice in dimensions $d = 2, \dots, 11$. These (D-finite) functions are defined by

$$G_d(z) := \frac{1}{\pi^d} \int_0^\pi \cdots \int_0^\pi \frac{d\theta_1 \cdots d\theta_d}{1 - z \binom{d}{2}^{-1} \sum_{1 \leq i < j \leq d} \cos \theta_i \cos \theta_j}.$$

When $d = 2$ and $d = 3$, we have hypergeometric expressions [48, Eq. (53)], [60, Eq. (3.6)]

$$G_2(z) = {}_2F_1 \left[\begin{matrix} \frac{1}{2} & \frac{1}{2} \\ 1 \end{matrix}; z^2 \right], \quad G_3(z) = {}_2F_1 \left[\begin{matrix} \frac{1}{6} & \frac{1}{3} \\ 1 \end{matrix}; \frac{z^2(z+3)}{4} \right]^2$$

that can be seen to be transcendental by the arguments of Section 1.6. For $d \geq 4$, no such closed formulas are known, and it is not clear how to decide transcendence of $G_d(z)$ using asymptotic arguments. Linear differential equations for $G_d(z)$ were first conjectured by Guttmann in dimension 4 [47] and by Broadhurst in dimension 5 [20], after intensive computations. They were then computed by Koutschan for $d = 3, 4, 5, 6$ using creative telescoping [66]. For $d = 7$, an equation was conjectured by Hassani, Maillard and Zenine [112] and then equations were conjectured up to $d = 11$ by Hassani, Koutschan, Maillard and Zenine [50].

While these differential equations have been obtained after days of computations, it is relatively simple for our code to prove that they are minimal. (Note that their minimality for $d \leq 11$, and even their irreducibility for $d \leq 7$, were claimed in [50, Section 2.2] and [112, Section 4], but the arguments there are based on heuristics.) Since all these equations are minimal and have a logarithmic singularity at 0, Proposition 3 implies that $G_d(z)$ is transcendental for $d \leq 11$ (assuming the conjectured differential equations to be correct for $d \geq 7$). Alternatively, one can use factorization of linear differential operators to prove that the differential equations for $G_d(z)$ are irreducible for $d \leq 8$, and use Proposition 2.5 in [96] (rather than Proposition 3) to deduce the transcendence of $G_d(z)$. Sizes and timings are reported in Table 1. (The column “transc” indicates the time taken by our code to prove transcendence of $G_d(z)$; the time taken by the factorization

program from [27] is given in the column “factor”. Maple’s native command `DFactor` was much slower.)

In practice, for all these examples, all the time is spent in the first step where a differential equation of certified minimality is computed.

5.2. Diagonals. A classical result by Pólya and Furstenberg states that the diagonal of any bivariate rational function is algebraic [87, 42]. In characteristic 0 and with three or more variables, diagonals need not be algebraic. The examples below illustrate how our methods can prove transcendence and can suggest algebraicity, which may then sometimes be established by a separate argument.

Example 1: algebraic case. Using Koutschan’s `HolonomicFunctions` package [65], the diagonal of

$$\frac{1}{(1 - 5x - 7yz - 13z^2)(1 - x - xy)}$$

is found to satisfy a linear differential equation of order 5 with coefficients of degree up to 24. This equation has logarithmic singularities at 0 and $1/140$. However, minimization shows that the diagonal actually satisfies an equation of order only 3 with coefficients of degree up to 13. From that equation, Algorithm 3 strongly suggests that the diagonal is algebraic (being a diagonal, it is globally bounded). Actually, the more general situation of the diagonal of

$$F(x, y, z) = \frac{1}{(1 - ax - byz - cz^2)(1 - x - xy)}$$

for arbitrary a, b, c can be seen to be algebraic by a direct residue computation: the diagonal is the residue at $x = y = 0$ of

$$G(x, y, z, t) = \frac{1}{xy} F(x, y, t/(xy)) = \frac{xy}{((x - ax^2 - bt)xy^2 - ct^2)(1 - x - xy)}$$

when $t \rightarrow 0$. The second factor of the denominator has a root in y that is away from 0 while the first factor has two roots in y that both tend to 0 when $t \rightarrow 0$. The sum of their residues is therefore a rational function in x, t . From there it follows that the sum of residues in x is algebraic. An explicit computation is possible: by a Rothstein-Trager resultant [43, Thm. 2.1], one obtains a polynomial cancelling these two residues in y , from whose coefficients their sum is deduced to be

$$\frac{x - 1}{ax^4 - 2ax^3 + btx^2 + ct^2x + ax^2 - 2btx - x^3 + bt + 2x^2 - x}.$$

Only one of the roots of the denominator tends to 0 with t . Again, a Rothstein-Trager resultant gives a polynomial of degree 4 cancelling the diagonal.

Example 2: a transcendental case. A very similar-looking diagonal $\Delta(z)$ is the diagonal of

$$\frac{1}{(1 - x - y - z^2)(1 - x - xy)}.$$

`HolonomicFunctions` gives a linear differential operator L_7 of order 7 for $\Delta(z)$ with coefficients of degree up to 19. This operator is not irreducible: it is the product $T_4 A_3$ of an operator T_4 that does not have nonzero algebraic solutions¹⁷

¹⁷This can be proved by showing that T_4 is irreducible and admits local logarithms at $z = 0$.

with an operator A_3 all of whose solutions are algebraic¹⁸. Our code proves that L_7 is minimal for the diagonal Δ , which is then proved transcendental by showing that L_7 has a logarithmic singularity at 0. Another proof (as in Section 4.3) is by observing that A_3 does not annihilate Δ , which implies that $A_3(\Delta)$ is a solution of T_4 and thus necessarily transcendental, and therefore so is Δ . That alternative way does not work on the adjoint L_7^* of L_7 . If one takes the solution $s(z)$ of L_7^* which is its unique power series solution with $s(0) = -1/2$, $s'(0) = 0$, $s''(0) = -21121726441/112000$, then again our program proves that it is transcendental. In this example, the factorization $L_7^* = A_3^* T_4^*$ has the left factor A_3^* whose solutions are all algebraic¹⁹, but now this factorization is not sufficient to conclude that $s(z)$ is transcendental.

Example 3: a transcendental case with a difficult factor. Finally, let us consider the diagonal of

$$\frac{1}{(1-x-y-z^2)(1-x-xy^2)}.$$

It is annihilated by a linear differential operator L_9 of order 9 and degree 60 that factors²⁰ as a product $L_4 A_5$. Our code detects that L_9 is minimal for the diagonal and proves its transcendence. Another transcendence proof is based on the method in Section 4.3, as above. By contrast, it is much more challenging to prove that all solutions of A_5 are algebraic. The heuristic based on p -curvatures in Section 4.1 suggests that this is indeed the case. The approach using guess-and-prove described in Section 4.2 is applicable in principle: the guessing part is feasible in practice, although computationally very expensive, but the proving step seems to be out of reach, even modulo a prime²¹.

5.3. Apéry-like series. For $p, q \in \mathbb{N} \setminus \{0\}$, we consider the following generalization of the power series in Example 5:

$$f_{p,q}(z) = \sum_{n \geq 0} \left(\sum_{k=0}^n \binom{n}{k}^p \binom{n+k}{k}^q \right) z^n.$$

In Appendix A we use Flajolet's criterion to prove that $f_{p,q}(z)$ is transcendental if and only if $(p, q) \neq (1, 1)$. It was conjectured by Sergey Yurkevich in [110, Conjecture 10.5] that the minimal number $\mu_{p,q}$ of variables needed to represent $f_{p,q}(z)$ as the diagonal of a rational power series is equal to $p + q$. Yurkevich

¹⁸This can be checked by observing that A_3 is irreducible and admits the solution

$$\frac{1}{z} \cdot {}_3F_2 \left[\begin{matrix} 1/6, 1/2, 5/6 \\ 1/4, 3/4 \end{matrix} \middle| \frac{27z^2}{256} \right],$$

which is algebraic by the interlacing criterion of Section 1.6.

¹⁹A very similar hypergeometric function appears, with parameters $\{1/6, 1/2, 5/6\}$, $\{5/4, 7/4\}$.

²⁰This operator and its factorization were communicated to us by Jean-Marie Maillard. Similar examples can be found in [51].

²¹Gilles Villard found a polynomial of degree 120 with coefficients of degree 460 that cancels the reduction modulo p of a power series with largest valuation among the solutions of A_5 , modulo a large prime $p = 892901$. Similar polynomials can be found for a basis of solutions mod p . Proving that the solutions of such a polynomial are a basis of the reductions modulo p of the solutions of A_5 is feasible in theory, but is extremely demanding computationally.

showed that $\mu_{p,q} \leq p + q$, by using the identity (due to Wadim Zudilin)

$$(3) \quad f_{p,q} = \text{Diag} \left(\frac{1}{\left(\prod_{j=1}^q (1 - y_j) - x_1 \right) \cdot \prod_{k=2}^p (1 - x_k) - \prod_{k=1}^p x_k \cdot \prod_{j=1}^q y_j} \right).$$

For $p + q \leq 10$, we checked the opposite inequality $p + q \leq \mu_{p,q}$ by using [49, Corollary 2.7] and the enhanced version of our Algorithm 3, mentioned in Remark 14. Indeed, in all these 45 cases, the indicial polynomial $I_{p,q}(z)$ of $L_{f_{p,q}}^{\min}$ at $z = 0$ equals $z^{p+q-1} \cdot R_{p,q}(z)$ where $R_{p,q}(0) \neq 0$, hence Yurkevich's conjecture holds. By looking closely at the indicial polynomials $I_{p,q}(z)$, we made a few additional observations. First, the degree of $I_{p,q}(z)$, that is the order of $L_{f_{p,q}}^{\min}$, is equal to $\lfloor (p + q)^2 / 4 \rfloor$ except when p and q are both even and equal, in which case it equals $p^2 - 1$. In particular, the order of $L_{f_{p,q}}^{\min}$ is equal to that of $L_{f_{q,p}}^{\min}$ in the 45 cases. Second, if p is odd or if $p > q$, then $I_{p,q}(z)$ factors $z^{p+q-1}(z - 1)^{p+q-3} \dots (z - r)^{p+q-2r-1}$, where $r = \lfloor (p + q)/2 \rfloor - 1$; when p and q are both even and equal, then $I_{p,q}(z)$ equals $z^{2p-1} \cdot (z - 1)^{2p-3} \dots (z - (p - 2))^3$. In the remaining case, the pattern may break; for instance $I_{2,4}$ has the extra exponent $1/2$. We leave the proof of these experimentally observed formulas, and their extension to arbitrary p, q , as open problems. (Similar considerations for the exponential generating versions of $f_{p,q}$ appear in Question 5.2 in [18].)

APPENDIX A. GENERALIZED APÉRY SERIES

A.1. A multiple binomial sum. A direct application of Flajolet's criterion proves the transcendence of the following family of power series, extending the one in Section 5.3.

Proposition 18. *Let $(p_0, \dots, p_m) \in \mathbb{N}^{m+1}$ with $p_0 \geq 1$. Then the power series*

$$f_{\mathbf{p}}(z) = \sum_{n \geq 0} \left(\sum_{k=0}^n \binom{n}{k}^{p_0} \binom{n+k}{k}^{p_1} \binom{n+2k}{k}^{p_2} \dots \binom{n+mk}{k}^{p_m} \right) z^n$$

is transcendental if and only if $p := p_0 + \dots + p_m > 2$.

Proof. The case $p > 2$ is covered by Flajolet's criterion. Asymptotically, the n th coefficient $S_{\mathbf{p}}(n)$ of $f_{\mathbf{p}}(z)$ behaves like

$$(4) \quad S_{\mathbf{p}}(n) \sim \gamma(\pi n)^{\frac{1-p}{2}} \beta^n, \quad n \rightarrow \infty$$

with γ and β algebraic [78]. If $p > 2$ is odd, the exponent of n is a negative integer making $f_{\mathbf{p}}(z)$ transcendental; if p is even, $\Gamma((3 - p)/2)$ is a rational multiple of $\sqrt{\pi}$ and thus $\gamma \Gamma((3 - p)/2) \pi^{(1-p)/2}$ is an algebraic multiple of $\pi^{1-p/2}$, which is not algebraic for $p > 2$.

For $p = 1$, the series $f_{(1,0,\dots,0)}$ is the rational $1/(1 - 2z)$.

Finally, for $p = 2$, either $p_0 = 2$ and all the other p_i are 0, in which case $f_{\mathbf{p}} = (1 - 4z)^{-1/2}$, or $p_0 = 1$, $p_j = 1$ for one $j \neq 0$ and all other p_i are 0. In that case, the power series can be seen to be algebraic as the diagonal of a bivariate rational function. Here, the rational function

$$\frac{1}{1 - z(1 + y)(y + (1 + y)^j)} = \sum_{n \geq 0} z^n \sum_{k=0}^n \binom{n}{k} y^{n-k} (1 + y)^{n+jk},$$

has a Taylor expansion whose coefficient of $z^n y^n$ is the desired

$$\sum_{k=0}^n \binom{n}{k} \binom{n+jk}{k}. \quad \square$$

A.2. Generic diagonals in more than 2 variables are transcendental. It is well known that the Apéry power series $f_{2,2}$ can be written as the diagonal of a rational power series in 4 variables, e.g., as the diagonal of $1/((1-x-y)(1-t-z)-xyz)$ [104, Thm. 1.1] or of $1/(((1-z)(1-t)-x)(1-y)-xyz)$ (see Section 5.3). From such a diagonal representation, the approach of analytic combinatorics in several variables (ACSV) allows to deduce the asymptotics of the n th Apéry number A_n as in Example 5, see e.g., [79, Example 1]. More generally, the generalized Apéry power series $f_{p,q}$ can be written as the diagonal of a rational power series in $m = p + q$ variables *with nonnegative coefficients*, e.g., via (3). Now, [79, Result 1] states that under certain assumptions that hold generically, the n th coefficient of the diagonal of a rational power series in m variables with nonnegative coefficients grows asymptotically like $\gamma \beta^n n^r$ where $r = (1-m)/2$ and $\gamma/(2\pi)^r \in \overline{\mathbb{Q}}$. (This is coherent with the asymptotics (4) in the proof of Proposition 18.) Combined with Flajolet’s criterion, the formula of [79, Result 1] implies that diagonals of “generic” rational functions with nonnegative coefficients are transcendental if $m > 2$; conversely, the bivariate case is algebraic by the result of Pólya and Furstenberg. Note that the genericity assumption is essential, as shown by the example of the diagonal of $1/(1-x-y-yz-xyz)$, equal to $f_{1,1}(z)$ which is algebraic by Proposition 18, or by the algebraic examples of Section 5.2.

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