Direct and Inverse Problems in Differential Galois Theory

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Abstract. This paper surveys recent work on the problems of calculating Galois groups of differential equations and of constructing differential equations with given groups as their Galois groups.

1. Introduction

In his address to the 1966 International Congress of Mathematicians [**87**], Ellis Kolchin described five problems with the hope that "my description of a few of the more challenging unsolved problems will lure some into accepting the challenge." The fourth problem, concerning the Picard-Vessiot theory, was presented in the following way:

> Now let $\mathfrak{F}$ be an ordinary differential field and denote its field of constants by $\mathfrak{C}$; denote the field of constants of $\mathfrak{U}$[1] by $\mathfrak{K}$. Consider a homogeneous linear differential polynomial $L = Y^{(n)} + a_1 Y^{(n-1)} + \ldots + a_n Y$ with coefficients in $\mathfrak{F}$. If $\mathfrak{C}$ is algebraically closed then L has a fundamental system of zeros $(\eta_1, \ldots, \eta_n)$ such that the extension $\mathfrak{G} = \mathfrak{F}\langle\eta_1, \ldots \eta_n\rangle$ of $\mathfrak{F}$ has the same field of constants $\mathfrak{C}$, and then the set of all the isomorphisms over $\mathfrak{F}$ of $\mathfrak{G}$ onto an extension of $\mathfrak{F}$ in $\mathfrak{U}$ has a natural group structure; by means of the fundamental system of zeros $(\eta_1, \ldots, \eta_n)$ this group can be identified with an algebraic subgroup G of $\mathrm{GL}_{\mathfrak{K}}(n)$ defined over $\mathfrak{C}$. This G can be called the *Galois group of L* over $\mathfrak{F}$. Of course, a different choice of fundamental systems of zeros in general gives a different Galois group; G is determined up to conjugation in $\mathrm{GL}(n)$ by a matrix that is rational over $\mathfrak{C}$. (This ambiguity can be removed by taking for G an algebraic subgroup of the group $\mathrm{GL}(V)$ of all automorphisms of the vector space V of zeros of L).
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> As in the ordinary Galois theory, there are two general problems:
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> 1^o *Given L, to determine G.*
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> 2^o *Given G, to find an L of which G is the Galois group.*

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[1]Kolchin previously defined $\mathfrak{U}$ to be a universal differential field, that is, a field "so large that it contains enough elements for every purpose we can have." Kolchin also assumes that this field is of characteristic zero and we shall make this assumption for all fields mentioned in this paper.

> Of course, the nature of these problems depends on the differential field $\mathfrak{F}$ which is given. For example, when $\mathfrak{F} = \mathfrak{C}$ then it is easy to see that, for every L, G is reducible to triangular form (and hence is solvable). A. Bialynicki-Birula [**25**] has shown that if $\mathfrak{F}$ has finite transcendence degree over $\mathfrak{C}$ and G is a connected and nilpotent algebraic subgroup of $\mathrm{GL}(n)$ defined over $\mathfrak{C}$, then there exists a homogeneous linear differential polynomial L with coefficients in $\mathfrak{F}$ such that the Galois group of L over $\mathfrak{F}$ is isomorphic to G. Aside from this and some work in a slightly different direction by Lawrence Goldman [**57**], even when $\mathfrak{F} = \mathbb{C}(x)$ very little of a general nature is known. For the second problem this is not unexpected, in view of the history of the analogous problem in classical Galois theory. The obstinacy of the first problem is somewhat of a surprise, and it now appears that we should be grateful for almost any kind of information about G; the difficulty stems in part from the fact that a zero of L that is not a zero of any linear differential polynomial over $\mathbb{C}(x)$ of lower order may very well be a zero of a nonlinear one.

We shall refer to problem 1^o as the *direct problem* and problem 2^o as the *inverse problem*. The aim of this paper is to describe some of the work that has been done on these problems since Kolchin's talk and to give a guide to the literature. Excellent guides to the 19^{th} Century literature are [**69**], [**131**], [**132**] and [**160**] as well as [**58**] and [**59**] which give historical insight into this era. A good recent survey of work on linear meromorphic differential equations is [**158**] (see also [**159**]). We refer to Borel's article [**26**] or any of [**22**], [**53**], [**54**], [**77**], [**88**], [**95**], [**100**] for the basic facts concerning differential Galois theory and [**27**] for the basic facts concerning linear algebraic groups.

2. The Direct Problem

At present, we do not know a general algorithm that will compute the Galois group of a linear differential equation with coefficients in a differential field k, even when $k = \bar{\mathbb{Q}}(x)$, where $\bar{\mathbb{Q}}$ is the algebraic closure of the rational numbers. In contrast, algorithms for calculating the Galois group of a polynomial with coefficients in $\mathbb{Q}$ or $\bar{\mathbb{Q}}(x)$ have been known for a long time [**39**], [**117**], [**163**]. The key idea behind these methods is to represent the splitting field of a polynomial in terms of generators and relations. The Galois group is then the set of permutations of the generators that preserve the relations. In the differential case, the Picard-Vessiot extension is the analogue of the splitting field. The obstruction to mimicking the ideas from the Galois theory of polynomials is that, at present, we do not know how to effectively present a general Picard-Vessiot extension in terms of generators and relations. Nonetheless, there are many *properties* of the Galois group that can be computed and, in some cases, one can even determine this group (see [**140**] for an introduction to differential Galois theory with an emphasis on computational aspects).

Let k be a differential field with algebraically closed field of constants C. Let $L(y)$ be a homogeneous linear differential polynomial of order n with coefficients in k, $K = k\langle y_1, \ldots , y_n\rangle$ the associated Picard-Vessiot extension, and G its Galois group. We will denote by V the solution space of $L(y) = 0$ in K. Note that V

is a finite dimensional C-vector space and a G-module. We shall now consider algorithms to determine various properties of G, V and L. Throughout we shall assume that the field of constants C is a computable field with a factorization algorithm, that is, a field whose field theoretic operations $+, -, \times, ^{-1}$ are recursive functions and over which we can effectively factor polynomials. For example, the algebraic closure of a finitely generated extension of $\mathbb{Q}$ is such a field [**163**].

2.1. Triviality of G. The Galois group is trivial if and only if $K = k$, so we wish to consider algorithms to decide if $L(y) = 0$ has a full set of solutions in k. In fact algorithms exist to find the space $W \subset V$ of solutions of $L(y) = 0$ in k for large classes of fields k. Consider first the simpler problem of finding solutions $a \in \mathbb{Q}$ of $p(x) = a_n x^n + \ldots a_0 = 0,\ p(x) \in \mathbb{Z}[x]$. One way to proceed is to note that factors of the denominator of a must divide a_n and the factors of the numerator must divide a_0 (assuming that a has relatively prime numerator and denominator). One could then try all possibilities. For differential equations $L(y) = a_n y^{(n)} + \ldots + a_0 y = 0$ with $a_i \in C(x),\ x' = 1$ one can attempt a similar approach. One can see that any *irreducible* factor q of the denominator of a solution $y \in C(x)$ must divide a_n, but the largest power q^m of q dividing the denominator of y may be greater than the largest power of q dividing a_n. For example, $y = x^m$ is a solution of $xy' - my = 0$. To bound m in general, one expands q-adically each $a_i = a_{i,n_i} q^{n_i} + a_{i,n_1+1} q^{n_i+1} + \ldots$, $deg(a_{i,j}) < deg(q)$, substitutes $y = A_m q^m + A_{m+1} q^{m+1} + \ldots$ into $L(y) = 0$ and notices that in order for equality to hold the exponent m must satisfy a polynomial equation over C (the *indicial equation at* q). The smallest integer root m_q of this equation will bound the possible m. Substituting $y = z \prod q^{m_q}$ into $L(y) = 0$, reduces the problem to finding polynomial solutions of $\tilde{L}(z) = L(z \prod q^{m_q}) = 0$. In order for equality to hold in $\tilde{L}(z) = 0$ after one substitutes $z = \alpha_N x^N + \ldots + \alpha_0$, one again sees that N must satisfy a polynomial equation (the indicial equation of $\tilde{L}$ at infinity). If N is the largest integer root of this polynomial, we can substitute $z = \alpha_N x^N + \ldots + \alpha_0$ into $\tilde{L}(z) = 0$ to yield a system of linear equations for the α_i. A basis of the solution space of this system will yield a basis of the solution space of $L(y) = 0$ in $C(x)$.

This naive algorithm can be improved in several ways. For example, one would like to work in the smallest field $C_0(x) \subset C(x)$ containing the coefficients of L and one would like to find efficient algorithms to find polynomial solutions of linear differential equations. These and related issues are discussed in [**1**], [**3**], [**4**], [**35**]. Before turning to other fields, we note that in many situations one is given a system $Y' = AY$ of differential equations where A is an $n \times n$ matrix with coefficients in $C(x)$ and asked to determine a basis for all solutions in $(C(x))^n$. In theory one can reduce this problem to finding all solutions of an associated scalar equation $L(y) = 0$ in $C(x)$ (this is described in Section 2.4.1) but finding this associated equation can be costly. An algorithm to find rational solutions of the system $Y' = AY$ directly has been given by M. A. Barkatou [**14**].

Algorithms to find solutions of $L(y) = 0$ in a given differential field k when k is an algebraic extension of $C(x)$, or more generally, certain types of liouvillian extensions (this term will be defined below) of $C(x)$ are discussed in [**141**]. Finally algorithms to find solutions of inhomogeneous linear differential equations in various fields are discussed in [**30**], [**31**], [**32**], [**33**], [**45**].

2.2. Reducibility of V. The G-module V contains a proper nontrivial G-module W of dimension r if and only if one can write the differential polynomial $L(y) = L_{n-r}(L_r(y))$ as a *composite* of homogeneous linear differential polynomials L_{n-r} and L_r of orders $n-r$ and r respectively with coefficients in k. To see this, let $\{y_1, \ldots, y_r\}$ be a basis of W and let $L_r(y) = \mathrm{Wr}(y, y_1, \ldots, y_r)/\mathrm{Wr}(y_1, \ldots y_r)$, where $\mathrm{Wr}(\ldots)$ denotes the wronskian determinant. Assume that W is a G-invariant subspace of V. The coefficients of L_r are left invariant by G since applying $g \in G$ to these wronskians multiplies each of these determinants by the determinant of the linear map induced by g on W. Writing $L(y) = L_{n-r}(L_r(y)) + R(y)$ where the order of $R(y)$ is less than r (c.f., [**118**], Ch. III) we see that $R(y) = 0$ has r independent solutions in K and therefore must be 0. Conversely, if $L(y) = L_{n-r}(L_r(y))$, the solution space W of $L_r(y) = 0$ is a G-invariant subspace of V.

This characterization of reducibility leads naturally to considering the ring $\mathcal{D} = k[D]$ of linear operators. This is the ring of noncommutative polynomials in D with coefficients in k where $D \cdot a = a \cdot D + a'$. To each homogeneous linear differential polynomial $L(y) = a_n y^{(n)} + \ldots + a_0 y$ we can associate the operator $L = a_n D^n + \ldots + a_0$ in $\mathcal{D}$. We then have that V is reducible as a G-module if and only if L factors in $\mathcal{D}$ as a product of operators of smaller positive order. Furthermore, to any operator $L \in \mathcal{D}$ one can associate the left $\mathcal{D}$-module $\mathcal{D}/\mathcal{D} \cdot L$. The operator L factors as a product of operators of smaller positive order if and only if the $\mathcal{D}$-module $\mathcal{D}/\mathcal{D} \cdot L$ contains a proper nontrivial $\mathcal{D}$-submodule (see [**143**] for a discussion of this ring and historical comments on the work of Frobenius, Jordan, Beke, Loewy, Ore and Jacobson concerning $\mathcal{D}$). Applying the Jordan-Hölder Theorem one sees that every operator $L \in \mathcal{D}$ can be written as a product of irreducible operators $L = L_1 \cdot \ldots \cdot L_r$. Furthermore, if $L = \tilde{L}_1 \cdot \ldots \cdot \tilde{L}_s$ with the $\tilde{L}_i$ irreducible, then $r = s$ and (after a possible renumbering) the $\mathcal{D}$-modules $\mathcal{D}/\mathcal{D}L_i$ and $\mathcal{D}/\mathcal{D}\tilde{L}_i$ are isomorphic or, in more pedestrian terms, the orders of L_i and $\tilde{L}_j$ are equal and there exist nonzero operators R and S whose orders are less than the order of L_i satisfying $L_i R = S\tilde{L}_i$.

Before we discuss algorithms to determine if V is reducible we investigate the more restrictive question of whether V has a one dimensional G-submodule. As we have seen this is equivalent to L having a right factor of order one. If we write $L = L_{n-1}L_1$ where $L_1 = D - r$, then one sees that L has a first order right factor if and only if $L(y) = 0$ has a solution of the form $y = \exp(\int r)$ with $r \in k$ (y is then called an *exponential over* k). Algorithms to find such solutions when $k = \overline{\mathbb{Q}}(x)$ are classical ([**131**], Section 177) and rest on the fact that at any point $x_0 \in \mathbb{C}$, one can formally find the solution space of $L(y) = 0$ (one can also consider the point at infinity by letting $x = 1/z$ and $d/dx = -z^2 d/dz$). If x_0 is a nonsingular point (i.e., a point where $a_n(x_0) \neq 0$), then all solutions are given by convergent power series. In general, if a linear differential equation has coefficients that are meromorphic at x_0, there exist n linearly independent solutions of the form

$$y_i = (x - x_0)^{\lambda_i} e^{Q_i(t)} (\phi_{i,0} + \phi_{i,1}(t) \log(x - x_0) + \ldots \phi_{i,s_i}(t)(\log(x - x_0))^{s_i}) \tag{2.1}$$

where $\lambda_i \in \mathbb{C}$, $t = (x - x_0)^{-1/q_i}$ for some positive integer q_i, Q_i is a polynomial in t^{-1} without constant term, and $\phi_{i,j} \in \mathbb{C}[[t]]$. At each point x_0 there are only a finite number of λ_i (called *exponents*) and Q_i (called *determining factors*) that can occur and these can be calculated directly from the operator L (or a system $Y' = AY$) ([**131**], Section 110, [**11**], [**13**], [**49**], [**68**], [**72**], [**94**], [**101**], [**148**], [**150**]). If $\exp(\int r)$ is a solution of $L(y) = 0$ and $r = \alpha_m (x - x_0)^{-m} + \ldots + \alpha_1 (x - x_0)^{-1} + \ldots$ is the

Laurent expansion of r at x_0, then $Q(x) = \frac{\alpha_m}{-m+1}(x-x_0)^{-m+1} - \ldots - \alpha_2(x-x_0)^{-1}$ is a determining factor and $\lambda = \alpha_1$ is an exponent at x_0. In particular, if x_0 is a nonsingular point, then $m = 1$ and α_1 is an integer between 0 and $n-1$. Therefore to find the possible $r \in \overline{\mathbb{Q}}(x)$ with $L(\exp(\int r)) = 0$, we write

$$r = p(x) + \sum_{\alpha\in\mathcal{S}}(\sum_{i=1}^{n(\alpha)} \frac{c_{\alpha,i}}{(x-\alpha)^i}) + \sum_{\beta\in\mathcal{N}} \frac{m_\beta}{(x-\beta)}$$

where $\mathcal{S}$ is the set of finite singular points, $\mathcal{N}$ is the set of finite non-singular points, the $c_{\alpha,i}$ are constants, the m_β are integers between 0 and $n-1$, and $p(x)$ is a polynomial. For $\alpha \in \mathcal{S}$, the possible terms $\sum_{i=1}^{n(\alpha)} \frac{c_{\alpha,i}}{(x-\alpha)^i}$ can be found by examining the exponents and determining factors at α. The possibilities for the polynomial $p(x)$ can be found by examining the determining factors at infinity. For each choice of these possibilities, we consider $z = y\exp(-\int \tilde{r})$ where $r = p(x) + \sum_{\alpha\in\mathcal{S}}(\sum_{i=1}^{n(\alpha)} \frac{c_{\alpha,i}}{(x-\alpha)^i})$. The element $z = \exp(\int \sum_{\beta\in\mathcal{N}} \frac{m_\beta}{(x-\beta)}) = \prod(x-\beta)^{m_\beta}$ satisfies the linear differential equation $\tilde{L}(z) = \exp(-\int \tilde{r})L(z\exp(\int \tilde{r})) = 0$. The original equation $L(y) = 0$ has an exponential solution if and only for some choice possibilities the equation $\tilde{L}(y) = 0$ has a polynomial solution. In the previous section we discussed how to decide this latter question. One can furthermore use this approach to find all such solutions. There are many places where this algorithm can be improved. For example, one would like to work in the coefficient field for as long as possible. One would like to efficiently find the exponents and determining factors. Also, it is not always necessary to try all possible combinations of the determining factors and exponents. These and other implementation issues as well as generalizations to other fields are discussed in [**34**], [**45**], [**71**], [**141**] . There are also algorithms to find exponential solutions of systems $Y' = AY$, that is solutions $Y = (y_1, \ldots, y_n)^T$ where each y_i is exponential over $\overline{\mathbb{Q}}(x)$ [**115**].

We now turn to the problem of deciding if V has invariant subspaces W of arbitrary dimension r. As noted in the beginning of this section, this is equivalent to finding a right factor of L of order r. Such an operator will be of the form $L_r = \mathrm{Wr}(y, y_1, \ldots, y_r)/\mathrm{Wr}(y_1, \ldots y_r)$ where $y_1, \ldots y_r$ forms a basis of W. The coefficients of L_r are quotients whose numerators are $r\times r$ subdeterminants of

$$\begin{pmatrix} y_1 & \ldots & y_r \\ y_1' & \ldots & y_r' \\ \ldots & \ldots & \ldots \\ y_1^{(r)} & \ldots & y_r^{(r)} \end{pmatrix} \tag{2.2}$$

and whose denominators are $\mathrm{Wr}(y_1, \ldots, y_r)$. For example the coefficient of $y^{(r-1)}$ in L_r is $\mathrm{Wr}(y_1, \ldots, y_r)'/\mathrm{Wr}(y_1, \ldots, y_r)$. Since each of these quotients is in k, one can show that each of these subdeterminants is an exponential over k. Furthermore, one can show ([**131**], Section 167) that there exists an operator $L^{\wedge r}$ of order at most $\begin{pmatrix} n \\ r \end{pmatrix}$ with coefficients in k, such that for any basis $y_1, \ldots, y_n$ of V, all $r\times r$ subdeterminants of $\mathrm{Wr}(y_1, \ldots, y_n)$ satisfy $L^{\wedge r}(y) = 0$. We again restrict ourselves to operators whose coefficients lie in the field $k = \overline{\mathbb{Q}}(x)$ and outline an algorithm originally due to Beke [**16**]. To find a right factor of order r of L, one first finds a maximal set $\{z_i\}$ of linearly independent solutions of $L^{\wedge r}(y) = 0$ that are exponential over k. One can then give a bound N on the degrees of the numerators and denominators of all elements of $\overline{\mathbb{Q}}(x)$ that equal a quotient $(\sum c_i z_i)/(\sum d_i z_i)$. One forms an operator $\tilde{L}_r$ of order r whose coefficients are quotients of polynomials of degree N in x with indeterminate coefficients. Formally dividing $\tilde{L}_r$ into L yields a remainder that is an operator R of order less than r. The coefficients of this remainder will be quotients of polynomials in x whose coefficients are polynomials in the indeterminates introduced above. The condition $R = 0$ is equivalent to systems of polynomial equalities and inequalities that the indeterminates must satisfy in order that $\tilde{L}_r$ divides L. One then decides if these equalities and inequalities can be satisfied in C (details and refinements of this basic algorithm are given in [**16**], [**36**], [**42**] Appendix, [**131**] Section 176, [**133**]). One can refine this algorithm to give a factorization into irreducible factors. Furthermore, one can enumerate all possible factorizations [**154**].

There have been other improvements in this algorithm. In [**63**] Grigoriev gives several improvements in the algorithm together with a careful analysis of the running time and the size of the resulting factors. He also gives a polynomial time algorithm to find the greatest common right divisor of a family of operators.

Another improvement comes from the observation that not all exponential solutions of $L^{\wedge r}(y) = 0$ yield information about factors. Generically, the solution space of $L^{\wedge r}(y) = 0$ is G-isomorphic to $\bigwedge^r V$ (although, there are situations when this solution space is isomorphic to a quotient of $\bigwedge^r V$). Not all G-invariant lines in $\bigwedge^r V$ correspond to decomposable vectors, i.e., vectors of the form $v_1 \wedge \ldots \wedge v_r$; the Plücker relations define those that do. Therefore not all exponential solutions of $L^{\wedge r}(y) = 0$ must correspond to subdeterminants of 2.2. Tsarev [**153**] has shown how one can use the Plücker relations to narrow down the possible exponential solutions of $L^{\wedge r}(y) = 0$ that enter into the factorization problem.

Recent work of van Hoeij [**71, 72**] has led to good implementations of algorithms to factor linear operators. In [**72**] van Hoeij takes the methods of [**101**] and turns them into efficient algorithms for factoring in $C((x))[D]$. In [**71**] he uses the local methods of [**72**] to give good ways to find exponential solutions of linear differential operators with the aim of efficiently determining the coefficients in a factorization. In addition he minimizes the need to go beyond the smallest field containing the coefficients of L.

When one considers restricted classes of differential operators, the problem of factorization becomes more tractable. We say that a differential operator L is *completely reducible* if it is the least common left multiple of irreducible operators. This is equivalent to any of the following: a) $\mathcal{D}/\mathcal{D}L$ is the sum of irreducible $\mathcal{D}$-modules; b) V is the sum of irreducible G-modules; c) G is a reductive group. In [**143**] algorithms are given to decide if an operator in $C(x)[D]$ is completely reducible (this is generalized to algebraic extensions of $C(x)$ in the appendix of [**42**]). Furthermore, once it is known that the operator is completely reducible, it can be factored by finding solutions *in* k of certain auxiliary systems; one does not need to search for elements that are exponential over k. Although this class seems to be restrictive, we shall see below that in other decision problems one is frequently given an operator that is known in advance to be completely reducible and for which one needs to determine a factorization. In [**70**], van Hoeij shows how his methods may be applied to this class of differential operators (as well as some more general classes).

For differential systems $Y' = AY$, the solution space has an invariant G-submodule of dimension s if and only if there is an invertible matrix $B \in \mathrm{GL}(n,k)$ such that

$$B'B^{-1} + BAB^{-1} = \begin{pmatrix} A_{1,1} & 0 \\ A_{2,1} & A_{2,2} \end{pmatrix}$$

where $A_{2,2}$ is in $M(s,k)$, [**62**]. In this latter paper, Grigoriev gives a procedure (with complexity analysis) for determining this property when $k = C(x)$.

Finally, we note that although Gauss's Lemma does not hold in $\mathcal{D}$, an Eisenstein-like criterion for irreducibility was developed for this ring in [**91**].

2.3. Solvability of the Identity Component. One consequence of Kolchin's work on the Galois theory of differential equations was to give a precise version of Vessiot's theorem concerning equations solvable by quadratures (see [**85**] for a discussion of Vessiot's original work). One says that an extension $E = k(t_1, \ldots t_r)$ of k is a *liouvillian extension* of k if for each i either:

1. t_i is primitive over $k(t_1, \ldots t_{i-1})$, i.e. $t_i' \in k(t_1, \ldots t_{i-1})$, or
2. t_i is exponential over $k(t_1, \ldots t_{i-1})$, i.e. $t_i'/t_i \in k(t_1, \ldots t_{i-1})$, or
3. t_i is algebraic over $k(t_1, \ldots t_{i-1})$.

We say that an element u of an extension of k is liouvillian over k if it belongs to a liouvillian extension of k. Kolchin showed [**85**] that a linear homogeneous differential equation is solvable in terms of liouvillian functions (i.e., its Picard-Vessiot extension lies in a liouvillian extension of k) if and only if the component G^0 of the identity of its Galois group G is solvable.

An algorithm to find a basis for the space of liouvillian solutions of a homogeneous linear differential equation $L(y) = 0$ with coefficients in an algebraic extension k of $C(x)$ was presented in [**136**]. To get a taste of the ideas involved, we will describe an algorithm to decide if all solutions of $L(y) = 0$ are liouvillian. An effective version of the Lie-Kolchin Theorem asserts that in this case G will have a subgroup H such that the elements of H can be simultaneously put in upper triangular form and such that the index of H in G is bounded by a computable function $I(n)$ of n, the order of L. If y is a common eigenvector of H, then $\sigma(y'/y) = cy'/cy = y'/y$ so y'/y is left fixed by all elements σ of H. This implies that y'/y is algebraic over k of degree bounded by $I(n)$ and so will be algebraic over $C(x)$ of degree bounded by $M = I(n)[k : C(x)]$. Therefore, if $L(y) = 0$ is solvable in terms of liouvillian functions over k, $L(y) = 0$ has a solution y such that y'/y is algebraic over $C(x)$ of degree bounded by M. To decide if $L(y) = 0$ has such a solution we will look for candidates for the minimal polynomial of $u = y'/y$. If $p(u) = u^N + b_{N-1}(x)u^{N-1} + \ldots + b_0(x)$, $N \leq M$, is the minimal polynomial of such a u, then one can show that there exist solutions $z_1, \ldots, z_N$ of $L(y) = 0$ such that each b_i is the i^{th} symmetric function of the z_i'/z_i. From the exponents and determining factors at the singular points of L, we can bound the degrees of the numerators and denominators of the b_i. Therefore, if $L(y) = 0$ has only liouvillian solutions, it will have a solution y such that $u = y'/y$ satisfies a polynomial over $C(x)$ of degree at most M whose coefficients have numerators and denominators of effectively bounded degrees. Elimination theory then allows us to decide if such a solution exists and produces u. We then use the change of variables $y = z\exp(\int u)$ to get a new equation $\tilde{L}(z) = 0$ of lower order and proceed by induction.

This algorithm is not a practical algorithm. For example when $n = 2$ and the equation has coefficients in $\mathbb{Q}(x)$, the bound for $I(n)$ given in [**136**] is $384,064$ while it is known that a second order linear differential equation with a liouvillian solution will have a solution whose logarithmic derivative is algebraic of degree at most 12 (see the description of Kovacic's algorithm below). In [**155**], Ulmer modified the algorithm in [**136**] by refining the group theoretic techniques involved. For second order equations, Ulmer's algorithm yields the correct bound. He was not only able to get significantly improved bounds in all cases but he developed conditions to further narrow down the set of possible degrees of the logarithmic derivative of an exponential solution of $L(y)$. Similar algorithms have been developed to find liouvillian solutions of inhomogeneous linear differential equations [**45**] and linear differential equations over more general fields [**141**]. A different approach to the general algorithm based on the fact that a linear differential equation has a liouvillian solution if and only if the Galois group has a semi-invariant linear form that factors into linear factors, is given in [**147**]. Finally, the substitution $y = z\exp(\int u)$ and other order reducing substitutions are discussed in [**2**], [**5**], and [**6**].

In 1979, previous to [**136**], Kovacic [**92**] discovered an algorithm to decide if a second order linear differential equation $L(y) = y'' + ay' + by = 0$ with coefficients in $C(x)$ has liouvillian solutions (see [**51**] or [**119**] for other expositions of this algorithm). With hindsight, this algorithm is also based on the idea that when such solutions exist, the equation will have a solution that is exponential over an algebraic extension of the base field. Yet Kovacic was able to produce an algorithm that is much more efficient than any general algorithm produced since because he also took advantage of knowledge of all the groups that can occur as Galois groups for second order equations. Kovacic's first observation is that after the substitution $y = \exp(-\frac{1}{2}\int a)z$ one has a differential equation of the form $z'' - rz = 0$ whose Galois group G is unimodular and has liouvillian solutions if and only if the original equation has solutions of this type. We will describe the rest of the algorithm using a trichotomy used by Ulmer in [**155**] to improve the results of [**136**]. If G is a group acting on a vector space V we say that G is *reducible* if V is reducible as a G-module. The group G is said to be *imprimitive* if G is not reducible and if there exist nontrivial, proper subspaces $V_1, \ldots, V_r$ of V such that $V = V_1 \oplus \ldots \oplus V_r$ and such that G permutes the V_i. We say G is *primitive* if it is not reducible and not imprimitive. If $G \subset \mathrm{SL}(2, C)$ is reducible then it must have an invariant one-dimensional subspace (so the linear differential equation will have a solution that is exponential over $C(x)$) and Kovacic gives a very explicit algorithm to decide this. If $G \subset \mathrm{SL}(2, C)$ is imprimitive then G^o is diagonalizable and of index 2 in G. Kovacic shows that this implies that the equation will have linearly independent solutions y_1, y_2 such that $y_1 y_2$ is exponential over a quadratic extension of $C(x)$ and shows how one can decide this. Finally, if $G \subset \mathrm{SL}(2, C)$ is primitive, Kovacic shows that G is either $\mathrm{SL}(2)$ or a specific finite group of order $24, 48,$ or 120. Furthermore, in each of these latter three cases he shows that there are linearly independent solutions y_1, y_2 and a homogeneous form f in y_1, y_2 with constant coefficients, of degree $4, 6,$ and 12 respectively, such that $f^n \in C(x)$ with $n = 3, 2,$ or 1 respectively. In particular f will be exponential over $C(x)$. Kovacic shows how one can decide each of these latter three cases. Since these are the only possibilities, we have that the equation has liouvillian solutions if and only if the Galois group is reducible, imprimitive or one of these three cases hold Furthermore, a decision procedure results from these considerations. Kovacic is able to construct a liouvillian solution when one exists and is also able to give very strong necessary conditions by looking at the possible exponents that can occur at singular points (these conditions are generalized to third order equations in [**146**]). We note that Kovacic's algorithm finds a solution of the form $\exp(\int u)$ where u is algebraic over $C(x)$ when the equation has liouvillian solutions. When the equation has only algebraic solutions, the algorithm does not find the minimal polynomials of such solutions, even when the Galois group is one of the finite primitive groups. For these groups this task was begun in [**55**] and [**56**] and completed and generalized to third order equations in [**144**] and [**145**].

Several improvements have been made on Kovacic's original algorithm. Duval and Loday-Richaud [**51**] have given a more uniform treatment of the considerations concerning singular points and have as well applied the algorithm to decide which parameters in the hypergeometric equations (as well as several other classes of second order equations) lead to liouvillian solutions. In [**119**], van der Put shows how the algorithm can be simplified if the equation has only one or two singular points. In [**157**], Ulmer and Weil improve the algorithm by using the observation that any homogeneous polynomial with constant coefficients of degree m in any number of solutions of the equation must satisfy an auxiliary equation $L^{\circledS m}(y) = 0$ called the m^{th} symmetric power of L (see also [**37**]). They then show that except in the reducible case, one can decide if there is a liouvillian solution (and find one) by looking for solutions of appropriate symmetric powers *that lie in* $C(x)$. This eliminates some of the nonlinear considerations of Kovacic's algorithm. A general necessary and sufficient condition for the existence of liouvillian solutions in terms of the existence of G-invariant homogeneous forms is given in [**147**]. If the equation has coefficients in $C_0(x)$ where C_0 is not algebraically closed, it is important to know in advance how large an algebraic extension of C_0 is required. In [**66**] and [**164**] sharp results are given for Kovacic's algorithm as well as a general framework for higher order equations. In [**156**], sharp results are given concerning what constant fields are needed for equations of all prime orders. Applications of the algorithm to questions concerning integrability of Hamiltonians are given in [**9**] and the references cited there. We finally note that it was recently discovered that an algorithm (with some mistakes) to find the minimal polynomial of an element u algebraic over $C(x)$ with $\exp(\int u)$ satisfying a given second order linear differential equation was found by Pepin [**113**] in 1881. The problem of deciding when an n^{th} order linear differential equation has a solution y with y'/y algebraic is also discussed by Marotte in [**102**] and concrete results are given when $n = 2, 3, 4$.

The question of deciding if a linear differential equation has only algebraic solutions (or even one nonzero algebraic solution) has a long history. In 1872, Schwarz [**134**] gave a list of those parameters for which the hypergeometric equation has only algebraic solutions (for higher hypergeometric functions this was done by Beukers and Heckman [**24**]). Using his algorithm, Pépin [**113**] was able to reproduce this list as well. Using invariant theory, Fuchs [**55**], [**56**] was able to find the minimal polynomial of an algebraic solution of a second order linear differential equation assuming that the Galois group is the finite imprimitive group of order 48 or 120 (this method is generalized in [**145**]). In [**83**], [**84**], Klein shows that any second order linear differential equation with only algebraic solutions comes from some hypergeometric equation via a rational change in the independent variable $x = r(z)$. This approach was turned into an algorithm by Baldassarri and Dwork

[**10**]. Jordan [**76**] considered the problem of deciding if a linear differential equation of order n has only algebraic solutions. He showed that a finite subgroup of $\mathrm{GL}(n)$ has an abelian normal subgroup of index bounded by a computable function $f(n)$ of n (this fact and its generalizations are at the heart of the algorithms for finding liouvillian solutions in [**136**] and [**155**] described above). This implies that such an equation has a solution whose logarithmic derivative is algebraic of degree at most $f(n)$. Jordan's approach was made algorithmic in [**135**] (see also [**28**] and [**112**] for similar but incomplete algorithms due to Boulanger and Painlevé). It should be noted that the algorithms of Boulanger, Klein, Painlevé, and Pépin, are all incomplete in at least one regard. Each of these algorithms, at one point or another, is confronted with the following problem: Given an element u, algebraic over $C(x)$, decide if $\exp(\int u)$ is also algebraic over $C(x)$. Boulanger refers to this as *Abel's Problem* ([**28**], p. 93) and none of these authors gives an algorithm to answer this question. In 1970, Risch [**130**] showed that this problem could be solved if one could decide if a given divisor on a given algebraic curve is of finite order. Risch showed how one could solve this latter problem by reducing the jacobian variety of the curve modulo two distinct primes and bounding the torsion of the resulting finite groups. For other work concerning Abel's Problem, see [**10**], [**31**] [**44**], [**151**], [**161**], [**162**]. The introduction to [**102**], the book [**59**] and the article [**58**] give historical accounts of work concerning algebraic solutions of linear differential equations.

Over the field $C((x))$ of formal power series, all linear differential equations are solvable in terms of liouvillian functions (see section 2.4.2). An algorithm to solve the direct problem for second order equations whose Galois group is a subgroup of $\mathrm{SL}(2)$ is given in [**61**].

The question of when a linear differential equation has liouvillian or algebraic solutions leads to the general question of when such an equation can be solved in terms of other special functions. An equivalent definition of liouvillian extension is the following; an extension $E = k(t_1, \ldots t_r)$ of k is a *liouvillian extension* of k if each t_i is either algebraic over $k(t_1, \ldots t_{i-1})$ or satisfies a first order linear differential equation $t_i' + at_i = b$ with $a, b \in k(t_1, \ldots t_{i-1})$. We can generalize this to define a linear differential equation of order n to be solvable in terms of lower order equations if its Picard-Vessiot field lies in a field of the form $k(t_1, \ldots t_r)$ where each t_i is either algebraic over $k(t_1, \ldots t_{i-1})$ or satisfies a linear differential equation of order *less than* n over $k(t_1, \ldots t_{i-1})$. In [**139**] it is shown that a linear differential equation of order n is *not* solvable in terms of lower order linear differential equations if and only if the Lie algebra $\mathfrak{g}$ of G has the following two properties:

- $\mathfrak{g}$ is simple, and
- $\mathfrak{g}$ has no nontrivial representation of degree less than n.

An algorithm for third order equations is described in [**137**] (see also [**138**] for related material). An algorithm to solve linear differential equations in terms of hypergeometric functions is given in [**114**].

2.4. Calculating Galois Groups.

2.4.1. *Algebraic Methods.* The key to understanding modern algebraic techniques for calculating Galois groups comes from the formulation of the Galois theory in terms of Tannakian Categories [**18**], [**19**], [**21**], [**46**], [**47**], [**48**], [**78**]. Instead of a linear differential equation we consider a *connection*, that is, a finite dimensional

k-space $\mathcal{M}$ with an operator $\nabla : \mathcal{M} \to \mathcal{M}$ satisfying

$$\begin{aligned}\nabla(u+v) &= \nabla(u) + \nabla(v)\\ \nabla(fu) &= f'u + f\nabla(u)\end{aligned}$$

for all $u, v \in \mathcal{M}$ and $f \in k$ (c.f., [**64**]). If $e_1, \dots, e_n$ is a k-basis of $\mathcal{M}$, we may write

$$\nabla e_i = -\sum_j a_{j,i} e_j \tag{2.3}$$

where $A = (a_{i,j}) \in M(n,k)$. If $u = \sum_i u_i e_i$, then $\nabla(u) = \sum_i (u_i' - \sum_j a_{i,j} u_j) e_i$. Therefore, once a basis of $\mathcal{M}$ has been selected and the identification $\mathcal{M} \simeq k^n$ has been made, we have that $u \in k^n$ satisfies $u' = Au$ if and only if $\nabla u = 0$. Conversely, given a system $Y' = AY$, $A \in M(n,k)$ one can use equation 2.3 to define a connection ∇_A on k^n.

All of the usual constructions of linear algebra can be carried out for connections. A connection $(\mathcal{N}, \nabla_N)$ is a subconnection of $(\mathcal{M}, \nabla)$ if $\mathcal{N} \subset \mathcal{M}$ and $\nabla_N = \nabla|_N$. Given a connection and a subconnection one can define a quotient connection and if $(\mathcal{M}_1, \nabla_1)$ and $(\mathcal{M}_2, \nabla_2)$ are two connections one can form the direct sum $(\mathcal{M}_1 \oplus \mathcal{M}_2, \nabla_1 \oplus \nabla_2)$ and the tensor product $(\mathcal{M}_1 \otimes \mathcal{M}_2, \nabla_1 \otimes 1 + 1 \otimes \nabla_2)$ in the obvious ways. A morphism $\phi : (\mathcal{M}_1, \nabla_1) \to (\mathcal{M}_2, \nabla_2)$ is a k-linear map $\phi : \mathcal{M}_1 \to \mathcal{M}_2$ such that $\phi \circ \nabla_1 = \nabla_2 \circ \phi$. If $\{e_1, \dots, e_n\}$ is a basis of $\mathcal{M}_1$ (resp. $\{f_1, \dots, f_m\}$ is a basis of $\mathcal{M}_2$) and $Y' = A_1 Y$ (resp. $Y' = A_2 Y$) is the equation associated with $(\mathcal{M}_1, \nabla_1)$ (resp. $(\mathcal{M}_2, \nabla_2)$) then $U \in \mathrm{Hom}_k(\mathcal{M}_1, \mathcal{M}_2)$ defines a morphism if and only if $U' = A_2 U - U A_1$. One can define a connection $(\mathrm{Hom}_k(\mathcal{M}_1, \mathcal{M}_2), \nabla_{\mathrm{Hom}})$ by the equation $\nabla_{\mathrm{Hom}} \phi(u) = \nabla_2(\phi(u)) - \phi(\nabla_1 u)$. One sees that $\phi \in \mathrm{Hom}_k(\mathcal{M}_1, \mathcal{M}_2)$ defines a morphism if and only if $\nabla_{\mathrm{Hom}} \phi = 0$. When $\mathcal{M}_2 = k$ and ∇_2 is the trivial connection, then we say that $(\mathrm{Hom}_k(\mathcal{M}_1, \mathcal{M}_2), \nabla_{\mathrm{Hom}})$ is the dual connection $(\mathcal{M}_1^*, \nabla_1^*)$. The differential equation associated with $(\mathcal{M}_1^*, \nabla_1^*)$ is $Y' = -A^T Y$. If $\mathcal{M}_1 = \mathcal{M}_2$ and U defines an isomorphism, we then have

$$A_2 = U' U^{-1} + U A_1 U^{-1} \tag{2.4}$$

We therefore define the systems $Y' = A_1 Y$ and $Y' = A_2 Y$ to be *equivalent* if there exists a matrix $U \in \mathrm{GL}(n,k)$ such that equation 2.4 holds. Finally we note that on any finite dimensional $\mathcal{D}$-module $\mathcal{M}$ one can define a natural connection via the equation $\nabla u = Du$.

The relationship with linear differential equations or, more specifically linear operators, is the following. Given an operator $L = D^n + a_{n-1} D^{n-1} + \dots + a_0 \in \mathcal{D}$ one can associate to it the system $Y' = A_L Y$ where

$$A_L = \begin{pmatrix} 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 1 & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots \\ -a_0 & -a_1 & \dots & \dots & -a_{n-1} \end{pmatrix} \tag{2.5}$$

We denote the associated connection (k^n, ∇_L). One can easily check that this $\mathcal{D}$-module is isomorphic to $(\mathcal{D}/\mathcal{D} \cdot L)^*$. It is well known [**7**], [**17**], [**20**], [**46**], [**75**], [**80**], [**123**], that if k contains a non-constant element then any connection $(\mathcal{M}, \nabla)$ is cyclic, that is, there exists an element $u \in \mathcal{M}$ such that the elements $u, \nabla u, \nabla^2 u, \dots, \nabla^{n-1} u$ form a k-basis of $\mathcal{M}$. Applying this fact to the dual $(\mathcal{M}^*, \nabla^*)$, we see that with respect to a basis of the form $v, \nabla^* v, \dots, (\nabla^*)^{n-1} v$, the connection will have a matrix of the form

$$\begin{pmatrix} 0 & 0 & 0 & \dots & -a_0 \\ 1 & 0 & 0 & \dots & -a_1 \\ 0 & 1 & 0 & \dots & -a_2 \\ \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & 1 & -a_{n-1} \end{pmatrix}$$

Therefore $(\mathcal{M}^{**}, \nabla^{**}) \simeq (\mathcal{M}, \nabla)$ is associated with the equation $Y' = -B^T Y$ where $-B^T = A_L$ for the operator $L = D^n + a_{n-1}D^{n-1} + \ldots + a_0$. Let $Y' = AY$ be a differential equation and (k^n, ∇_A) the associated connection. We shall refer to an operator L so that $(k^n, \nabla_A) \simeq (k^n, \nabla_L)$ as an operator *equivalent to the system* $Y' = AY$ or *equivalent to the connection* ∇_A. Finding an equivalent linear operator allows one to reduce questions concerning systems (e.g., existence of rational or liouvillian solutions) to the corresponding questions for linear operators.

The Picard-Vessiot extension K associated to a system $Y' = AY$ is a field of the form $k(z_{1,1}, \ldots z_{n,n})$ having no new constants where $Z = (z_{i,j})$ is an invertible matrix satisfying $Z' = AZ$. The columns of Z span the solution space V of this equation. Equivalent equations have isomorphic Picard-Vessiot extensions and so we can define the Picard-Vessiot extension K corresponding to a connection. If $(\tilde{\mathcal{M}}, \tilde{\nabla})$ is a connection gotten from $(\mathcal{M}, \nabla)$ via iteration of the operations of direct sums, tensor products and duals, we say that $(\tilde{\mathcal{M}}, \tilde{\nabla})$ is a *construction* of $(\mathcal{M}, \nabla)$. If V is the solution space in K^n of the connection $(\mathcal{M}, \nabla)$ we can also consider all iterations of the operations of direct sums, tensor products and duals on V and call a resulting vector space $\tilde{V}$ a *construction* of V. One can show that if the same sequence of operations are applied to both $(\mathcal{M}, \nabla)$ and V, resulting in $(\tilde{\mathcal{M}}, \tilde{\nabla})$ and $\tilde{V}$, then the solution space of $(\tilde{\mathcal{M}}, \tilde{\nabla})$ is $\tilde{V}$. Let $\mathbf{M}$ be the collection of all subconnections of all constructions of $(\mathcal{M}, \nabla)$ and let $\mathbf{V}$ be the collection of all solutions spaces of elements of $\mathbf{M}$. Clearly, $\mathbf{V}$ is a collection of certain subspaces of constructions of V. For any construction $\tilde{V}$ of V there is a natural representation $\mathrm{GL}(V) \hookrightarrow \mathrm{GL}(\tilde{V})$. One can show [**47**], [**48**], [**78**], that the group

$$G = \{g \in \mathrm{GL}(V) \mid g(W) = W \text{ for all } W \in \mathbf{V}\}$$

is the Galois group of K over k. In more pedestrian terms, reducibility properties of the solutions spaces of associated differential equations determine the Galois group of a given differential equation.

This idea is at the heart of many results determining the Galois groups of linear differential equations. One distinguishes a certain group from other groups by the manner in which constructions decompose into invariant subspaces. One then gives criteria (or uses the tests described in Section 2.2) for the associated connections to decompose into the subconnections in the same way. For example, Beukers, Brownawell and Heckman show in [**23**] that for a linear algebraic group $G \subset \mathrm{GL}(V)$, G modulo its center is infinite and the second symmetric power S^2V is irreducible under G if and only if G is an extension by scalars of $\mathrm{SL}(V)$ or $\mathrm{Sp}(V)$. Using this they were able to show that large classes of hypergeometric equations have Galois groups of this type (for related work see also [**65**]). Katz and Pink [**82**] give a sufficient condition for the Galois group of a linear differential equation to contain $\mathrm{SL}(n), \mathrm{Sp}(n)$ or $\mathrm{SO}(n)$. The differential Galois groups of all irreducible confluent hypergeometric differential equations were determined by Katz and Gabber,[**79**], [**81**] using representation theoretic characterizations of semisimple

Lie algebras. An algebraic study of the Galois groups of reducible hypergeometric equations was made by Boussel in [**29**]. We note that many of the calculations were motivated by questions in the theory of transcendental numbers where one needs to know if certain functions satisfying differential equations and their derivatives are algebraically independent. This can be translated into questions concerning the dimension of certain Galois groups. Kolchin also wrote on this question. In [**86**], Kolchin calculated the Galois groups of Picard-Vessiot extensions generated by sets of Bessel functions but he relied on simple group theoretic considerations rather than the representation theoretic approach described above.

Algorithms based on this philosophy for computing Galois groups and their properties are given in [**144, 145, 146**]. For example, it is shown that $L(y) = y'' + ry = 0$ has liouvillian solutions if and only if $L^{\circledS 6}(y)$ is reducible, where $L^{\circledS 6}(y) = 0$ is the seventh order equation whose solution space is the sixth symmetric power of the solution space of $L(y) = 0$. We note that if L is irreducible its Galois group is reductive. This implies that $L^{\circledS 6}$ will be completely reducible and so its reducibility can be tested using the results of [**143**]. Similar criteria are given for properties of third order equations as well as criteria to determine if a particular group is the Galois group of such an equation. Fourth order equations have been treated by Hessinger in [**67**].

If one knows in advance that a linear differential equation has a reductive Galois group (that is, if the equation is completely reducible), then one can calculate a presentation of the associated Picard-Vessiot extension and its Galois group [**43**]. In [**41, 40**], Compoint showed that if a Picard-Vessiot extension has a reductive unimodular Galois group then the relations defining the Picard-Vessiot extension come from the invariants of the Galois group. To be more specific, let k be a differential field of characteristic zero with algebraically closed field C of constants and let $Y' = AY$ be a differential equation where A is an $n \times n$ matrix with entries in k. Let $G \subset \mathrm{SL}(n)$ be the Galois group and let its action on the polynomial ring $C[Y_{1,1}, \dots, Y_{n,n}]$ be defined by letting each element of G act on the $n \times n$ matrix $[Y_{i,j}]$ by multiplication on the left. Since G is reductive, the ring $C[Y_{1,1}, \dots, Y_{n,n}]^G$ of invariants is finitely generated. Compoint showed that if this ring is generated by polynomials of degree at most m, then the Picard-Vessiot extension is the quotient field of a ring $k[Y_{1,1}, \dots, Y_{n,n}]/I$, where I is an ideal generated by polynomials of degree at most m as well. It furthermore is known that given such an m, one can calculate the generators of I directly from the equation $Y' = AY$, without a priori knowledge of the Galois group [**73**]. Therefore the question of determining the Galois group of an equation $Y' = AY$ with reductive unimodular group is reduced to the question of finding a bound on the generators of the ring of invariants. The main result of [**43**] is that there is an effective method to find such a bound. It is also shown how this result can be extended to equations with arbitrary (not necessarily unimodular) reductive Galois groups as well as be used to calculate G/G^o, that is, the group of connected components of the Galois group of a linear differential equation with arbitrary Galois group.

Reduction modulo a prime can also be used to get information concerning the Galois group. Let $L(y) = 0$ be a linear differential equation with coefficients in $k_0(x)$ where k_0 is a number field. For almost all primes $\mathfrak{p}$ of k_0, it makes sense to reduce the coefficients of L modulo $\mathfrak{p}$ and get an operator $L_{\mathfrak{p}}$ with coefficients in $k_{\mathfrak{p}}(x)$, where $k_{\mathfrak{p}}$ is the residue field at $\mathfrak{p}$. A conjecture of Grothendieck states: *All solutions of $L(y) = 0$ are algebraic over $k_0(x)$ if and only if, for almost all $\mathfrak{p}$, the* *equation* $L_{\mathfrak{p}}(y) = 0$ *has a full set of solutions in* $k_{\mathfrak{p}}(x)$, [**19**], [**38**], [**74**], [**78**], [**120**]. In [**122**], van der Put uses the classification of differential equations in positive characteristic and ideas around the Grothendieck conjecture to factor differential operators and find symbolic solutions over $\mathbb{Q}(x)$.

2.4.2. *Analytic Methods.* Classically [**59**], analytic/topological techniques were used to associate a group with a differential equation $Y' = AY$ having coefficients in $\mathbb{C}(x)$. This group is defined as follows [**93**]. Let $\{\alpha_1, \ldots, \alpha_m\}$ be the singular points of the equation on the Riemann sphere S^2, let $\alpha_0 \in S^2 - \{\alpha_1, \ldots, \alpha_m\}$ and let $\{y_1, \ldots, y_n\}$ be a fundamental set of solutions, analytic in a neighborhood of α_0. If $[\gamma] \in \pi_1(S^2 - \{\alpha_1, \ldots, \alpha_m\}, \alpha_0)$, one can analytically continue each solution y_i along γ. The resulting function y_i^γ is independent of the homotopy class representative γ used and is again a solution of $Y' = AY$. From this one sees that there is an invertible matrix A_γ such that $(y_1^\gamma, \ldots, y_m^\gamma) = (y_1, \ldots y_n)A_\gamma$. Furthermore, the map $[\gamma] \mapsto A_\gamma$ is a group homomorphism of $\pi_1(S^2 - \{\alpha_1, \ldots, \alpha_m\}, \alpha_0)$ into $\mathrm{GL}(n, \mathbb{C})$. Selecting a different fundamental set of solutions at α_0 will give a conjugate homomorphism. The conjugacy class of this homomorphism is called the *monodromy representation* of the differential equation and the conjugacy class of the image is called the *monodromy group* [**118**]. The monodromy group is contained in the Galois group, but the Galois group can be strictly larger. For example the equation $y' - y = 0$ has trivial monodromy group since $\pi_1(S^2 - \infty, 0) = 0$ while its Galois group is $\mathbb{C}^*$. We say that a singular point α of the operator L is a *regular* singular point if for any open sector Ω at α, all solutions y, analytic in Ω, satisfy $\lim_{x\to\alpha} x^N y = 0$ for some $N \geq 0$. An equation is called *regular* if all of its singular points are regular. Fuchs showed that for a differential operator, $\alpha \in \mathbb{C}$ is a regular singular point if and only if the coefficient of D^{n-i} has a pole of order at most i. For a system $Y' = AY$, $\alpha \in \mathbb{C}$ is a regular singular pount if and only if there is a $U \in \mathrm{GL}(n, C((x-\alpha)))$ such that $U'U^{-1} + UAU^{-1} = (x-\alpha)^{-1}A_0$ for some constant matrix A_0 (this latter property is decidable [**12**]). It was known classically ([**131**], Section 160) that if all singular points are regular singular points then the monodromy group is Zariski dense in the Galois group. In general this is not true as the above example shows. For arbitrary singular points Ramis [**103**], [**124**], [**126**], [**125**] was able to define analytically a subgroup of each local Galois group (defined below) that is dense in this latter group.

To understand Ramis's results, we begin with some definitions (we shall follow the presentation in [**107**]; see also [**121**]). Let $\alpha \in S^2$ and let t_α be the local coordinate of S^2 at α. Let $\mathbb{C}(\{t_\alpha\})$ be the field of fractions of the ring of convergent power series at α and let $\mathbb{C}((t_\alpha))$ be the field of fractions of the ring of formal power series. Any differential equation $Y' = AY$ with $A \in M_n(\mathbb{C}(x))$ can be thought of as a differential equation over $\mathbb{C}(\{t_\alpha\})$ or $\mathbb{C}((t_\alpha))$ as well. The Galois group of $Y' = AY$ over $\mathbb{C}(\{t_\alpha\})$ is called the *local Galois group at* α and the Galois group over $\mathbb{C}((t_\alpha))$ is called the *formal Galois group at* α. It is known [**107**] that the Zariski closure of the group generated by all local Galois groups is the Galois group of the equation over $\mathbb{C}(x)$. For simplicity of notation, we shall restrict ourselves to the situation where $\alpha = 0$.

A differential equation $Y' = AY$ with $A \in \mathbb{C}(\{x\})$ admits a formal solution

$$\hat{Y} = \hat{H}(x)x^L e^{Q(t)} \tag{2.6}$$

where $t = x^{1/r}$, r a positive integer , $L \in M_n(\mathbb{C})$, $\hat{H} \in \mathrm{GL}(n, \mathbb{C}((x)))$ and $Q = diag(q_1, \ldots q_n)$ where $q_i \in t^{-1}\mathbb{C}[t^{-1}], i = 1, \ldots, n$ (this is the matrix version of formula 2.1). Furthermore, one may assume that the polynomials q_i are permuted by the map $t \mapsto te^{2i\pi/r}$.

We now define two groups. The first is the *formal monodromy group* G_M. This is the group generated by the matrix $\hat{M}$ defined $\hat{Y}(te^{2i\pi/r}) = \hat{Y}\hat{M}$. The second is the *exponential torus* $\mathcal{T}$. This is the Galois group of $\mathbb{C}((t))\langle e^{q_1}, \dots, e^{q_n}\rangle$ over $\mathbb{C}((t))$. Both of these groups may be identified with subgroups of the local Galois group of $Y' = AY$ at 0 and both of these groups may be calculated once one has the formal solution 2.6. It can be shown that the Zariski closure of the group generated by G_M and $\mathcal{T}$ is precisely the formal Galois group at 0 [**19**], [**107**]. Furthermore, when 0 is a regular singular point, the local and formal Galois groups coincide. For general singular points, this is not true and one needs additional elements to generate G topologically.

To define these elements, Ramis shows that one can canonically select open sectors $\mathcal{S}_1, \dots, \mathcal{S}_m$ at 0 that fill out a neighborhood of 0 and canonically (in fact, functorially) select fundamental solution matrices $\mathcal{Y}_i$ of $Y' = AY$, holomorphic in $\mathcal{S}_i$ (this latter process is called *multisummation* [**97**], [**99**], [**121**]). On the intersection $\mathcal{S}_i \cap \mathcal{S}_{i+1}$ the fundamental solution matrices must differ multiplicatively by a constant matrix S_i, $\mathcal{Y}_i = \mathcal{Y}_{i+1}S_i$. The matrices S_i are called the *Stokes matrices.* The fundamental result of Ramis is that the Zariski closure of the group generated by the formal monodromy, the exponential torus and the Stokes matrices is the local Galois group ([**103**] or [**98**] for a proof based on Tannakian considerations). When the differential equation has at most two singular points, at worst a regular one at 0 and a possibly irregular one at ∞, then one also knows that the Zariski closure of the local Galois group at infinity is the entire Galois group of the equation over $\mathbb{C}(x)$.

In general, there is no effective procedure to compute the Stokes matrices. There are numerical schemes to approximate the canonically selected holomorphic functions on the sectors (and so approximate the Stokes matrices) [**111**] as well as to directly approximate the Stokes matrices for systems of dimension 2 [**96**]. For large classes of differential equations, one has integral representations of the solutions and this does allow one to calculate the Stokes matrices. This has been done by Duval and Mitschi [**50**], [**52**] and applied to calculate Galois groups of some confluent hypergeometric differential equations. This work has been extended and generalized by Mitschi [**106, 107**] to calculate the Galois groups (over $\mathbb{C}(x)$) of large classes of confluent hypergeometric differential equations. Toulouse [**149**] uses this approach to calculate Galois groups of classes of equations of the form $Y' = (A_0 + xA_1)Y$ where $A_i \in M_n(\mathbb{C})$ with $n = 2$ or 3.

3. The Inverse Problem

The inverse problem is: *Given a differential field k, what groups are Galois groups of Picard-Vessiot extensions of k?* As Kolchin already noted in the extract quoted at the beginning of this article, the answer depends both on k and G. Expanding on Kolchin's comments, let $k = \mathbb{C}$ and let $Y' = AY$ be a differential equation with $A \in M_n(\mathbb{C})$. Classically, it is known that by reducing the matrix A to rational normal form one can reduce the problem of finding solutions of this equation to the problem of finding solutions of constant coefficient linear differential equations $L(y) = 0$ of order possibly greater than one. All solutions of such equations are of the form $y = \sum p_i(x)e^{\alpha_i x}$ where $p_i \in \mathbb{C}(x)$ and the $\alpha_i \in \mathbb{C}$.

Therefore the associated Picard-Vessiot extension K lies in a field of the form $F = \mathbb{C}(x, e^{\alpha_1 x}, \ldots, e^{\alpha_t x})$. One sees that F is a Picard-Vessiot extension of $\mathbb{C}$ with Galois group $G = \mathbb{C} \times \mathbb{C}^* \times \ldots \times \mathbb{C}^*$. Therefore the Galois group of K over $\mathbb{C}$ is a quotient of such a group, which can be shown to be of the form $\mathbb{C}^s \times \mathbb{C}^{*t}$ where s and t are nonnegative integers and $0 \leq s \leq 1$. Conversely, one can adjoin appropriate elements to $\mathbb{C}$ and show that any such group is a Galois group. Group theoretically, one can characterize such groups as linear algebraic groups that are (i) connected, (ii) abelian and (iii) whose largest unipotent subgroup has dimension at most one (a linear group is unipotent if all of its elements are unipotent). These three properties therefore characterize the groups that can appear as Galois groups of Picard-Vessiot extensions of $\mathbb{C}$. Once again, one can approach the general problem using algebraic and analytic techniques.

3.1. Algebraic Methods. Preceding Kolchin's 1966 talk, Bialynicki-Birula [**25**] had already shown that, for any differential field k of characteristic zero with algebraically closed field of constants C, if the transcendence degree of k over C is finite and nonzero then any connected nilpotent group is a Galois group of a Picard-Vessiot extension of k. Goldman [**57**] developed the notion of a generic differential equation with group $G \subset \mathrm{GL}(n)$, that is, a linear differential operator L with parameters such that any linear differential operator with Galois group contained in G is a specialization of L (this is analogous to the notion E. Noether developed for algebraic equations). In his thesis [**105**], Miller developed the notion of a differentially hilbertian differential field and gave a sufficient condition for the generic equation of a group to specialize over such a field to an equation having this group as a Galois group.

A breakthrough was made by Kovacic [**89**], [**90**] who proposed an ultimately successful program to solve the inverse problem for connected groups (over fields of finite, nonzero transcendence degree over their algebraically closed field of constants) and solved the problem for connected solvable groups. A key element of Kovacic's program is an emphasis on the role of the Lie algebra of the Galois group. Let G be a connected linear algebraic group defined over the algebraically closed field of constants C of a differential field k and let $\mathfrak{g}$ be its Lie algebra. We denote by $\mathfrak{g}_k$ the k-points in the Lie algebra, that is, $\mathfrak{g}_k = \mathfrak{g}_C \otimes_C k$. Kovacic shows that if $A \in \mathfrak{g}_k$ then the Picard-Vessiot extension of k corresponding to $Y' = AY$ has Galois group H that is a subgroup of G. Furthermore, for $k = C(x)$ (or any cohomologically trivial field), if the Galois group $H \subset G$ of $Y' = AY$ with $A \in \mathfrak{g}_k$ is connected, then there exists an invertible $g \in G(k)$ such that $g'g^{-1} + gAg^{-1} \in \mathfrak{h}_k$, where $\mathfrak{h}$ is the Lie algebra of H. In particular, to construct a differential equation $Y' = AY$ with coefficients in $C(x)$ having a given connected Galois group G one must find a matrix $A \in \mathfrak{g}_k$ such that:

1. the Galois group of $Y' = AY$ is connected and
2. for any $g \in G(k)$, $g'g^{-1} + gAg^{-1}$ does not lie in the Lie algebra of a smaller connected group.

Let us consider an example. Let

$$G = \{ \begin{pmatrix} 1 & c \\ 0 & 1 \end{pmatrix} \mid c \in C\}$$

G is isomorphic to the additive group C. The Lie algebra of this group is

$$\mathfrak{g} = \{\begin{pmatrix} 0 & c \\ 0 & 0 \end{pmatrix} \mid c \in C\}$$

If we start with a matrix

$$A = \begin{pmatrix} 0 & f \\ 0 & 0 \end{pmatrix}$$

with $f \in C(x)$ then the equation $Y' = AY$ will have a Galois group that is an algebraic subgroup of the additive group C. The only such groups are the trivial group and C itself, so the Galois group will be connected. If the Galois group is trivial then there is an element

$$g = \begin{pmatrix} 1 & h \\ 0 & 1 \end{pmatrix} \in G(C(x))$$

such that $g'g^{-1} + gAg^{-1}$ is the zero matrix. This is equivalent to the existence of an element $h \in C(x)$ such that $h' + f = 0$, that is, f has an antiderivative in $C(x)$. On can easily select an f so that this is not the case (e.g., $f = 1/x$) and so find a differential equation whose Galois group is the additive group C.

This criterion can be thought of in more geometric terms. Kolchin showed ([**88**] or the paper of Borel [**26**] in this volume) that a Picard-Vessiot extension of a differential field k with Galois group G is the function field of a principal homogeneous G-space. If $k = C(x)$ or any other cohomologically trivial field, all principal homogeneous spaces for a connected group G are isomorphic to G. Therefore, a Picard-Vessiot extension K of $C(x)$ can be thought of as the function field of a principal G-bundle over the projective line. The derivation on K gives a connection on this bundle and the condition that for any $g \in \mathrm{GL}(n, k)$, $g'g^{-1} + gAg^{-1}$ does not lie in the Lie algebra of a smaller connected group is equivalent to the differential geometric condition that the connection is not reducible to a connection on an H-bundle for some connected subgroup H of G.

Another key element of Kovacic's program was to give a procedure to inductively apply the above criterion to lift a solution of the inverse problem for G/R_u to a solution for the group G (R_u is the unipotent radical of G, that is, the largest normal subgroup of G all of whose elements are unipotent). Using this, Kovacic showed that to give a complete solution of the inverse problem for connected groups, one needed only to solve this problem for reductive groups (since G/R_u is reductive). He was able to solve the problem for tori and so could give a solution when G/R_u is a torus, that is, when G is a solvable group. He furthermore showed that one could reduce the problem to the case when G is the power of a simple connected linear algebraic group (see also [**60**] for extensions and modifications of Kovacic's methods to partial differential fields).

The solution of the inverse problem for such groups (or more generally semisimple groups) is one of the main contributions of [**108**] (see [**109**] for a simplified exposition). In this paper, the authors use an idea of Kovacic to show that to solve the inverse problem for connected groups over a differential field k of finite transcendence degree over its algebraically closed field of constants C it is enough to solve this problem for connected groups over $C(x)$. For this field and connected linear algebraic groups G, the authors show there are integers $d(G)$ and $e(G)$ depending only on G such that G is the Galois group of an equation of the form

$$Y' = (\frac{A_1}{x-\alpha_1} + \ldots + \frac{A_{d(G)}}{x-\alpha_{d(G)}} + A_\infty)Y$$

where $A_i,\ i = 1,\ldots,d(G)$ are constant matrices and A_∞ is a matrix with polynomial entries of degree at most $e(G)$. In particular, the only possible singularities of this system are $d(G)$ regular singular points in the finite plane and a (possibly irregular) singular point at infinity. The integers $d(G)$ and $e(G)$ are defined as follows. Let G be a connected linear algebraic group and let R_u be its unipotent radical and P a Levi factor, that is, a reductive group P such that $G = R_u \rtimes P$. The group $R_u/(R_u,R_u)$, where (R_u,R_u) is the commutator subgroup, is a commutative unipotent group and so isomorphic to a vector group C^n. The group P acts on R_u by conjugation and this action factors to an action on $R_u/(R_u,R_u)$. Therefore we may write $R_u/(R_u,R_u) = U_1^{n_1} \oplus \ldots \oplus U_s^{n_s}$, where each U_i is an irreducible P-module. We shall assume that U_1 is the trivial one dimensional P-module (and so allow the possibility that $n_1 = 0$). We may write $P = T \cdot H$, where T is a torus and H is a semisimple group. Let $m_i = n_i$ if the action of H on U_i is trivial and let $m_i = n_i + 1$ if the action of H on U_i is nontrivial. Let $N = 0$ if H is trivial and $N = 1$ if H is nontrivial. We define the *defect* $d(G)$ of G to be the number n_1 and the *excess* $e(G)$ of G to be $\max\{N, m_2, \ldots, m_s\}$. We note that any two Levi factors are conjugate so that these numbers are independent of the choice of P. Furthermore, one can show that $d(G)$ is the dimension of $R_u/(G,R_u)$

For example, for any connected reductive group G, we have that $R_u(G) = (0)$. Therefore $d(G) = 0$ and $e(G) = 1$ so there exists constant matrices A and B such that G is the Galois group of an equation of the form $Y' = (A + xB)Y$. It is also shown in [**108**] that the bound on the number of singular points given by $d(G)$ is sharp: there are examples of groups G that cannot be Galois groups of differential equations with fewer than $d(G) + 1$ singular points. Finally, the proof in [**108**] is constructive.

Kovacic also considered fields that are not finitely generated over their constant subfields. He showed that a solvable connected linear algebraic group G is a Galois over $C((x))$ if and only if it is commutative and its unipotent radical R_u is of dimension at most one. Equation 2.1 implies that the Picard-Vessiot extension of a linear differential equation over $C((x))$ lies in liouvillian extension of $C((x))$. Therefore, the identity component G^o of the Galois group G is solvable (in fact, G/G^o is cyclic so G is solvable). Therefore Kovacic's result implies that a connected group is a Galois group over $C((x))$ if and only if it is commutative and its unipotent radical is of dimension at most 1. Kovacic also considers the field $C(\{x\})$ of convergent power series and shows that a connected solvable linear algebraic group is a Galois group over this field if and only if the unipotent radical of the center of $G/(R_u,R_u)$ has dimension at most one. Using the notation of the previous paragraphs, this is equivalent to the condition that $d(G) \leq 1$. The complete characterization of Galois groups over $C(\{x\})$ when $C = \mathbb{C}$ rests on analytic techniques due to Ramis and will be described in the next section.

It should be noted also that Kovacic [**89**], [**90**] considered the inverse problem for arbitrary connected algebraic groups (not just linear groups), that is, classifying those groups that arise as Galois groups of strongly normal extensions. He showed that one can reduce this problem to the inverse problem for connected linear algebraic groups and for abelian varieties. In [**89**] he solved the problem for abelian varieties over fields of finite nonzero transcendence degree over their algebraically closed constant subfields. In [**116**], Pillay develops a Galois theory of differential fields where the Galois groups are differential algebraic groups. He also considers inverse problems in this context.

In the soon to be published second edition of [**100**], Magid uses geometric techniques to solve the inverse problem for several classes of groups. The proof given in the first edition of [**100**] to demonstrate that all connected groups having no subgroups of codimension one are Galois groups (Theorem 7.13) is flawed.

It should be finally noted that algebraic techniques have not yet been successful in attacking the inverse problem for groups that are not connected. At present, one must rely on analytic techniques.

3.2. Analytic Methods. Analogous to the inverse problem in differential Galois theory, there is an inverse problem in the analytic theory of linear differential equations. This is the *Riemann-Hilbert Problem.* In its weakest form the problem asks if, for any representation $\rho : \pi_1(S^2 - \Sigma) \to \mathrm{GL}(n, \mathbb{C})$ of the Riemann Sphere S^2 minus a finite set Σ of points, does there exist a differential equation $Y' = AY, A \in M_n(\mathbb{C}(x))$ with regular singular points at the points of Σ whose monodromy representation is given by ρ. In this form, the answer is always yes. This positive answer can be used to show that there is an equivalence of categories between the category of connections having only regular singular points in some set $\Sigma \subset S^2$ and the category of finite dimensional representations of $\pi_1(S^2 - \Sigma)$. This equivalence is called the *Riemann-Hilbert Correspondence* and, in this form, was established and generalized by Deligne [**46**]. A more restrictive form of the Riemann-Hilbert problem is to ask if there is a differential equation of the form

$$Y' = \sum_{\alpha \in \Sigma} \frac{A_\alpha}{x - \alpha} Y$$

(with $A_\alpha \in M(n, \mathbb{C})$ and $\sum A_\alpha = 0$) having ρ as its monodromy representation. Bolibruch has shown that the answer to this question is no for certain representations (for a history of this problem and its solution, see [**15**] or [**8**]).

C. and M. Tretkoff [**152**] used the solution of the weak Riemann-Hilbert Problem to solve the inverse problem for all linear algebraic groups G over the field $\mathbb{C}(x)$. They showed that any such G contains a Zariski - dense finitely generated subgroup H. One can select generators $h_1, \ldots h_m$ of H such that $\prod_{i=1}^{m} h_i = 1$ and therefore there exists a representation ρ of the fundamental group of the Riemann Sphere minus m points onto H. Using the solution of the weak Riemann-Hilbert Problem, there is a differential equation $Y' = AY$ with regular singular points whose monodromy representation is given by ρ. Since, for equations with regular singular points, the monodromy group is Zariski dense in the Galois group, the equation has Galois group G.

Ramis [**103**], [**104**] has shown how one can use analytic methods to completely solve the inverse problem over $\mathbb{C}(\{x\})$. To do this he defined a *wild monodromy group* $\pi_{1,S}$ and showed that it has the following two properties:

1. If $Y' = AY$ is a differential equation with $A \in \mathrm{GL}(n, \mathbb{C}(\{x\})$, there is a representation $\rho : \pi_{1,S} \to \mathrm{GL}(n, \mathbb{C})$ whose image is the group generated by the formal monodromy, exponential torus and Stokes matrices of this equation.
2. Any representation $\rho : \pi_{1,S} \to \mathrm{GL}(n, \mathbb{C})$ can be realized as in 1.

These two facts yield a Riemann-Hilbert Correspondence between the categories of local connections (i.e., connections over the field $\mathbb{C}(\{x\})$) and finite dimensional representations of $\pi_{1,S}$. To define the wild monodromy group, one notes that the possible formal monodromy groups are all homomorphic images of $\mathbb{Z}$. Furthermore, the possible exponential tori form an inverse system and we let $\mathcal{T}$ denote their inverse limit. There is a natural action of $\mathbb{Z}$ on $\mathcal{T}$ and so we can form their semi-direct product $\pi_{1,Sf} = \mathbb{Z} \ltimes \mathcal{T}$. The group $\pi_{1,Sf}$ is called the wild formal fundamental group and any formal Galois group is a quotient of this group. One can furthermore define a group $\mathcal{R}$ such that any group generated by Stokes matrices is a homomorphic image of $\mathcal{R}$ and an action of $\pi_{1,Sf}$ on $\mathcal{R}$ [**104**]. The group $\pi_{1,S}$ is defined to be $\pi_{1,Sf} \ltimes \mathcal{R}$.

We have already noted that the group generated by the formal monodromy, exponential torus and Stokes matrices is Zariski dense in the Galois group of the equation over $\mathbb{C}(\{x\})$. Therefore to solve the inverse problem over this field it is enough to identify those linear algebraic groups G containing a Zariski dense homomorphic image of $\pi_{1,S}$. Ramis gave necessary and sufficient conditions on the Lie algebra of G for this to be true and equivalent group theoretic conditions were given in [**110**]. The final result is the following [**128**] (see also [**127**], [**129**]): a linear algebraic group G with identity component G^o and unipotent radical R_u is a Galois group of a Picard-Vessiot extension of $\mathbb{C}(\{x\})$ if and only if

1. the finite group G/G^o is cyclic,
2. the dimension of $R_u/(R_u, G^o)$ is at most one, and
3. the group G/G^o acts trivially on $R_u/(R_u, G^o)$

Ramis is furthermore able to show that these three conditions are equivalent to the quotient $G/L(G)$ being the Zariski closure of a cyclic group, where $L(G)$ is the subgroup of G generated by all the tori of G. Ramis took this local result and combined it with a glueing technique to show that any linear algebraic group is the Galois group of a differential equation $Y' = AY$ over $\mathbb{C}(x)$ (or even over an algebraic extension of $\mathbb{C}(x)$) . Furthermore he can bound the number and type of singular points in terms of the number of generators of any Zarisk-dense subgroup of $G/L(G)$.

Analytic methods only work when the field of constants is $\mathbb{C}$. Nonetheless, some transfer techniques have been developed to allow one to draw conclusions about $C(x)$ where C is any algebraically closed field. Let $\mathcal{L}(n, m)$ be the vector space of differential operators

$$L = \sum_{i=0}^{n} \sum_{j=0}^{m} a_{i,j} x^j D^i$$

of order at most n whose coefficients are polynomials of degree at most m with coefficients in $\mathbb{C}$. Let G be a linear algebraic group, V a finite dimensional G-module and let $\mathcal{L}(n, m, G, V)$ be the set of operators in $\mathcal{L}(m, n)$ with Galois group G and solution space G-isomorphic to V. In general, $\mathcal{L}(n, m, G, V)$ is not a Zariski-closed subset (or even a constructible subset) of $\mathcal{L}(n, m)$. For example, an operator $L_c = xD - c$, $c \in \mathbb{C}$, has its solution space spanned by x^c. Therefore it has Galois group $\mathbb{C}^*$ if and only if c is not rational. This implies that $\mathcal{L}(n, m, \mathbb{C}^*, \mathbb{C})$ where $\mathbb{C}^*$ acts on $\mathbb{C}$ by multiplication, cannot be constructible. Note that c is an exponent at the singular point 0. This suggests that one fix a certain amount of information at the singular points. Let $\mathcal{S}$ be a finite subset of $k_0[x]$, $k_0 \subset \mathbb{C}$ and let $\mathcal{L}(n, m, \mathcal{S}, G, V)$ be the set of $L \in \mathcal{L}(n, m, G, V)$ such that at any singular point the exponents and determining factors at that point lie in $\mathcal{S}$ (c.f., the discussion following equation 2.1). In [**142**], large classes of groups are determined for which this set is constructible. To be more precise, let G be a linear algebraic group with identity component G^o. Let $\mathrm{Ker}\chi(G)$ be the intersection of the kernels of all characters $\chi : G^o \to \mathbb{C}^*$. The group $G^o/\mathrm{Ker}\chi(G^o)$ is abelian (in fact it is a torus) and so one can use the sequence

$$1 \to G^o/\mathrm{Ker}\chi(G^o) \to G/\mathrm{Ker}\chi(G^o) \to G/G^o \to 1$$

to define an action of G/G^o on $G^o/\mathrm{Ker}\chi(G^o)$. The main result of [**142**] is: if the action of G/G^o on $G^o/\mathrm{Ker}\chi(G^o)$ is trivial, then for any fixed G-module V and fixed n, m and $\mathcal{S}$, the set $\mathcal{L}(n, m, \mathcal{S}, G, V)$ is constructible and defined over k_0. For example, connected groups, finite groups, and groups where G^o is semisimple all satisfy these hypotheses. It is also shown in [**142**] that the assumption of triviality of the action is crucial; there is a family of second order differential operators with regular singular points and fixed exponents such that their Galois groups are all subgroups of $\mathbb{C}^* \rtimes \mathbb{Z}/2\mathbb{Z}$ and such that the set of parameters for which the Galois group is all of $\mathbb{C}^* \rtimes \mathbb{Z}/2\mathbb{Z}$ is not constructible.

If G is a group satisfying the hypothesis of this result, one can use the results of the Tretkoffs or Ramis to show that for any faithful G-module V and for some fixed n, m and $\mathcal{S}$, the set $\mathcal{L}(n, m, \mathcal{S}, G, V)$ contains a $\mathbb{C}$-point. The Hilbert Nullstellensatz then implies that it will contain a point in any algebraically closed field containing k_0. This allows one to solve the inverse problem for many classes of groups over $C(x)$ where C is an arbitrary algebraically closed field of characteristic zero.

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