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Comments on *"Über algebraisch integrirbare lineare Differentialgleichungen"* by F.G. Frobenius

In the paper [1], Frobenius considers homogeneous linear differential equations $L(y) = 0$ with coefficients in $\mathbb{C}(x)$ such that every solution can be expressed as a rational function in one of the solutions. Among other things, he shows that if the equation is irreducible and of order greater than or equal to 3, then all solutions are algebraic. He is motivated by the fact that when $L(y) = 0$ (of arbitrary order and not necessarily irreducible) has only algebraic solutions, then all solutions can be expressed as a rational function in one of them. Frobenius only considers those equations with regular singular points because he relies heavily on arguments using analytic continuation and the monodromy group. Replacing the monodromy group with the differential Galois group allows one to remove this restriction and Frobenius's arguments transfer *mutatis mutandis* to the general situation.

After reading Frobenius's paper, I realized that some of his arguments could be replaced with results from [3]. In this note, I will begin by using those results and some of Frobenius's arguments to prove his result. I will then discuss Frobenius's arguments.

1 Frobenius's Theorem

Let k be a differential field of characteristic 0 with algebraically closed field of constants C and let $L(y) = 0$ be a homogeneous linear differential equation of order n with coefficients in k. Let K be the associated Picard-Vessiot extension of k and let G be its differential Galois group. Following Frobenius, we first show

Proposition 1 *If K is an algebraic extension of k, then there exists a solution $z \in K$ of $L(y) = 0$ such that all solutions in k of this equation can be expressed as a rational function of z.*

Proof. Let $\{z_1, \ldots, z_n\}$ be a basis of the solution space of $L(y) = 0$. Since derivations extend to algebraic extensions, we have $K = k(z_1, \ldots, z_n)$. Standard proofs of the Primitive Element Theorem (PET) imply that there exist *constants* $c_1, \ldots, c_n$ such that $K = k(z)$ where $z = c_1 z_1 + \ldots + c_n z_n$, which is also also a solution of $L(y) = 0$. ∎

Frobenius includes a proof of the PET in his proof of this Proposition. Frobenius then showed the following partial converse.

Theorem 2 *If $L(y) = 0$ is an irreducible equation*[1] *of order $n \geq 3$ having a solution $z \in K$ such that all solutions in K lie in $k(z)$, then all solutions are algebraic.*

[1] Frobenius defines *irreducible* to mean that no solution satisfies a homogeneous linear differential equation of order $t < n$ where n is the order of $L(y)$. This can be seen to be equivalent to saying that the associated operator L has no right factor of order t for $t < n$.

Frobenius notes that the condition on the order is necessary by noting that for $v, w \in \mathbb{C}(x)$,

$$y = \sqrt{v}e^{\int \sqrt{w}dx}$$

satisfies

$$\frac{d^2y}{dx} - (\frac{v'}{v} + \frac{1}{2}\frac{w'}{w})\frac{dy}{dx} + (\frac{3}{4}(\frac{v'}{v})^2 - \frac{1}{2}\frac{v''}{v} + \frac{1}{4}\frac{v'w'}{vw} - w)y = 0.$$

Furthermore, the condition on irreducibility is also necessary. Let $P(X)$ be the monic polynomial whose roots are $\{1, \dots, n\}$ and $L(y) = P(\frac{d}{dx})(y)$. The associated Picard-Vessiot extension of $k = \mathbb{C}(x)$ is $k(e^x, e^{2x}, \dots, e^{nx}) = k(e^x)$.

I will show the following, from which Theorem 2 easily follows.

Proposition 3 *Let $L(y) = 0$ be a differential equation such that no nonzero solution of $L(y) = 0$ satisfies a homogeneous linear differential equation of order 1 or 2* [2]*. If $L(y) = 0$ has a solution $z \in K$ such that all solutions of this equation lie in $k(z)$, then all solutions are algebraic.*

To prove Proposition 3, it is enough to show that z is algebraic since all solutions of $L(y) = 0$ are in $k(z)$. Therefore, from now on, I shall assume the hypotheses of Proposition 3 and that z is transcendental over k and derive a contradiction.

For the proof of Lemma 5 below, I will need the following result from [3, Corollary 3.4].

Proposition 4 *Let $F \subset E$ be differential fields with F algebraically closed and having the same subfield of constants. Let $f, g \in E$ satisfy some nonzero homogeneous linear differential equations over F. Assume that $f \in F(g)$. Then,*

(i) either $f \in F[g]$, or

(ii) there exists $\theta \in K$ such that $\theta'/\theta \in F$ and $g = C + D\theta$ and $f \in F[\theta, \theta^{-1}]$ for some $C, D \in F$.

Lemma 5 *There exists an element $v \in k$ such that for any $\sigma \in G$ there exists a constant e such that $\sigma(z) = ez$ or $\sigma(z) = \frac{ev}{z}$.*

Proof. Note that if $\sigma \in G$, then σ acts as an automorphism of $k(z)$ and so

$$\sigma(z) = \frac{az + b}{cz + d} \tag{1}$$

for some $a, b, c, d \in k$. We have two cases:

Case 1, $c = 0$. We then have $\sigma(z) = ez + f, e, f \in k$. For any positive integer m, we have $\sigma^m(z) = e^m z + (e^{m-1} + e^{m-2} + \dots 1)f$. Since, for n equaling the order of $L(y)$, the elements $\sigma^n(z), \sigma^{n-1}(z), \dots, z$ must be linearly dependent over C, we have that there exist $c_n, \dots, c_0 \in C$ such that $(c_n e^n + c_{n-1}e^{n-1} + \dots + c_0)z +$ an element of $k = 0$. Since z is transcendental over k we have $c_n e^n + c_{n-1}e^{n-1} + \dots + e_0 = 0$ and so $e \in C$. This

[2]Once again, this is equivalent to saying that the associated operator has no right factor of order 1 or 2.

implies that ez is again a solution of the linear differential equation and so $L(f) = 0$. Assuming f is nonzero it satisfies a first order linear differential $y' - \frac{f'}{f}y = 0$ contradicting our assumption. Therefore $f = 0$ and $\sigma(z) = ez$. (This is exactly the argument that Frobenius gives in this case.)

Case 2, $c \neq 0$. We then may write $\sigma(z) = s + \frac{v}{z-u}$ for some $s, u, v \in k, v \neq 0$. To apply Proposition 4 we need to work over an algebraically closed base field and so we make the following construction. Let k_1 be the relative algebraic closure of k in K and let F be the algebraic closure of k. We then have that $k \otimes_{k_1} F$ is a domain and we let E denote the quotient field of this ring. Furthermore we have $k \subset F \subset E$ and $K \subset E$. Note that the derivations of k and K extend uniquely to derivations on F and E and these differential fields satisfy the hypotheses of Proposition 4.

We now apply Proposition 4 with $g = z$ and $f = \phi(z)$. Clearly (i) does not apply so by (ii) we have that $z = t + w\theta \in F(\theta)$, for some $t, w \in F$. If $\theta \in F$, then z would be algebraic over k, a contradiction and θ is transcendental over k. Therefore,for any $c \neq 0$ the map $\theta \mapsto c\theta$ induces a differential automorphism of $F(\theta)$ and so $t + cw\theta$ is again a solution of $L(y) = 0$. Subtracting we get $w\theta$ and therefore t are solutions of $L(y) = 0$. The Picard-Vessiot extension K of k contains a basis of the solution space of $L(y) = 0$. E has the same field of constants as k so $t \in K$ and therefore in $k(z)$. Since z is transcendental, any element of this field that is algebraic over k is in k. Therefore $t \in k$ and, if nonzero, satisfies the first order differential equation $y' - (t'/t)y = 0$, contradicting the hypotheses. Therefore $t = 0$ and $z = w\theta$.

We also have that $f = \sigma(z) = s + \frac{v}{z-u} = s + \frac{v/w}{\theta-(u/w)} \in F[\theta, \theta^{-1}]$. This implies that $u/w = 0$. Note that $\sigma^{-1}(z) = u + \frac{v}{z-s}$. If we apply Proposition 4 to $g = z$ and $f = \phi^{-1}(z)$, we then have $s/w = 0$ as well. Therefore, $\sigma z = \frac{v/w}{\theta} = \frac{v}{z}$. Furthermore, since $\sigma(z) \in k(z)$, we have $v \in k$. For any other $\phi \in G$, we will also have $\phi(z) = \frac{\bar{v}}{z}$ for some $\bar{v} \in k$. We also have $\sigma(\phi(z)) = \frac{\bar{v}}{z}$ so by Case 1, we have $\bar{v} = ev$ for some constant e. ∎

We can now finish the proof of Proposition 3. Lemma 5 implies that the C-vector space V spanned by z and $\frac{v}{z}$ is set-wise invariant under the action of G. Lemma 2.17 of [2] states that this implies that there is a linear differential equation $L(y) = 0$ with coefficients in k such that its solution space is V. Since $\dim(V) \leq 2$ and z is a solution, we have a contradiction.

2 Frobenius's Proof

I will not give a complete exposition of Frobenius's description of the form of solutions of linear differential equations having the property that all solutions are rational functions of this one. Instead I will follow Frobenius's ideas to prove Proposition 3 from which Theorem 2 easily follows. As I mentioned above, Frobenius restricted himself to linear differential equations with regular singular points so that he could use monodromy arguments. In particular he used the fact that if a rational function of solutions of the linear differential equation is left invariant by the monodromy group, if must be a rational function of x and also that a finite monodromy group implies that the solutions are algebraic. I will replace this with arguments involving the Galois group but this will not change the essential features of Frobenius's argument. Frobenius's argument is clear but often not parsed into Lemmas, Propositions, etc. but I isolate parts of the arguments using these structures.

Once again, I shall assume the hypotheses of Proposition 3 and that z is transcendental over k and derive a contradiction. As above we know that if $\sigma \in G$, then σ acts as an automorphism of $k(z)$ and so

$$\sigma(z) = \frac{az+b}{cz+d}$$

for some $a, b, c, d \in k$. Frobenius then

Lemma 6 *If $\sigma \in G$ and $\sigma(z) = az + b,\ a, b \in k$ then $a \in C$ and $b = 0$.*

The is the same as Case 1 of Lemma 5 and as I mentioned the proof is essentially the same as Frobenius's.

If for all $\sigma \in G$ we have $\sigma(z) = a_\sigma z + b_\sigma, a_\sigma, b_\sigma \in k$, then the above lemma implies that $\frac{z'}{z}$ is left fixed by G and so must lie in k. This would imply that z would satisfy a first order linear differential equation, a contradiction. Therefore, there exists a $\sigma \in G$ such that

$$\sigma(z) = \frac{az+b}{cz+d}$$

for some $a, b, c, d \in k$ *and* $c \neq 0$. In this case we will write

$$\sigma(z) = s + \frac{v}{z-u}$$

for some $s, v, u \in k$. Let n be the order of L.

Lemma 7 *There exist finite sets $U, S \subset k$ with $|U| \leq n, |S| \leq n$, such that if $\sigma \in G$ and $\sigma(z) = s + \frac{v}{z-u}$ then $u \in U$ and $s \in S$.*

Proof. Assume that there are distinct $u_1, \ldots, u_m$ as above with $m > n$. We must then have that the $\sigma_i(z) = s_i + \frac{v_i}{z-u_i}$ must be linearly dependent over C since they are all solutions of $L(y) = 0$. Uniqueness of the partial fraction decomposition implies that $u_i = u_j$ for some i, j, a contradiction.

Noting that $\sigma^{-1}(z) = u + \frac{v}{z-s}$ yields the same result for s. ∎

Lemma 8 *There exist an infinite number of $c \in C$ for which there exists a $\sigma \in G$ with $\sigma(z) = cz$.*

Proof. Assume not, that is, that the set of such c is finite.

If for each $u \in U$ there are only finitely may $\sigma \in G$ such that $\sigma(z) = s_\sigma + \frac{v_\sigma}{z-u}$, then G would be finite and z would be algebraic over k. Therefore we can assume we have $\sigma_1 \neq \sigma_2 \in G$ such that

$$\sigma_1(z) = s_1 + \frac{v_1}{z-u} \quad \text{and} \quad \sigma_2(z) = s_2 + \frac{v_2}{z-u}.$$

We then have that

$$\sigma_2\sigma_1^{-1}(z) = s_2 + \frac{v_2}{u + \frac{v_1}{z-s_1} - u} = \frac{v_2}{v_1}z - \frac{v_2}{v_1}s_1 + s_2.$$

Lemma 6 implies that $c = \frac{v_2}{v_1} \in C$, so $v_2 = cv_1$ and $\frac{v_2}{v_1}s_1 = s_2$ so $s_2 = cs_1$. As we are assuming the set of such c is finite, we would have that the set of automorphisms of the form $\sigma(z) = s + \frac{v}{z-u}$ would be finite. Therefore G would be finite and z would be algebraic, a contradiction. ∎

Lemma 9 *(cf. Lemma 5 above) There exists a $v \in k$ such that for all $\sigma \in G$ either $\sigma(z) = cz$ for some $c \in C$ or $\sigma(z) = \frac{cv}{z}$ for some $c \in C$.*

Proof. We know that for any $\sigma \in G$, either $\sigma(z) = cz$ for some $c \in C$ (and there are an infinite number of these) or $\sigma(z) = s + \frac{v}{z-u}$. Let $c \in C$ be such that it is not a t^{th} root of unity for any $t, 1 \leq t \leq n$ and let $\psi(z) = cz$. For $i = 0, \ldots, n$ we have that

$$\psi^i\sigma(z) = s + \frac{c^{-i}v}{z - c^{-i}u}.$$

Lemma 5 implies that $c^{-i} = c^{-j}$ with $0 \leq i, j \leq n$, contradicting the choice of c, unless $u = 0$. Arguing in the same way for σ^{-1} we have that $s = 0$ as well.

If $\tau(z) = \frac{w}{z}$, then $\psi(\tau(z)) = \frac{w}{v}y$ so $\frac{w}{v} = c \in C$. ∎

Lemma 9 implies that the vector space spanned by z and $\frac{v}{z}$ is invariant under the action of G and so spans the solution space of a homogeneous linear differential equation of order at most 2. This contradicts the assumption that no solution of $L(y) = 0$ satisfies such an equation. Therefore our assumption that z is transcendental over k cannot be true.

References

[1] Frobenius, F.G., *Uber algebraisch integrirbare lineare Differentialgleichungen*, Journal für die reine und angewandte Mathematik, 80, pp. 183–193, 1875.

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[3] Roques, J. and Singer, M.F., *On the Algebraic Dependence of Holonomic Functions*, Annales Henri Lebesgue, (5), pp. 141–177, 2022.