Integrals of Solutions of Linear Differential Equations

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In this note, I give an explication of some results of Daniel Bertrand and in particular the following

Theorem 1 *Let k be an ordinary differential field with algebraically closed constants and let K be a Picard-Vessiot extension of k. Assume that $Gal(K/k)$ is reductive. Let $y \in K$ satisfy a homogeneous linear differential equation $L(y) = 0$ of order m with coefficients in k. If y satisfies no homogeneous linear differential equation of smaller order (for example, if L is irreducible), then there exists a u algebraic over K such that $u' = y$ if and only if $L^*(z) = 1$ has a solution $z \in k$, where $L^*(z)$ is the adjoint equation of $L(y) = 0$. In this case, $u = -\pi(y, z)$ is a solution of $u' = y$ where $\pi(y, z)$ is the Lagrange bilinear concomitant.*

Note that if the operator L is irreducible over k, then $Gal(K/k)$ is reductive and y satisfies no homogeneous linear differential equation of smaller order

The above follows from Theorem 1 and Lemma 4 of Daniel Bertrand's paper [3]. I will try to unravel his proof and (I hope) make things more transparent. To motivate the proof of the general result I will first explain in the next section what happens when L has order 1 and K is the associated Picard-Vessiot extension. Following this I will give a proof of the full theorem and finally apply this result to the Airy equation and Bessel functions. All fields in this paper are of charactersitic zero.

1 Order 1 Equations

Let k be a differential field with algebraically closed constants C. Let

$$L(y) = \frac{dy}{dx} - ay \;\; = \;\; 0 \tag{1}$$

*Most of this appeared in a March 10, 2012 letter to Bruno Salvy be a first order linear differential equation with $a \in k$ and (abusing notation) let $K = k(y)$ be the Picard-Vessiot extension corresponding to this equation. Let $G \subset \mathrm{GL}_1(C)$ denote the Galois group, where the action of $\sigma \in G$ is given by $\sigma(y) = c_\sigma y$ for some $c_\sigma \in C$.

Assume that an integral of y, denoted by $\int y$, is algebraic over K. One first notes that $\int y$ must actually lie in $k(y)$. Too see this, let $P(z) = z^n + a_{n-1}z^{n-1} + \ldots + a_0$ be the minimal monic polynomial satisfied by $z = \int y$. Differentiating $P(\int y) = 0$ and using minimality one sees that $y = (a_{n-1}/nI)'$ and so $\int y = a_{n-1}/n + c$ for some constant $c \in k$.

The Galois group acts on $\int y$ and, for any $\sigma \in G$ we wish to see what σ does to $\int y$. We would like to say that $\sigma(\int y) = c_\sigma \int y$ but this may not be true. Nonethless, we will see how we can arrange for this to be true.

Differentiating $\sigma(\int y)$, we have

$$(\sigma(\int y))' = \sigma(y) \tag{2}$$
$$= c_\sigma y \tag{3}$$

so

$$\sigma(\int y) - c_\sigma \int y = \xi_\sigma \in C. \tag{4}$$

The map $\sigma \mapsto \xi_\sigma$ is not a homomorphism from G to the additive group C but rather satisfies the following identity for $\sigma, \tau \in G$

$$\xi_{\sigma\tau} = \xi_\sigma + c_\sigma \xi_\tau. \tag{5}$$

Now assume the following lemma (which I will prove below).

Lemma 2 *If $\sigma \mapsto \xi_\sigma$ is a map from a subgroup $G \subset GL_1(C) = C\backslash\{0\}$ such that equation (5) holds, then there exists a $\beta \in C$ such that*

$$\xi_\sigma = c_\sigma \beta - \beta \tag{6}$$

for all $\sigma \in G$.

Let β be as in the lemma and let $z = \int y + \beta$. For any $\sigma \in G$, we have

$$\begin{aligned} \sigma(z) &= \sigma(\int y + \beta) \\ &= c_\sigma \int y + \xi_\sigma + \beta \text{ (by the definition of } \xi_\sigma) \\ &= c_\sigma \int y + c_\sigma \beta - \beta + \beta \\ &= c_\sigma z. \end{aligned}$$

Therefore the action of $\sigma \in G$ on y and on z are the same and so

$$\sigma(\frac{z}{y}) \;=\; \frac{z}{y} \text{ for any } \sigma \in G.$$

The Galois theory implies that $\frac{z}{y} = v \in k$. Therefore $z = vy$. Differentiating, we have

$$z' = v'y + vy' = v'y + vay.$$

Dividing the equality $y = v'y + avy$ by y we get $1 = v' + av$, that is, $L^*(v) = 1$ where L^* is, by the definition below the adjoint of L.

Conversely if there is a $v \in k$ such that $1 = v' + av$, then vy is an integral of y. Therefore the result above is proved in this case.

Let me now give a proof of Lemma 2.

<u>Proof of Lemma 2.</u> Regretably, this is just a calculation. We follow the proof of Lemma 10.2, Ch. VI§10 of [9]. First note that if G is the trivial group, then we may take $\beta = 0$. Therefore we may assume that there is some element τ of the group such that $c_\tau \neq 1$. Note that for id, the identity element, $\xi_{id\cdot id} = \xi_{id} + \xi_{id}$ so $\xi_{id} = 0$. Since

$$0 = \xi_{id} = \xi_{\tau\tau^{-1}} = \xi_\tau + c_\tau \xi_{\tau^{-1}}$$

we have

$$c_\tau \xi_{\tau^{-1}} = -\xi_\tau.$$

Since G is commutative, we have for any $\sigma \in G$

$$\begin{aligned} \xi_\sigma = \xi_{\tau\sigma\tau^{-1}} &= \xi_\tau + c_\tau \xi_{\sigma\tau^{-1}} \\ &= \xi_\tau + c_\tau(\xi_\sigma + c_\sigma \xi_{\tau^{-1}}) \\ &= \xi_\tau + c_\tau \xi_\sigma + c_\sigma c_\tau \xi_{\tau^{-1}} \\ &= \xi_\tau + c_\tau \xi_\sigma - c_\sigma \xi_\tau \end{aligned}$$

so

$$\xi_\sigma - c_\tau \xi_\sigma \;=\; \xi_\tau - c_\sigma \xi_\tau$$

We therefore have

$$\xi_\sigma = c_\sigma(-\frac{\xi_\tau}{1 - c_\tau}) - (-\frac{\xi_\tau}{1 - c_\tau}).$$

Letting $\beta = -\frac{\xi_\tau}{1-c_\tau}$ yields the conclusion of the Lemma. ∎

2 Equations of any order.

There were two key parts to the above proof. The first was changing the given integral $\int y$ into $z = \int y + \beta$ on which the Galois group acted in the same way as it acts on y. This is the point of Lemma 2. This lemma is a statement from group cohomology. In particular, it says something about the first cohomology group. The first cohomology group measures “extensions” of groups and naturally arises here because when we adjoin an integral of an element, we are possibly extended the Galois group by a copy of $(C, +)$. I do not want to make this precise but this is the heart of Bertrand’s argument in his paper [3]. I will state the generalization of Lemma 2 without proof but use this in a way similar to the way I used it above. The second part was noticing that since the Galois group acted on z and y in the same way we could write z in terms of y and then easily deduce the conclusion. This also needs to be generalized but one can easily do this.

2.1 A Little Bit of Group Theory.

In the case of order 1 equations, the Galois group is a subgroup of $\mathrm{GL}_1(C) = C\backslash\{0\}$. Such groups have very special properties but the one that is most relevant is that if we represent such a group as matrices acting on a (possibly) larger vector space V and U is a subspace of W that is left invariant under the action of these matrices, then there is a complementary invariant subspace V such that $V = U \oplus W$. In general, we have the following definition.

Definition 3 *A linear algebraic group $G \subset \mathrm{GL}_n(C)$ (i.e. a Zariski closed subgroup of $\mathrm{GL}_n(C)$) is said to be* reductive *if for any G-invariant subspace U of C^n, then there exists another G-invariant subspace $W \subset C^n$ such that $C^n = U \oplus W$.*

Note that if G acts irreducibly on C^n (i.e., no invariant subspaces) then G is reductive. This is not the usual definition but, in characteristic zero, it is equivalent to the usual definition (see p.216, ex. 14 of [6]). For such groups we have an analog of Lemma 2. We need some definitions to state this.

Let G be a group and V a $G-$module, that is a finite dimensional vector space on which G acts as linear transformations. A map $\xi : G \to V$ where $\xi(\sigma) = \xi_\sigma$ is a *1-cocycle* if $\xi_{\sigma\tau} = \sigma(\xi_\tau) + \xi_\sigma$. A 1-cocycle is called a *1-coboundary* if there exists a $v \in V$ such that $\xi_\sigma = \sigma(v) - v$. The 1-cocycles form a group (under addition) designated by $Z^1(G, V)$ and the 1-coboundaries form a subgroup designated by $B^1(G, V)$. Let $H^1(G, V) = Z^1(G, V)/B^1(G, V)$. The following is proved in [5], p. 194. An elementary proof may be deduced from Proposition 2 of [3].

Lemma 4 *Let G be a reductive group. Then for any $G-$module V, $H^1(G, V) = 0$.*

The following is the generalization of Lemma 2 that we need.

Corollary 5 *Let G be a reductive group. Assume $G \subset GL(V)$ where V is a finite dimensional vector space and write $[\sigma]$ for the matrix representing $\sigma \in G$. Let $\xi : G \to V$ where $\xi(\sigma) = \xi_\sigma$. If $\xi_{\sigma\tau} = \xi_\sigma \cdot [\tau] + \xi_\tau$, then there exists a $v \in V$ such that $\xi_\sigma = v \cdot [\tau] - v$.*

Proof: ξ does not satisfy the cocycle condition so we cannot apply Lemma 4 immediately. Let G^{opp} be the opposite group of G, that is the group defined on the elements of G using the multiplication $\sigma \star \tau = \tau\sigma$. G is isomorphic to G^{opp} via the map $\sigma \mapsto \sigma^{-1}$. ξ is a cocycle for G^{opp} and now the result follows from Lemma 4. ∎

In Lemma 2, the group was a commutative group so we did not need to worry about left or right multiplication, but in the above we do.

2.2 A Little Bit of Galois Theory.

I will now prove the result which corresponds the fact used above allowing us to write $z = vy$ in Section 1.

Lemma 6 *Let K be a Picard-Vessiot extension of k with constants C. Let $V \subset K$ and $W \subset K$ be finite dimensional $Gal(K/k)$-invariant vector spaces over C. If $\{v_1, \ldots, v_n\}$ and $\{w_1, \ldots, w_n\}$ are bases of V and W such that for any $\sigma \in Gal(K/k)$ the matrices of σ with respect to $\{v_1, \ldots, v_n\}$ and $\{w_1, \ldots, w_n\}$ are the same, then there exist $\alpha_0, \ldots, \alpha_{n-1}$ in k such that $u_i = \sum_{j=0}^{n-1} \alpha_j v_i^{(j)}$ for $i = 1, \ldots, n$.*

Proof: It is well known ([7], p. 21) that elements $z_1, \ldots, z_n$ of K are linearly independent over C if and only if the determinant of the wronskian matrix det($Wr(z_1, \ldots, z_n)) \neq 0$, where $Wr(z_1, \ldots, z_n)$ is the $n \times n$ matrix $(z_i^{(j)})$. Let $\{v_1, \ldots, v_n\}$ and $\{w_1, \ldots, w_n\}$ be as above and let $[\sigma]$ be the matrix of $\sigma \in Gal(K/k)$. We then have $\sigma(Wr(v_1, \ldots, v_n)) = Wr(v_1, \ldots, v_n) \cdot [\sigma]$ and $\sigma(Wr(w_1, \ldots, w_n)) = Wr(w_1, \ldots, w_n) \cdot [\sigma]$ so $A = Wr(v_1, \ldots, v_n) \cdot Wr(w_1, \ldots, w_n)^{-1}$ is left invariant by $Gal(K/k)$. Therefore the entries of A are left invariant by $Gal(K/k)$. The first column of A gives the desired elements α_i. ∎

2.3 Integrals in Picard-Vessiot Extensions

We now combine Lemma 6 and Corollary 5 to show:

Proposition 7 *Let K be a Picard-Vessiot extension of k. Assume that $Gal(K/k)$ is a reductive group. Let $y \in K$ satisfy a homogeneous linear differential equation*

$$L(y) = a_0 y^{(m)} + a_1 y^{(m-1)} + \ldots + a_m y = 0 \tag{7}$$

with coefficients in k.

1. *If there exists a $u \in K$ such that $u' = y$, then there exist $\alpha_i \in k$ and $c \in C$ such that*

$$u = c + \alpha_0 y + \alpha_1 y' + \ldots + \alpha_{m-1} y^{(m-1)} \tag{8}$$

2. *If, in addition, y satisfies no linear homogeneous differential equation of order smaller than m, then $L^*((-1)^{(m+1)}(1/a_0)\alpha_{m-1}) = 1$, where L^* is the adjoint equation of L.*

Proof: (1) Let V be the $C-$span of $\{\sigma(y) \mid \sigma \in Gal(K/k)\}$. Since y satisfies a homogeneous linear differential equation of order m, the dimension of V is at most m. V has a basis of the form $y_1 = y, y_2 = \sigma_2(y), \ldots, y_t = \sigma_t(y)$ for some $\sigma_i \in Gal(K/k)$. For $\sigma \in Gal(K/k)$, let $[\sigma]$ be the matrix such that $\sigma(y_1, \ldots, y_t) = (y_1, \ldots, y_t)[\sigma]$. We first note that $\sigma_i(u)' = \sigma_i(u') = \sigma_i(y) = y_i$. Let $u_i = \sigma_i(u)$. Using this notation, we have for any $\sigma \in Gal(K/k)$,

$$\begin{aligned}(\sigma(u_1), \ldots, \sigma(u_t))' &= (\sigma(y_1), \ldots, \sigma(y_t)) \\ &= (y_1, \ldots, y_t) \cdot [\sigma] \\ &= ((\sigma(u_1), \ldots, \sigma(u_t)) \cdot [\sigma])'\end{aligned}$$

Therefore, $\xi_\sigma = (\sigma(u_1), \ldots, \sigma(u_t)) - (\sigma(u_1), \ldots, \sigma(u_t)) \cdot [\sigma] \in C^n$. Since

$$\begin{aligned}(\sigma(\tau(u_1)), \ldots, \sigma(\tau(u_t))) &= \sigma((y_1, \ldots, y_t) \cdot [\tau] + \xi_\tau) \\ &= ((y_1, \ldots, y_t) \cdot [\sigma] + \xi_\sigma) \cdot [\tau] + \xi_\tau) \\ &= ((y_1, \ldots, y_t) \cdot [\sigma][\tau] + \xi_\sigma \cdot [\tau] + \xi_\tau\end{aligned}$$

we have $\xi_{\sigma\tau} = \xi_\sigma \cdot [\tau] + \xi_\tau$. Therefore, Corollary 5 implies that there exists a $\overline{c} = (c_1, \ldots, c_t) \in C^t$ such that $\xi_\sigma = \overline{c}[\sigma] - \overline{c}$. Let $(z_1, \ldots, z_t) = (\sigma(u_1), \ldots, \sigma(u_t)) + \overline{c}$. One easilly checks that $\sigma((z_1, \ldots, z_t)) = (z_1, \ldots, z_t) \cdot [\sigma]$. Therefore Lemma 6 implies that there exist $\alpha_0, \ldots, \alpha_{t-1} \in k$ such that $z_1 = u + c_1 = \alpha_0 y + \ldots + \alpha_{t-1} y^{(t-1)}$ and the conclusion of part (1) follows.

(2) Before we can prove assertion (2), we need to review certain facts about the adjoint equation (c.f., [10, Section 10]). Recall that the *adjoint equation* of $L(y) = 0$ is defined as:

$$L^*(y) = (-1)^m (a_0 y)^{(m)} + (-1)^{m-1} (a_1 y)^{(m-1)} + \ldots + a_m y \tag{9}$$

One can write equation (2) in matrix form as $Y' = AY$, where

$$A = \begin{pmatrix} 0 & 1 & 0 & \ldots & 0 & 0 \\ 0 & 0 & 1 & \ldots & 0 & 0 \\ 0 & 0 & 0 & \ldots & 0 & 0 \\ . & . & . & . & . & . \\ 0 & 0 & 0 & \ldots & 1 & 0 \\ 0 & 0 & 0 & \ldots & 0 & 1 \\ -\frac{a_m}{a_0} & -\frac{a_{m-1}}{a_0} & -\frac{a_{m-2}}{a_0} & \ldots & -\frac{a_2}{a_0} & -\frac{a_1}{a_0} \end{pmatrix}$$

We now write equation (8) as $u - c = \overline{\alpha} \cdot Y$, where

$$\overline{\alpha} = (\alpha_0, \ldots, \alpha_{m-1}), \quad and \;\; Y = \begin{pmatrix} y \\ y' \\ . \\ . \\ y^{(m-1)} \end{pmatrix}$$

Differentiating $u - c = \overline{\alpha} \cdot Y$, we get $y = u' = \overline{\alpha}' \cdot Y + \overline{\alpha} \cdot Y' = (\overline{\alpha}' + \overline{\alpha} \cdot A) \cdot Y$. Writing $y = (1, 0, \ldots, 0) \cdot Y$ and noting that $y, y', \ldots, y^{(n-1)}$ are linearly independent over k (since y satisfies no linear differential equation of order lower than n), we have that $(\overline{\alpha}')^T = -A^* \cdot \overline{\alpha}^T + (1, 0, \dots, 0)^T$. This implies

$$\begin{array}{rcl} \alpha'_{m-1} & = & -\alpha_{m-2} + \dfrac{a_1}{a_0}\alpha_{m-1} \\ \alpha'_{m-2} & = & -\alpha_{m-3} + \dfrac{a_2}{a_0}\alpha_{m-1} \\ \dots & \dots & \dots \\ \alpha'_1 & = & -\alpha_0 + \dfrac{a_{m-1}}{a_0}\alpha_{m-1} \\ \alpha'_0 & = & 1 + \dfrac{a_m}{a_0}\alpha_{m-1} \end{array}$$

Eliminating the $\alpha_i, 0 \leq i \leq m-1$, we get

$$\alpha_{m-1}^{(m)} - (\frac{a_1}{a_0}\alpha_{m-1})' + \cdots + (-1)^{m-1}\frac{a_m}{a_0}\alpha_{m-1} = (-1)^{m-1}$$

Letting $z = (-1)^{m+1}(1/a_0)\alpha_{m-1}$, we get $L^*(z) = 1$. ∎

Before we state our main result, we need to recall one more fact concerning adjoint equations. One defines the *Lagrange bilinear concomitant* as:

$$\pi(u, v) = \sum_{i=0}^{m}\sum_{j=0}^{i-1}(-1)^j(va_{m-i})^{(j)}u^{i-1-j}$$

One then gets *Lagrange's identity*, (c.f., [10, Section 10]):

$$vL(u) - uL^*(v) = (\pi(u, v))'$$

Proof of Theorem 1: Proposition 7 implies that if there exists a $u \in K$ such that $u' = y$ then $L^*(z) = 1$ has a solution $z \in k$. Conversely, if $L^*(z) = 1$ has a solution $z \in k$, then Lagrange's identity gives the other implication. ∎

Note that Proposition 7 and Theorem 1 are not true without the assumption that $Gal(K/k)$ is reductive. For example, let $k = \mathbb{C}$ and $L(y) = y'' = 0$. The corresponding Picard-Vessiot extension of $\mathbb{C}$ is $\mathbb{C}(x), x' = 1$. Note that $(x^2)' = 2x$ is a solution of $L(y) = 0$, but $x^2 \neq \alpha_0 x + \alpha_1 1$ for any $\alpha_0, \alpha_1 \in \mathbb{C}$. Furthermore, $L^*(y) = y'' = 1$ has no solution in $k = \mathbb{C}$.

3 Examples

Let $k = \mathbb{C}(x)$. As mentioned following the statementof Theorem 1, this theorem applies if the operator L is irreducible over k.

The **Airy Equation** $L(y) = y'' - xy$.

Let K be the Picard-Vessiot extension corresponding to this equation. This equation is irreducible over k so we can apply our results. In this case $L^*(y) = y'' - xy$. If $z \in k$ is a solution of $L^*(z) = 1$, then z would have no poles and so would be a polynomial. Comparing the degrees of z'' and xz shows that this is impossible. It is known that if y_1 and y_2 are linearly independent solutions of the Airy equation, then y_1, y_2, y_1' are algebraically independent [4, p. 180]. Therefore we can conclude that these latter elements and the integral of any nonzero solution of the Airy equation are algebraically independent over $\mathbb{C}(x)$.

The **Bessel Equations** $L_\nu(y) = y'' + \frac{1}{x}y' + (1 - \frac{\nu^2}{x^2})y$

The standard basis for the solution space of $L_\nu(y) = 0$ is usually denoted by $\{J_\nu, Y_\nu\}$. Let $K = k(J_0, Y_0, J_0', Y_0')$ be the Picard-Vessiot extension associated with $L_0(y)$. The recursion formulas for Bessel functions [2, p. 361] imply that for any integer ν, we have $J_\nu, Y_\nu \in K$. In [8], Kolchin showed (among other things) that the Galois group of $L_\nu(y) = 0$ is $\mathrm{SL}_2(\mathbb{C})$ if $\nu - 1/2 \notin \mathbb{Z}$. In particular, the Galois group of K over k is reductive. and we can apply the Theorem. Here are two simple examples:

$L_0(y) = y'' + \frac{1}{x}y' + y$: The adjoint of this equation is $L_0^* = y'' - \frac{1}{x}y' + (1 + \frac{1}{x^2})y$. A rational function solution of $L^*(y) = 1$ can only have finite poles at $x = 0$ and a local computation shows this is impossible. Furthermore, $L^*(y) = 1$ has no polynomial solutions so $\int J_0(x)$ is transcendental over K.

$L_1(y) = y'' + \frac{1}{x}y' + (1 - \frac{1}{x})y$: The adjoint of this equation is $L_1^*(y) = y'' - \frac{1}{x}y' + y$. The equation $L_1^*(y) = 1$ has a solution $y = 1$ so $\int J_1(x) = -\pi(J_1(x), 1) = -(J_1'(x) + \frac{1}{x}J_1(x))$.

A more complicated example is given when one considers $\int J_1^3(x)$. This function satisfies

$$L_1^{\circledS 3}(y) = y^{(iv)} + \frac{6}{x}y''' + \frac{10x^2 - 3}{x^2}y'' + \frac{30x^2 - 9}{x^3}y' + \frac{9x^4 - 6x^2 + 9}{x^4}y = 0$$

and no equation of smaller order. The adjoint of this equation is

$$(L_1^{\circledS 3})^*(y) = y^{(iv)} - \frac{6}{x}y''' + \frac{10x^2 + 15}{x^2}y'' - \frac{30x^2 + 15}{x^3}y' + \frac{9x^2 + 24}{x^2}y.$$

Considering local expansions, one sees that $(L_1^{\circledS 3})^*(y) = 1$ has no solution in k.

In [1], Abramov and van Hoeij consider the problem of finding the integral (or sum, in the difference case) of a solution of an Ore equation. They show that if the integral/sum satisfies an Oe equation of the same order, then one can find it using a Ore-generalization of the adjoint operator.

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