Comments on Rosenlicht's *Integration in Finite Terms*

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Author Proof

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In 1968, Maxwell Rosenlicht [Ros68] published the first purely algebraic proof of Liouville's Theorem on Integration in Finite Terms (which we will simply refer to as "Liouville's Theorem") . This paper, together with Robert Risch's paper [Ris69], stimulated renewed interest in both the mathematical and algorithmic aspects of this area. The paper *Integration in Finite Terms* [Ros72] appearing in this volume presents the material of [Ros68] in a simplified form, suitable for an advanced undergraduate. It is a beautiful paper and is the best introduction to the subject. In this commentary, I will review the history of Liouville's Theorem prior to these papers, discuss Rosenlicht's and Risch's contributions and then describe the related results that have appeared

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subsequently. I will focus mainly on the theoretical aspects since the algorithmic aspects are well described in Clemens Raab's and Barry Trager's commentaries to this volume. Although I will almost exclusively describe results related to elementary functions, I will occasionally mention results related to Liouvillian functions (those built up using exponentials, *arbitrary integrals* and algebraic functions) when this appears in the work of researchers concerned with elementary functions but I will not delve deeper into the problem of solving differential equations in terms of these.

Several books and articles present the history of Liouville's Theorem and related topics. In particular Jesper Lützen's thoughtful and detailed book [Lüt90] thoroughly discusses the contributions of Laplace, Abel, Liouville and others and I have relied heavily on this book in my summaries below. Anyone interested in the subject should go directly to [Lüt90] to get a much more complete picture. Hardy has also given a pleasantly excursive presentation of these contributions in [Har71]. Much of the theoretical and computational aspects of this subject and related results are to be found in Bronstein's book [Bro97]. Finally, the articles of Kasper [Kas80] and Marchisotto and Zakeri [MZ94] give concise histories of some aspects of these results as well.

I will assume the reader is familiar with [Ros72] and will not restate the definitions of differential field, elementary extension, etc.

1 Before Rosenlicht and Risch

Initial glimmerings of phenomena related to Liouville's Theorem can be found in the writings of Fontaine and Condorcet (cf. [Lüt90, pp. 352–357]). A clear statement of the principle underlying Liouville's Theorem was given by Laplace in [Lap12, pp. 4–5]:

> Thus, since the differentiation lets the exponential and the radical quantities subsist and only makes the logarithmic quantities disappear when they are multiplied by constants, one may conclude that the integral of a differential function cannot include any other exponentials and radicals than those already included in this function.[2]

but no rigorous proof seems to have been published by him [Lüt90, p. 358].

1.1 Abel

Abel stated several results related to Liouville's Theorem and stated that he had a general theory of integration of algebraic functions aimed at reducing these objects using algebraic and logarithmic functions (cf. [Lüt90, pp. 358–369], [Rit48, pp. 28–31]). Part of Abel's research has reached modern times as half of the Abel–Jacobi Theorem (see [Gri76, Gri04]) and part has a direct bearing on the subject of this volume. First of all, Abel stated Liouville's Theorem, although any proof that he had now seems lost. He showed that certain elliptic integrals could not be expressed in the form prescribed by this theorem (as did Chebyshev, Zolotarev and others [Lüt90, pp. 367, 415–417]). Concerning Liouville's Theorem, Abel presented an argument

[2] I am using the translation that appears in [Lüt90, p. 358].

that if an algebraic function y has an integral of the form $v+\sum_{i=1}^{n} c_i \log v_i$, where v and the u_i are algebraic functions and the c_i are complex numbers, then v and the u_i may be taken to be rational functions of x and y (cf. [Lüt90, pp. 363–364] and [Rit48, pp. 28–31]; note that Littlewood criticized this proof [Lüt90, p. 365] and [Har71, Preface, pp. 38–43]). In Rosenlicht's paper [Ros68] this fact results from taking norms and traces. An even more precise statement of where the c_i, v, and u_i lie can be found in [Ris69, p. 171] or [Bro97, p. 141].

Abel also stated generalizations of Liouville's Theorem. For example, he stated that if one has a relation of the form $F(x, \int y_1 \mathrm{d}x, \int y_2 \mathrm{d}x, \ldots, \int y_n \mathrm{d}x) = 0$, where F is an elementary function of $n+1$ variables and $y_1, \ldots, y_n$ are algebraic functions, then some constant linear combination of the $\int y_i \mathrm{d}x$ is of the form $u+\sum_{i=1}^{m} \log v_i$ with c_i constants and u, v_i algebraic functions. We do not have Abel's proof but a modern statement and proof of this result appears in [PS83, Corollary 2].

Another generalization of Liouville's Theorem stated by Abel is: if the integral of an algebraic function can be expressed in terms of implicitly or explicitly defined elementary functions then it is elementary and so satisfies the conclusion of Liouville's Theorem. Lützen remarks that it is not clear what Abel meant by implicit elementary functions as no proof survives, but Ritt [Rit48, pp. 71–98] and Risch [Ris76] proved precise versions of this result.

1.2 Liouville

Lützen [Lüt90, Chapter IV] and Ritt [Rit48] give detailed analyses of Liouville's papers and methods, so I will only give an overview.

Liouville's first paper on integration in finite terms appeared in 1832 (when he was 23!) and by 1840 he had essentially finished his work on the subject. Liouville began by considering the question of integrating an algebraic function in terms of algebraic functions, obtaining many of Abel's results. Liouville then turned to the problem of expressing integrals of algebraic functions in terms of more general elementary functions. He began by examining when such an integral is of a special form (e.g., when the integral of an expression involving a square root is the sum of a similar expression and logarithms of similar expressions). He then turned to prove what we now know as Liouville's Theorem in the special case when the integrand is an algebraic function and published this result in 1834.

To prove this result, Liouville first gave a classification of elementary functions. Briefly, algebraic functions are called functions of order zero. A function is elementary of order at most n if it is an algebraic function of elementary functions of order at most $n-1$ and logarithms and exponentials of elementary functions of order at most $n-1$. The smallest n such that an elementary function is of order at most n is called its order. Since Liouville dealt with actual functions, one needs to take into account branches of these functions and this led to criticisms of imprecision, especially by Ritt (see below). Nonetheless, the concept of order is key to Liouville's proofs. He relies heavily on the following observation (called *Liouville's principle* by Lützen [Lüt90, p. 387] and Ritt [Rit48, p. 16]):

> Let f be an elementary function of order n and assume that the number r_n of logarithms and exponentials of order n appearing in its definition is as small as possible. Any algebraic relations among these r_n quantities and elementary functions of order at most $n-1$ must hold when any quantity is substituted for any of these r_n quantities.

Liouville then proceeds as follows. Assume that one expresses the integral of an algebraic function as an elementary function of order n with r_n as above minimal. Differentiating one then has a relation as in the statement of Liouville's principle. Liouville then uses this principle to show that exponentials cannot appear and logarithms must only appear linearly in the expression of the integral. To get a feeling of Liouville's techniques (following closely Lützen's exposition [Lüt90, p. 388]), assume $n = 1$ and

$$\int y \mathrm{d}x = \phi(x, \theta),$$

where y is algebraic, ϕ is an elementary function of order n and $\theta = \log u$ is a logarithm of order n. Differentiating, we have

$$y = \frac{\partial\phi(x,\theta)}{\partial x} + \frac{\partial\phi(x,\theta)}{\partial\theta}\left(\frac{\mathrm{d}u}{\mathrm{d}x}/u\right).$$

From Liouville's principle, this expression remains unchanged when we replace θ with $\theta + c$ and equating the two expressions for y, we get

$$\frac{\partial\phi(x,\theta+c)}{\partial x} + \frac{\partial\phi(x,\theta+c)}{\partial\theta}\left(\frac{\mathrm{d}u}{\mathrm{d}x}/u\right) = \frac{\partial\phi(x,\theta)}{\partial x} + \frac{\partial\phi(x,\theta)}{\partial\theta}\left(\frac{\mathrm{d}u}{\mathrm{d}x}/u\right).$$

Therefore

$$\phi(x, \theta + c) = \phi(x, \theta) + \text{constant}$$

so

$$\frac{\partial\phi(x,\theta)}{\partial\theta} = A,$$

where A is a constant. Liouville's principle implies we can replace θ with a variable ζ in this equation. Doing this and solving the resulting differential equation we have

$$\phi(x, \zeta) = A\zeta + \phi(x, \zeta_0) - A\zeta_0,$$

where ζ_0 is some arbitrary value of ζ. We then have

$$\int y \mathrm{d}x = A\theta + \phi(x, \zeta_0) - A\zeta_0,$$

so $\theta = \log u$ appears linearly. Arguing analogously, Liouville showed that for $n = 1$ and $\theta = \mathrm{e}^u$, ϕ will be independent of θ. When $n > 1$, an induction and similar use of Liouville's principle yields the result.

In 1835, Liouville published the general version of his theorem, allowing y to be an algebraic function of $u_1(x), \ldots, u_n(x)$, where each u_i satisfies an equation of the form $\frac{\mathrm{d}u_i}{\mathrm{d}x} = p_i(u_1, \ldots, u_n)$, p_i an algebraic function (in modern terms, y belongs to a differential of finite transcendence degree over the constants; a precursor of

Ostrowski's approach [Ost46b]). In this paper he also proved the result concerning the elementary integrability of $f e^g$ proven in Rosenlicht's paper. In 1837 he published a paper continuing his research into the structure of elementary functions. He was able to show that e^x cannot be expressed in terms of logarithms and algebraic functions and $\log x$ cannot be expressed in terms of exponential and algebraic functions and that Kepler's equation $x = y - a \sin y$, $a \in \mathbb{C}$, has no elementary solution y (a modern proof is given in [Ros69]). Furthermore, Lützen cites some unpublished 1840 notes where Liouville claims to show that if the integral of an algebraic function is among a set of functions that are defined implicitly by elementary functions of several variables, then it must be elementary. Lützen outlines Liouville's argument but concludes that it does not represent a proof in the modern sense. Such a result has been proven by Risch [Ris76] and in a slightly weaker form in [PS83].

Liouville also considered solving differential equations in terms of (what we now call) *Liouvillian functions*, that is, functions expressible in terms of exponentials, arbitrary integrals and algebraic functions (cf. [Lüt90, pp. 401–411]). His main result is that if a second-order linear differential equation $y'' - Py = 0$, P a polynomial, has a Liouvillian solution then the associated Riccati equation $u' + u^2 = P$ has an algebraic solution. He was able to use this to show for which parameters the Bessel equation has Liouvillian solutions. Questions of this nature can now be handled using differential Galois theory [vdPS03] and I will not describe this further.

1.3 The Russian School

As already noted, Chebyshev and Zolotarev considered the question of elementary integrability of elliptic integrals and this, together with work of Ostrogradsky and Dolbnia, is discussed in [Lüt90, pp. 414–418].

Mordukhai-Boltovskoi produced a significant body of work around the problem of integration in finite terms. Most of his papers are not easily available. The book [MB10] is available online and the library at the Research Institute for Symbolic Computation, Johannes Kepler University Linz has photocopies (due to Robert Risch) of several of Mordukhai-Boltovskoi's articles. In addition [MB09] is a translation of a much cited article and [DLS98] also mentions the work of Mordukhai-Boltovskoi and related works. Regrettably, I do not read Russian and so most of what I know of this author is what is explained in [Rit48].

In [MB13] (cf. [Rit48, p. 52]) Mordukhai-Boltovskoi describes methods for deciding if elementary functions have elementary integrals, although no general algorithm is given. In [MB10] (cf. [Rit48, p. 76]) Mordukhai-Boltovskoi describes methods for solving linear differential equations in terms of Liouvillian functions. These are only briefly mentioned in [Rit48] but Ritt does go into greater detail concerning two other works of Mordukhai-Boltovskoi in [Rit48, Ch. VII].

Ritt first presents a result of [MB09] concerning elementary first integrals of first-order differential equations $y' = f(x, y)$, f an algebraic function. Mordukhai-Boltovskoi showed, using analytic techniques in the spirit of Liouville, that if there exists an elementary function $g(x, y)$ of two variables such that $g(x, y)$ is constant on solutions of this equation (i.e., g is a first integral) then there is a first integral of the form $\phi_0(x,y)+\sum_{i=1}^m c_i \log\phi_i(x,y)$, where the c_i and ϕ_i are algebraic functions of two variables. This result is given a modern, purely algebraic proof in [PS83]. A result dealing with Liouvillian first integrals appears in [Sin92]. There is a very large literature concerning elementary and Liouvillian first integrals and this deserves its own survey, but that will not be given here.

The second result of Mordukhai-Boltovskoi presented by Ritt is from [MB37] and concerns explicit elementary solutions of first-order differential equations $F(x,y,y')=0$, F a polynomial. Mordukhai-Boltovskoi showed, again using analytic techniques, that if such a differential equation has a nonalgebraic elementary solution, then it has a one-parameter family of solutions of the form

$$y=G(x,\sum_{i=1}^m c_i\log(\phi_i(x))+c) \text{ or } y=G(x,\mathrm{e}^{\phi_0(x)+\sum_{i=1}^m c_i\log(\phi_i(x))+c}),$$

where G is an algebraic function of two variables, c is the parameter, c_i are fixed constants and the ϕ_i are algebraic functions. This follows from a result characterizing differential subfields of elementary extensions presented and proven algebraically in [Sin75],[RS77] and [Ris79] (a result anticipating these results can be found in [Koe87]). A result characterizing differential subfields of Liouvillian extensions is given in [Sri20] (see also [Sri17, Sri18]).

1.4 Hardy

Hardy's book [Har71, pp. 38–44] contains a proof of Abel's Theorem on the algebraic integrability of algebraic functions. In addition, he describes techniques for integrating various kinds of elementary functions and outlines a method to determine if the integral of an algebraic function is algebraic. A large part of the book is devoted to integrals of algebraic functions and how the theory of algebraic curves informs this. Hardy also gives an exposition of Liouville's theorem (without proof) and some of its consequences. He also mentions two problems [Har71, p. 7]: *(i) if f(x) is an elementary function, how can one determine whether its integral is also an elementary function?* and *(ii) if the integral is an elementary function, how can we find it?*. He then makes the regrettable prognostication *Complete answers to these questions have not and probably never will be given.*

1.5 Ostrowski

In [Ost46b], Ostrowski introduced what he called a *corps liouvillien* (and what is now called a differential field) into the study of integration in finite terms. He showed that Liouville's Theorem holds for *functions* $f(x)$ lying in such a field and $\int f(x)\mathrm{d}x$ lying in an elementary extension of this field. Outside of this formalism, Ostrowski's approach is still analytic and follows Liouville's general approach, making several simplifications and clarifications. Ostrowski also published [Ost46a] in which he showed if the integrals of elements of a differential field are algebraically dependent over this field then a constant linear combination of these will lie in the field. This, together with a statement concerning algebraic dependence among exponentials, is now known as the Kolchin–Ostrowski Theorem [Kol68] (cf. [Ax71, Theorem 4], [Ros76, Corollary])).

1.6 Ritt

In [Rit48], Ritt describes the work of Abel, Liouville, Mordukhai-Boltovskoi and his own contributions. A main feature of his presentation is his emphasis on dealing with the function-theoretic properties of elementary functions. As he says in his introduction.

> There are, however, certain questions connected with the many-valued character of the elementary functions which could be pressed back behind the symbols in Liouville's time but which have since learned to assert their rights. Such matters are mulled over in the first chapter. The mulling is inescapable. It might be great fun to talk just as if the elementary functions were one-valued. I might even sound convincing to some readers; I certainly could not fool the functions. [Rit48, p. vi]

The first three chapters of [Rit48] focus on giving rigorous foundations to the functional properties of elementary function and presenting Liouville's Theorem taking this (and Ostrowski's work) into account. Askold Khovanskii discusses some of these issues and also generalizes this approach in his appreciation of Ritt and Ritt's book in his article in this volume.

In Chapter IV, Ritt shows that Kepler's Equation (mentioned above) has no elementary solution and briefly alludes to two of his papers, [Rit25] and [Rit29]. In [Rit25], Ritt shows: *if a function $F(z)$ and its inverse are both elementary, then there exist n functions $\phi_1(z),\dots,\phi_n(z)$ where each $\phi_i(z)$ with odd index is algebraic, and each $\phi_i(z)$ with even index is either* e^x *or* $\log x$ *such that $F(x) = \phi_n\phi_{n-1}\dots\phi_2\phi_1(x)$ each $\phi_i(z)$, $(i < n)$, being substituted for z in $\phi_{i+1}(z)$*. To prove this, Ritt gives a careful analysis (very much in the spirit of Liouville) as to how the order of a composite of two elementary functions depends on the orders of each of these functions. A modern algebraic proof of this result is given in [Ris79].

The paper [Rit29] briefly alluded to in [Rit48, p. 59] describes the interaction of analytic and algebraic properties of finite exponential sums $c_1\mathrm{e}^{a_1x}+\dots+c_r\mathrm{e}^{a_rx}$, where the c_i and the a_i are complex numbers. Using properties of Dirichlet series, Ritt shows that if w is a solution of a polynomial equation $a_nw^n+\dots+a_1w+a_0=0$, where the a_i are finite exponential sums and w is analytic in a sector of angle greater than π, then w is a finite exponential sum. He also shows that if the quotient of two finite exponential sums is analytic in such a sector, then this quotient equals a finite exponential sum. In an earlier paper [Rit27a], Ritt gave a factorization theorem for finite exponential sums. Finite exponential sums form a $\mathbb{C}$-algebra and one can talk about division, irreducibility, etc in this ring. Ritt showed that any finite exponential sum can be written uniquely as a product of irreducible finite exponential sums and simple finite exponential sums, where these latter objects have the property that their exponents are integer multiples of a fixed complex number. A recent proof and generalization can be found in [EvdP97]. Other properties of finite exponential sums can be found in [HRS89]. Ritt's Factorization Theorem also relates to the model theory of fields with exponentiation (cf. [Mac16, p. 921]).

In Chapter V, Ritt develops properties of fractional power series and uses these to prove, in Chapter VI, Liouville's result that if the equation $y' + y^2 = P(x)$ has a Liouvillian solution then it has an algebraic solution. In particular, if a second-order homogeneous linear differential equation with algebraic function coefficients has a Liouvillian solution, it will have a solution y such that y'/y is algebraic. This result (in fact for a homogeneous linear differential equation of arbitrary order) can now be derived using differential Galois theory. A proof of this and related results using power series techniques (in the guise of valuation theory) can be found in [Ros73] and [Sin76].

In Chapter VII, Ritt presents the results of Mordukhai-Boltovskoi mentioned above and in Chapter VIII he presents the results from his papers [Rit23] and [Rit27b]. In [Rit23], Ritt proves that if the integral of an elementary function satisfies an elementary relation, then the integral is actually elementary. As mentioned above, Risch proved algebraically a generalization of this result in [Ris76]. In [Rit27b], Ritt proved that if a solution of a second-order homogeneous linear differential equation satisfies a Liouvillian relation then all solutions of the linear equation are Liouvillian. This latter result is proven algebraically in [Sin92]. Ritt's proofs of these results depend on analytic considerations and proceed using an induction on the elementary order or Liouvillian order of the functions used in building the elementary or Liouvillian relations.

Author Proof

2 The Fundamental Papers of Rosenlicht and Risch

2.1 Rosenlicht

In [Ros68], Rosenlicht gave the first purely algebraic statement and proof of Liouville's Theorem. The proof relies on little more than a knowledge of partial fraction decomposition and a few simple facts from Galois theory. This is reproduced in [Ros72], the paper presented in this volume, with an even simpler presentation and additional comments. A proof of Abel's Theorem concerning algebraic integrals is embedded in the proof of Liouville's Theorem (see the argument on lines 2–15 on page 969). Using an induction argument, the proof is then reduced to showing that if K is a differential field, $\alpha \in K$ and

$$\alpha = \sum_{i=1}^{n} c_i \frac{u_i'}{u_i} + v', \tag{1}$$

where the c_i are constants and $u_i, v \in K(t)$, t transcendental over K and t an exponential or a logarithm of an element in K, then one can find a similar expression with $u_i, v \in K$. The underlying idea is to compare the partial fraction decompositions of the u_i'/u_i and v' and to understand what cancelations occur in order for the right-hand side of (1) to be an element of K. Rosenlicht explains this clearly and I have nothing to add. Besides proving Liouville's Theorem, Rosenlicht reproves another result of Liouville:

If f and g are algebraic functions, then $\int f e^g dx$ is an elementary function if and only if there is an $a \in \mathbb{C}(x)$ such that $f = a' + ag'$. Using this he can easily show that $\int e^{x^2} dx$ is not elementary. Rosenlicht's other papers concerning elementary functions will be discussed in Section 3.2.

2.2 Risch

Risch published his first contribution to the algorithmic aspects of Liouville's Theorem in [Ris69]. Risch's algorithm will be more fully discussed in Clemens Raab's commentary in this volume but I want to discuss his more theoretical contributions here. In [Ris69], Risch gave a refinement of Liouville's Theorem. In Rosenlicht's version, it was assumed that the constants were algebraically closed and that the elementary tower containing the integral had no new constants. Risch did away with these two restrictions and showed that if the integral of an element α lies in an elementary extension of a differential field k containing α then there are $v \in k$, $c_i \in \overline{C}$ (the algebraic closure of the constants C of k), $v_i \in \overline{C}k$ such that $\alpha = v' + \sum_{i=1}^n c_i v_i'/v_i$ and every automorphism of $\overline{C}k$ over k permutes the terms of the sum. Further results concerning the precise subfield of $\overline{C}$ needed in the above refinement are due to Lazard, Riboo, Rothstein and Trager and are described in [Bro97, Chapter 5.6]. A finer description of the u, v_i that can appear is also given in [Bro07].

The preprint [Ris67] (contemporaneous with the preprint version of [Ris69]) contains a treatment of elementary integrals of real functions. Risch showed that if an element α in a real differential field k has an elementary integral then $\alpha = w_0' + \sum_{i=1}^s c_i w_i'/w_i + \sum_{i=s+1}^n c_i w_i'/(w_i^2 + 1)$, where $w_0 \in k$, the c_i are in the real closure $\tilde{C}$ of the constant field C (with respect to some fixed order on C) of k and $w_1, \ldots, w_n \in \tilde{C}k$, that is, the integral of α can be expressed as an element of k and a sum of logarithms and arctangents of elements of $\tilde{C}k$. Risch also discusses related algorithmic questions.

3 The Aftermath

Two interrelated themes have dominated the topic of expressing integrals in finite terms after the papers of Rosenlicht and Risch: generalizations of Liouville's Theorem and understanding the algebraic relations that can occur among elementary functions and other special functions.

3.1 Generalizations of Liouville's Theorem

A multivariate generalization of Liouville's Theorem was presented by Caviness and Rothstein in [CR75]. Its proof is very much in the spirit of [Ros68]. Another proof of this result was presented by Rosenlicht in [Ros76, Theorem 3] based on his reworking of techniques introduced by Ax [Ax71] (more about this in Section 3.2).

In [Ris76], Risch presented a geometric approach to Liouville's Theorem and in the process showed that if the integral of a function is among a set of functions that are defined implicitly by elementary functions of several variables, then it must be elementary. As mentioned above, this generalizes Liouville's and Ritt's results of a similar nature.

An early generalization of Liouville's Theorem appears in [MZ79] (see also [ND79]). The authors consider the problem of expressing the antiderivative of an elementary function in terms of elementary functions and special functions defined by indefinite integrals, such as the Spence function or error function. The natural conjecture to make is that if an integral can be expressed in this way, the special functions must appear linearly. The authors show that this conjecture is false in general. They also show that if the special functions involved are integrals of elements in the field of definition of the integrand and functional composition is not allowed, then this conjecture is true.

The paper [SSC85] begins to consider what happens when one expresses integrals in terms of compositions of algebraic functions, elementary functions, and special functions. The authors define a class of special functions, the so-called $\mathcal{E}\mathcal{L}$-elementary functions, which include the error function

$$\operatorname{erf}(x) = \frac{2}{\sqrt{\pi}} \int_0^x \mathrm{e}^{-t^2} \mathrm{d}t$$

and the logarithmic integral

$$\operatorname{li}(x) = \int_0^x \frac{1}{\log t} \mathrm{d}t$$

but does not include the dilogarithm

$$\ell_2(x) = -\int_0^x \frac{\log(1-t)}{t} \mathrm{d}t$$

or the exponential integral

$$\operatorname{Ei}(x) = -\int_x^\infty \frac{\mathrm{e}^{-t}}{t} \mathrm{d}t$$

(the form of these functions is slightly different in [SSC85] but differ from these by additive or multiplicative constants and so the results still hold). The authors give a Liouville-type Theorem for integration in terms of these special functions. Rather than state the general result, I will restrict myself to the question of integration in terms of error functions. One then defines an *erf-elementary extension of a differential field* k as a differential field K such that there is a tower $k = K_0 \subset \ldots \subset K_n = E$ where each $K_i = K_{i-1}(\theta_i)$ and θ_i is either algebraic over K_{i-1}, or there exist $u_i, v_i \in K_{i-1}$ such that $\theta_i' = u_i'/u_i$, $\theta_i'/\theta_i = u_i'$, or $\theta_i' = u_i' v_i$ where $v_i'/v_i = -2u_i u_i'$. One says that an element $\alpha \in k$ is *erf-elementary integrable* if there is an element y in an erf-elementary extension of k such that $y' = \alpha$. The main result of [SSC85] is that, assuming the constants C of k are algebraically closed, then such an α has an erf-elementary integral if and only if there are constants $a_i, b_i \in C$, $w_i \in k$ and u_i, v_i algebraic over k such that

$$\alpha = w_0' + \sum_{i=1}^{n} a_i \frac{w_i'}{w_i} + \sum_{i=1}^{m} b_i u_i' v_i,$$

where $v_i'/v_i = -2u_i u_i'$ and u_i^2, v_i^2 and $u_i' v_i$ are in F. In other words, if α is erf-elementary integrable then its integral can be expressed as a constant linear combination of an element in k, logarithms of elements in k and error functions of elements algebraic over k. The authors give an example to show that one cannot strengthen the conclusion to conclude that the u_i, v_i are actually in k. They also give an example to show that such a result guaranteeing that the logarithms appear linearly in the integral does not hold when one integrates in terms of the dilogarithm. They present a procedure to decide if an element in a purely exponential extension of $C(x)$ has an erf-elementary integral and find this if it does. The procedure proceeds by induction on the defining tower of the integrand and an interesting aspect of this is that one must rewrite this tower (if necessary) so that the elements at each stage are selected so as to be independent in a certain way from the previous ones.

A similar result for logarithmic integrals can be deduced from the general result of [SSC85] and Cherry gave in [Che85] a procedure to determine if an element of a transcendental elementary extension of $C(x)$, $x' = 1$, C algebraically closed, can be integrated in terms of logarithmic integrals. In [Che86] the procedure of [SSC85] is extended to apply to a more general class of elementary functions. In [Kno92, Kno93], Knowles gives a procedure dealing with an even further extended class of elementary functions that allows one to decide if they are erf-elementary integrable. Cherry [Che89] also gave a procedure to determine if $f e^g$, $f, g \in C(x)$, can be expressed in terms of a class of special functions called the special incomplete gamma functions. This class of special functions includes the exponential integral, the error function, the sine and cosine integrals, and the Fresnel integrals.

The results of [SSC85] do not apply to integration in terms of the exponential integral $\mathrm{Ei}(x)$ and elementary functions. In [LL02] the authors sharpen the results of [Ros76] and extend the results of [SSC85] to include these functions as well as the exponential integrals and special cases of incomplete gamma functions. The conclusion is, as in [SSC85], if an integral can be expressed in terms of these functions, then they will appear in the integral in constant linear combinations composed with algebraic functions. In [Heb15], the author considers integration in terms of elementary functions, exponential integrals and general incomplete gamma functions and derives a similar result together with more precise structural information aimed at efficient algorithms and implementations which have been included in the CAS computer algebra system (see also [KS19]). In [Heb21], the author extends Liouville's Theorem to include integration in terms of elementary functions and elliptic integrals.

The results of [SSC85] also do not apply to integration in terms of dilogarithms and elementary functions. This case of dilogarithms was taken up by Baddoura [Bad94, Bad11]. A key element in these results is a characterization of the algebraic identities among the dilogarithm and elementary functions. More formally, a *transcendental-dilogarithmic-elementary-extension* of a differential field k is a field E for which there exists a tower of fields $k = K_0 \subset \ldots \subset K_n$ where for each $i > 0$, $K_i = K_{i-1}(\theta_i)$, θ_i transcendental over K_{i-1} and there exist $u_i, v_i \in K_{i-1}$ such that either $\theta_i' = u_i'/u_i$, $\theta_i' = u_i'\theta_i$, or $\theta_i' = -v_i/u_i$, where $v_i' = -u_i'/(1-u_i)$, that is θ_i is a logarithm, exponential or dilogarithm of an element in K_{i-1}. Baddoura shows that if k is a Liouvillian extension of an algebraically closed field of constants C and $\alpha \in k$ and there exists an element y in a transcendental-dilogarithmic-elementary-extension such that $y' = \alpha$ then there exist $c_0, \ldots, c_m, v_0, \ldots, v_m, w_1, \ldots, w_n \in k$ and constants $d_1, \ldots, d_n \in C$ such that in a suitable extension of k we have

$$\int \alpha = v_0 + \sum_{i=1}^{m} c_i \log(v_i) + \sum_{i=1}^{n} d_i (\ell_2(w_i) + \frac{1}{2}\log(w_i)\log(1-w_i)).$$

Notice that, although the dilogarithms appear linearly, we have introduced products of logarithms into the expression. Reworkings, refinements, and extensions of these results appear in [KS19, KS21] and [Heb18].

The dilogarithm is one of a sequence of functions referred to as polylogarithms $\ell_n(x)$, which are defined inductively for $n > 2$ as

$$\ell_n(x) = \int_0^x \frac{\ell_{n-1}(t)}{t} \mathrm{d}t.$$

In [Bad94], Baddoura considers integration in terms of polylogarithms and elementary functions and made a conjecture concerning the form that such an integral must take. In [Bad11], an argument is presented to verify this conjecture in a special case.

A problem that arises when one deals with integration in terms of nonelementary functions is the following: *Given $y_1, \ldots, y_n$ in a differential field K determine the set of constants $c_1, \ldots, c_n$ such that $c_1y_1 + \cdots + c_ny_n$ has an integral elementary over K*. Risch solved[3] the related question of determining the set of constants $c_1, \ldots, c_n$ such that $c_1y_1 + \cdots + c_ny_n$ has an integral in K, when K is an elementary extension of $C(x)$, C algebraically closed and $x' = 1$. He showed how one can construct a system of linear equations over C such that the c_i satisfy these equations if and only if this expression has an integral in K. In [Bro97, Chapter 7], the author considers a similar question concerning determining if $c_1y_1 + \cdots + c_ny_n$ has an integral in K when K is a transcendental Liouvillian extension of the constants (or even a more general extension), as well as related problems. When K is a purely transcendental Liouvillian extension of an algebraically closed field of constants, Mack ([Mac76]; see also [SSC85, Appendix] and [Raa12, Chapters 3 and 4]) showed that one could effectively find a system of linear equations with constant coefficients such $c_1, \ldots, c_n$ such that $c_1y_1 + \cdots + c_ny_n$ has an integral elementary over K if and only if the c_i satisfy this system This leads to the general question of describing the set of

[3] Setting $f = 0$ in Theorem 1(b) of *On the Integration of Elementary Functions which are Built Up Using Algebraic Operations* in this volume yields this result. When K is a purely transcendental elementary extension of $C(x)$, this already appears in [Ris69].

parameters for which a parameterized expression has an elementary integral. The first case of this question is *Let $f(x,t)$ be an element of an algebraic extension of $C(x,t)$ where C is algebraically closed, $t' = 0$ and $x' = 1$. Describe the set $\{c \in C \mid f(x,c)$ has an integral elementary over $C(x)\}$.* In [Dav81, p. 90] Davenport asserted that if $f(x,t)$ is not generically integrable in elementary terms then there are only finitely many values of $c \in C$ such that $f(x,c)$ has an elementary integral over $C(x)$. Recently, Masser and Zannier [MZ20] (see [Zan14] for earlier results) have shown that this assertion is false (see [Mas17] for an elementary introduction to their results). For example, the function

$$\frac{x}{(x^2 - t^2)\sqrt{x^3 - x}}$$

is not generically integrable but is integrable for infinitely many algebraic values of t. Furthermore, in [MZ20] they can characterize those algebraic functions which satisfy Davenport's assertion. From the paper of Risch and the thesis and commentary of Trager in this volume, one sees that integration of algebraic functions is intimately connected with the question of whether integer multiples of certain divisors on a curve are principal or not, that is, whether points on the associated Jacobian are torsion or not. In [MZ20] the authors study how points on a Jacobian become torsion under specialization in the parameterized case and this allows them to deduce their results on elementary integrability in this case.

Another generalization of Liouville's Theorem appears in [DS86]. One can consider Liouville's Theorem as describing elementary solutions of first-order inhomogeneous linear differential equations of the form $y' = a$. In [DS86, Theorem 2], the authors show that if an n^{th} order linear differential equation $L(y) = b$, with coefficients in a differential field k, has a solution elementary over k, then it has a solution of the form $P(\log u_1, \ldots, \log u_n)$, where P is a polynomial with coefficients algebraic over k whose degree is at most equal to n, and the u_i are algebraic over k. Furthermore, if P_n is the homogeneous term of P of degree n and L has no order zero term, then the coefficients of P_n are constant. References to other papers and other results concerning elementary and Liouvillian solutions of linear differential equations also appear in [DS86].

Finally, as mentioned in Section 1.3, the result of Mordukhai-Boltovskoi on elementary first integrals has given rise to a large field aimed at understanding rational, algebraic, elementary and Liouvillian first integrals, a field too large to be surveyed here.

3.2 Structure Theorems

In algorithmic considerations, it is useful to know when two expressions represent the same object. For elementary functions this question was considered in [Cav70] and the references given in this paper (see also [Eps79]). Key to this is an understanding the algebraic relations that can occur among a set of elementary functions. The structure theorems developed by Epstein, Caviness, Risch, Rosenlicht, and Rothstein address this issue. These results now follow from results of Rosenlicht based on ideas of Ax and I will begin by describing these.

A fundamental conjecture in transcendental number theory is Schanuel's Conjecture: *Let* $\alpha_1,\dots,\alpha_n$ *be* $\mathbb{Q}$*-linearly independent complex numbers. Then* $\mathrm{tr.deg}_{\mathbb{Q}}\mathbb{Q}(\alpha_1,\dots,\alpha_n,\mathrm{e}^{\alpha_1},\dots,\mathrm{e}^{\alpha_n}) \geq n$. This conjecture implies many of the known transcendence facts (the transcendence of π, Lindemann's Theorem, ...). In [Ax71], Ax proved function-theoretic and differential algebraic analogues of this conjecture. Among other results, he gave a new proof of the Kolchin–Ostrowski Theorem [Kol68, Ost46a]: *Let* $k \subset K$ *be differential fields with commuting derivations* $\{D\}_{D\in\Delta}$ *of characteristic* 0 *with the same field of constants* $C = \cap_{D\in\Delta}\mathrm{Ker}D$ *and let* $y_1,\dots,y_n,z_1,\dots,z_n \in K^*$ *satisfy* $Dy_i, Dz_i/z_i \in k$ *for* $1 \leq i \leq n$ *and* $D \in \Delta$*. If the* y_i *are* C*-linearly independent modulo* k *and no non-trivial power product of the* z_i *is in* k*, then* $y_1,\dots,y_n,z_1,\dots,z_n$ *are algebraically independent over* k. Ax's approach to this problem is to linearize the property of algebraic dependence. This is done by considering the module of differentials $\Omega_{K/k}$ (see [Ros76] or [Bro97, Chapter 9.1] for an exposition of this object aimed at its use in these kinds of questions). Ax's proof technique for his general results depends on the fact that if $k \subset K$ are fields then $u_1,\dots,u_m \in K$ are algebraically dependent over k if and only if their differentials $\mathrm{d}u_1,\dots,\mathrm{d}u_m$ are k-linearly dependent in $\Omega_{K/k}$. In addition, following earlier results of Johnson [Joh69a, Joh69b], one can put a differential structure on $\Omega_{K/k}$ to take into account the differential relations among elements in a differential field K. Finally residue calculations allow Ax to restrict the kind of algebraic relations that can occur. The earlier work of Kolchin is based on his differential Galois theory and general criteria for solutions of differential equations to be algebraically dependent. He can additionally deduce a statement about dependence among integrals, exponentials and Weierstrass functions as well as criteria for Bessel functions. Algebraic independence statements for Weierstrass functions, using refinements of Ax's techniques, are also shown in [BK77].

In [Ros76], Rosenlicht gives a simplified presentation of some of the basic results of [Ax71]. He proves results concerning algebraic dependence among elementary and Liouvillian functions. From these he not only deduces the multivariate generalization of Liouville's Theorem mentioned above but can use these new techniques to recover and generalize work from his earlier paper [Ros69]. In this latter paper Rosenlicht uses valuation theory and the associated power series techniques to show that if k is a differential field and $y_1,\dots,y_n,z_1,\dots,z_n$ are elements of a Liouvillian extension of k such that $y_i'/y_i = z_i', i = 1,\dots,n$ and that $k(y_1,\dots,y_n,z_1,\dots,z_n)$ is algebraic over each of $k(y_1,\dots,y_n)$ and $k(z_1,\dots,z_n)$ then $y_1,\dots,y_n,z_1,\dots,z_n$ are all algebraic over k. Using this he can show that $y = \mathrm{e}^{y/x}$ and Kepler's equation have no Liouvillian solutions. Rosenlicht's results were also used in [BCDJ08] to show that the Lambert W-function is not Liouvillian.

The results of [Ros76] can be used to prove the structure theorems of Epstein–Caviness [EC79], Risch [Ris79] and Rothstein–Caviness [RC79]. These structure theorems describe the relations that occur between an exponential or integral in an elementary or Liouvillian extension and the elements used to build the tower defining such an extension. For example, the Risch Structure Theorem is formulated in the following way. Let K be an elementary extension of $k = C(x)$, $x' = 1$, with the same field of constants C. We may write $K = k(t_1, \ldots, t_n)$ where, for each i, t_i is algebraic over $K_{i-1} = k(t_1, \ldots, t_{i-1})$, or there exists a $u_i \in K_{i-1}$ such that $t_i' = u_i' t_i$ or $t_i' = u_i'/u_i$. Let $E_{K/k} = \{i \in \{1, \ldots, n\} \mid t_i$ transcendental over K_{i-1} and $t_i' = u_i' t_i, u_i \in K_{i-1}\}$ and $L_{K/k} = \{i \in \{1, \ldots, n\} \mid t_i$ transcendental over K_{i-1} and $t_i' = u_i'/u_i, u_i \in K_{i-1}\}$. The Risch Structure Theorem states that for $u, v \in K$, if $v' = u'/u$ then there are $r_i \in \mathbb{Q}$ such that

$$v + \sum_{i \in L_{K/k}} r_i t_i + \sum_{i \in E_{K/k}} r_i u_i \in C,$$

where $t_i' = u_i' t_i$ for $i \in E_{K/k}$. The results of Epstein–Caviness give a more precise special version of this latter result and the result of Rothstein–Caviness generalizes Risch's result to include Liouvillian extensions. The importance of these results lies in the fact that they lead to algorithms to determine the structure of elementary/Liouvillian towers as one builds these towers. An excellent exposition of the various structure theorems is given in [Bro97, Chapter 9]. Related questions are considered in [CDL18], where the authors define a canonical decomposition of an element in an extension field generated by an integral. This is useful in determining if the integral of an element already lies in this field.

Structure theorems also play a key role in the ongoing work concerning a parallel approach to algorithms determining if an elementary function has an elementary integral. The original algorithm of Risch proceeded in a recursive manner, working down the elementary tower that defines the given elementary function. Let us restrict to functions defined by elementary towers of the form $K = C(x, \theta_1, \ldots, \theta_n)$, where $x' = 1$ and, for each i, θ_i is transcendental over $K_{i-1} = C(x, \theta_1, \ldots, \theta_{i-1})$ and either $\theta_i' = u_i'/u_i$ or $\theta_i' = u_i'\theta_i$ for some $u_i \in K_{i-1}$. The parallel approaches to determining if an element $y \in K$ has an elementary integral proceed by determining, all at once, the rational functions of $x, \theta_1, \ldots, \theta_n$ $v, u_1, \ldots, u_m$ that can occur in the expression

$$y = v' + \sum_{j=1}^{m} c_j \frac{u_j'}{u_j}$$

predicted by Liouville's Theorem. This approach is clearly described in [Bro07] where references to previous work is given. At present the parallel approach still contains heuristic elements but results concerning degree bounds and other information about the possible v and u_i are given in this latter work.

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