Errors in Chapter 3 of
"Galois Theory of Difference Equations"

In Chapter 3 of [PS97], we claim to give a proof of

Theorem 3.1 *Let C be an algebraically closed field of characteristic zero and $k = C(x), \phi(x) = x+1$. Any connected linear algebraic group G defined over k is the difference Galois group of a difference equation $\phi(Y) = AY$ over $k = C(x)$.*

The purported proof of this result depends on

Proposition 3.2. *Consider the difference equation $\phi(Y) = AY$ with $A \in \mathrm{GL}_d(k)$ where $k = C(x)$ and let $G \subset \mathrm{GL}_d(C)$ be the Galois group of $\phi(Y) = AY$. Let T denote the smallest algebraic subgroup of $\mathrm{GL}_d(C)$ such that $A \in T(k)$. Then:*

1. *Let U denote the open subset of $\mathbf{P}^1(C)$ consisting of the elements a with $A(a) \in \mathrm{GL}_d(C)$. Then T is generated as an algebraic subgroup by $\{A(a) \mid a \in U\}$.*
2. *G is (after conjugation) a subgroup of T and $\dim T \leq 1 + \dim G$.*

Ruyong Feng sent me two examples, presented below, showing that Proposition 3.2(2) of this book is incorrect. **This invalidates the proof of Theorem 3.1 presented in Chapter 3.**

Nonetheless, in [Feng21] Feng shows that Theorem 3.1 is true. He does this by noting that Theorem 8.11 of [PS97] (whose proof uses analytic techniques) states that Theorem 3.1 is true when $C = \mathbb{C}$, the complex numbers, and then applying the powerful techniques of his paper which allow one to specialize linear difference equations defined over a field to yield difference equations over a smaller field while preserving the Galois groups.

Finally, before presenting Feng's examples, we note that the alleged proof of Theorem 3.1 in [PS97] proceeds by reducing the general case to the case when $G = H^m$ where H is a simple and simply connected noncommutstive algebraic group over C. The proof of this reduction does not use Proposition 3.2(2) and is correct. The authors use Proposition 3.2(2) in their fallacious proof of this special case. It would be of interest to present a correct purely algebraic proof of this special case.

The following are the two examples as Feng presented them (with small changs in notation to match the notation of [PS97]).

Example 1 *Let $C, k = C(x)$, and $\phi(x) = x + 1$ as above. Consider the difference equation*

$$\phi(Y) = \mathrm{diag}(x+2, x+1, x)Y.$$

Then $(C^*)^3$ is the smallest algebraic subgroup T such that $\mathrm{diag}(x+2, x+1, x) \in T(k)$ for if htere are integers z_0, z_1, z_2 such that $(x+2)^{z_2}(x+1)^{z_1}x^{z_0} = 1$ then $z_0 = z_1 = z_2 = 0$. On the other hand, the Galois group G of the above equation over k is

$$\{(c, c, c)\} \mid c \in C^*\},$$

because the above equation is equivalent to $\phi(Y) = \mathrm{diag}(x, x, x)Y$ under the transformation $\mathrm{diag}(x(x+1), x, 1)$. So G is a subgroup of codimension 2 in T.

In Feng's note to me, he presented the following more detailed investigation of Example 1 by following the proof of Proposition 3.2(2) step-by-step. Let $B_0 = \mathbf{diag}(x(x+1), x, 1)$. Then $Z = B_0 G(k)$ is a minimal τ-invariant Zariski-closed subset of $\mathrm{GL}_3(k)$. Consider the morphism $\pi : (C^*)^3 \to (C^*)^2$ given by

$$(c_1, c_2, c_3) \mapsto \left(\frac{c_1}{c_3}, \frac{c_2}{c_3}\right).$$

The map π is surjective and the kernel of π is G. Hence T/G is isomprhic to $(C^*)^3$. τ operates on T/G in the following way:

$$\tau((b_1, b_2)) = \left(\frac{x\phi(b_1)}{x+2}, \frac{x\phi(b_2)}{x+1}\right).$$

Note that $\pi(B_0) = (x(x+1), x)$ and $\tau(\pi(B_0)) = \pi(B_0)$. Let Y denote the smallest Zariski-closed subset such that $\pi(B_0) \in Y(k)$. One then sees that Y is defined by the equation $t_1 - t_2^2 - t_2 = 0$ where (t_1, t_2) are the coordinates on $(C^*)^2$. Let $W = \pi^{-1}(Y)$. Then W is defined by the equation

$$y_2^2 + y_2 y_3 - y_1 y_3 = 0$$

where (y_1, y_2, y_3) are the coordiantes on T. One sees that $Z \subset W(k)$ but $W(k)$ does not seem to be τ-invariant. Let $B = (1/2, 1, 1)$ and $\overline{W} = WB$. Then $1 \in \overline{W}$ and $\overline{W}$ is defined by the equation

$$y_2^2 + y_2 y_3 - 2y_1 y_3 = 0.$$

One sees that $(x+2, x+1, x) \notin \overline{W}(k)$.

Example 2. *Consider the difference equation $\phi(Y) = AY$ where*

$$A = \begin{pmatrix} 0 & A_1 \\ A_2 & 0 \end{pmatrix}, \quad A_1 = \begin{pmatrix} 0 & -1 \\ 1 & x \end{pmatrix}, \quad A_2 = \begin{pmatrix} 0 & -1 \\ 1 & x^2 \end{pmatrix}.$$

A similar argument as in the proof of Lemma 3.9 of [PS97] implies that the smallest algebraic subgroup T such that $A \in T(k)$ is

$$\left\{ \begin{pmatrix} B_1 & 0 \\ 0 & B_2 \end{pmatrix} \,\middle|\, B_i \in \mathrm{SL}_2(C) \right\} \bigcup \left\{ \begin{pmatrix} 0 & B_1 \\ B_2 & 0 \end{pmatrix} \,\middle|\, B_i \in \mathrm{SL}_2(C) \right\}.$$

Set $Z = \mathbf{diag}(A_2, 1)Y$. Then

$$\phi(Z) = \begin{pmatrix} 0 & \phi(A_2)A_1 \\ I_2 & 0 \end{pmatrix} Z \quad \text{and} \quad \phi^2(Z) = \begin{pmatrix} \phi^2(A_2)\phi(A_1) & 0 \\ 0 & \phi(A_2)A_1 \end{pmatrix} Z.$$

Note that the Galois group of the second equation is the Galois group of $\phi^2(Y) = \phi(A_2)A_1 Y$ which is of dimension at most 4. Hence the Galois group of $\phi(Y) = AY$ is of dimension at most 4 while $\mathbf{dim}(T) = 6$.

References

[Feng21] R. Feng *Difference Galois Groups Under Specialization*. Transactions of the American Mathematical Society, Vol.374, No. 1, January 2021, pp. 61-96.

[PS97] M. van der Put, M.F. Singer *Galois Theory of Difference Equations,* Lecture Notes in Mathematics, 1666 Springer-Verlag, 1997