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Linear algebraic groups as parameterized Picard–Vessiot Galois groups

Michael F. Singer [1]

Department of Mathematics, North Carolina State University, Box 8205, Raleigh, NC 27695-8205, United States

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ABSTRACT

We show that a linear algebraic group is the Galois group of a parameterized Picard–Vessiot extension of $k(x)$, $x' = 1$, for certain differential fields k, if and only if its identity component has no one-dimensional quotient as a linear algebraic group.

1. Introduction

In the usual Galois theory of polynomial equations, one starts with a polynomial having coefficients in a field[2] k, forms a splitting field K of this polynomial and then defines the Galois group of this equation to be the group of field automorphisms of K that leave k element-wise fixed. A natural inverse question then arises: *Given the field k, which groups can occur as Galois groups*. For example, if $k = C(x), C$ an algebraically closed field and x transcendental over C, any finite group occurs as a Galois group [27, Corollary 7.10]. In the Galois theory of linear differential equations, one starts with a homogeneous linear differential equation with coefficients in a differential field k with algebraically closed constants C, forms a Picard–Vessiot extension K (the analogue of a splitting field)

E-mail address: singer@math.ncsu.edu.

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and defines the Galois group of the linear differential equation to be the differential automorphisms of K that leave k element-wise fixed. This Galois group is a linear algebraic group defined over C and one can again seek to determine which groups occur as the Galois group of a homogeneous linear differential equation over a given differential field. For example, if $k = C(x)$, C an algebraically closed field, $x' = 1$ and $c' = 0$ for all $c \in C$, then any linear algebraic group occurs as a Galois group of a Picard–Vessiot extension of k ([9,10] for proofs of this as well as references to earlier work). Besides putting the Picard–Vessiot theory on a firm modern footing, Kolchin developed a generalization of Picard–Vessiot extensions called *strongly normal extensions* and developed a Galois theory for these fields (see [12] for an exposition and references to the original articles and [16] for a reworking of this theory in terms of differential schemes). The Galois groups of these extensions can be arbitrary algebraic groups. Kovacic [14,15] studied the general inverse problem in the context of strongly normal extensions and showed that this problem can be reduced to the inverse problem for linear algebraic groups and for abelian varieties. If $k = C(x)$ as above, Kovacic showed that any abelian variety can be realized and, combining this with the solution for linear algebraic groups described above, one sees that any algebraic group defined over C can be realized as a Galois group of a strongly normal extension of $C(x)$ (Kovacic also solved the inverse problem for connected solvable linear algebraic groups and laid out a general plan for attacking the inverse problem for linear groups over arbitrary fields).

In [17], Landesman developed a new Galois theory generalizing Kolchin's theory of strongly normal extension to include, for example, certain differential equations that contain parameters. The Galois groups appearing here are differential algebraic groups (as in [13]). A special case was developed in [4] where the authors consider parameterized *linear* differential equations and discuss various properties of the associated Galois groups, named parameterized Picard–Vessiot groups or PPV-groups for short. These latter groups are linear differential algebraic groups in the sense of Cassidy [2], that is, groups of matrices whose entries belong to a differential field and satisfy a fixed set of differential equations. The inverse problem in these theories is not well understood. Landesman showed that any connected differential algebraic group is a Galois group in his theory over *some* differential field that may depend on the given differential algebraic group [17, Theorem 3.66]. The analogue of the field $C(x)$ mentioned above is a field $k = k_0(x)$ with commuting derivations $\Delta = \{\partial_x, \partial_1, \dots, \partial_m\}$, $m \geqslant 1$, where k_0 is a differentially closed (see the definition below) $\Pi = \{\partial_1, \dots, \partial_m\}$-differential field, x is transcendental over k_0, $\partial_i(x) = 0$ for $i = 1, \dots, m$ and ∂_x is defined on k by setting $\partial_x(a) = 0$ for all $a \in k_0$ and $\partial_x(x) = 1$. In what follows, the symbol k will always be used refer to a differential field of this type. It is not known, in general, which differential algebraic groups appear as Galois groups in Landesman's theory over this field. In [17] and [4], it is shown that the additive group $\mathbb{G}_a(k_0)$ cannot appear while any proper subgroup of these groups does appear as a Galois group (the same situation for $\mathbb{G}_m(k_0)$ is also described in [17]).[3] More recently, the results of [21] and [6] give necessary and sufficient conditions in *topological terms* for a linear differential algebraic group to be a PPV-group of a PPV-extension of $k_0(x)$ for certain k_0 (see below for a precise statement of their results).

A goal of this paper, is to make progress in finding purely *algebraic* necessary and sufficient conditions. In the following, we give an algebraic characterization of those linear *algebraic* groups, considered as linear *differential* algebraic groups, that can occur as PPV-groups of PPV-extensions of k (under suitable hypotheses concerning k). Before we state the main result of this paper, we will recall some definitions. Although these definitions may be stated in more generality, we will state them relative to the field k defined above.

The parameterized Picard–Vessiot theory (PPV-theory) considers linear differential equations of the form

$$\partial_x Y = AY \tag{1.1}$$

[3] There are other Galois theories of differential equations due primarily to Malgrange [18], Pillay [22,20,23] and Umemura [26]. In particular, inverse problems are addressed in [20]. We will not consider these theories here.

where $A \in \mathrm{gl}_n(k)$. In analogy to classical Galois theory and Picard–Vessiot theory, we consider fields, called *PPV-extensions of k*, that act as "splitting fields" for such equations. A PPV-extension K of k for (1.1) is a Δ-field K such that

1. $K = k\langle Z\rangle$, the Δ-field generated by the entries of a matrix $Z \in \mathrm{gl}_n(K)$ satisfying $\partial_x Z = AZ$, $\det(Z) \neq 0$.
2. $K^{\partial_x} = k^{\partial_x} = k_0$, where for any Δ-extension F of k, $F^{\partial_x} = \{c \in F \mid \partial_x c = 0\}$.

A Π-field E is said to be *differentially closed* (also called *constrainedly closed*, see, for example §9.1 of [4]) if for any n and any set $\{P_1(y_1,\dots,y_n),\dots,P_r(y_1,\dots,y_n), Q(y_1,\dots,y_n)\} \subset E\{y_1,\dots,y_n\}$, the ring of differential polynomials in n variables, if the system

$$\{P_1(y_1,\dots,y_n)=0,\ \dots,\ P_r(y_1,\dots,y_n)=0,\ Q(y_1,\dots,y_n)\neq 0\}$$

has a solution in some differential field F containing E, then it has a solution in E. In [4] (and more generally in [8]), it is shown that under the assumption that k_0 is differentially closed, then PPV-extensions exist and are unique up to Δ-k-isomorphisms. This hypothesis has been weakened to non-differentially closed k_0 in [7] and [29]. In these papers the authors give conditions weaker than differential closure for the existence and uniqueness of PPV-extensions and discuss the corresponding Galois theory. Although some of our results remain valid under these weaker hypotheses, we will assume in this paper that k_0 is Π-differentially closed. The set of field-theoretic automorphisms of K that leave k element-wise fixed and commute with the elements of Δ forms a group G called the *parameterized Picard–Vessiot group* (PPV-group) of (1.1). One can show that for any $\sigma \in G$, there exists a matrix $M_\sigma \in \mathrm{GL}_n(k_0)$ such that $\sigma(Z) = (\sigma(z_{i,j})) = ZM_\sigma$. Note that ∂_x applied to an entry of such an M_σ is 0 since these entries are elements of k_0 but that such an entry need not be constant with respect to the elements of Π. In [4], the authors show that the map $\sigma \mapsto M_\sigma$ is an isomorphism whose image is furthermore a *linear differential algebraic group*, that is, a group of invertible matrices whose entries satisfies some fixed set of polynomial *differential* equations (with respect to the derivations $\Pi = \{\partial_1,\dots,\partial_m\}$) in n^2 variables. We say that a set $X \subset \mathrm{GL}_n(k_0)$ is *Kolchin-closed* if it is the zero set of such a set of polynomial differential equations. One can show that the Kolchin-closed sets form the closed sets of a topology, called the *Kolchin topology* on $\mathrm{GL}_n(k_0)$ (*cf.* [2–4,13]).

As mentioned above, the papers [21] and [6] give necessary [6, Corollary 2.18 and Theorem 3.10] and sufficient [21, Corollary 5.2] conditions for a linear differential algebraic group to be a PPV-group over $k_0(x)$, as above, assuming that k_0 is "sufficiently large". The notion of "sufficiently large" is given in the following definition. A Π-field F is a *Π-universal field* if for any Π-field $E \subset F$, finitely differentially generated over $\mathbb{Q}$, any Π-finitely generated extension of E can be differentially embedded over E into F [12, p. 133]. Note that a universal field is differentially closed. The results of [21] and [6] mentioned above combine to give

Theorem 1.1. *Let $k = k_0(x)$ be as above with k_0 a Π-universal field and G a linear differential algebraic group defined over k_0. The group $G(k_0)$ is a PPV-group over k if and only if $G(k_0)$ is the Kolchin-closure of a finitely generated subgroup.*

In Section 2, we will show

Proposition 1.2. *Let k_0 be a differentially closed field and G be a linear algebraic group defined over k_0. The group $G(k_0)$ contains a Kolchin-dense finitely generated subgroup if and only if the identity component of G has no quotient (as an algebraic group) isomorphic to the additive group $\mathbb{G}_a$ or the multiplicative group $\mathbb{G}_m$.*

Theorem 1.1 and Proposition 1.2 combine to immediately give the result mentioned in the abstract:

Theorem 1.3. *Let $k = k_0(x)$ be as above with k_0 a Π-universal field and let $G(k_0)$ the group of k_0-points of a linear algebraic group G defined over k_0. The group $G(k_0)$ is a PPV-group of a PPV-extension of k if and only* *if the identity component of G has no quotient (as an algebraic group) isomorphic to the additive group $\mathbb{G}_a$ or the multiplicative group $\mathbb{G}_m$.*

Although the proof of Proposition 1.2 is algebraic, the proof of Theorem 1.3 outlined above depends heavily on Theorem 1.1 whose proof is analytic. Nonetheless, part of Theorem 1.3 can also be given an algebraic proof. In Section 3, we give an algebraic proof of the fact that a linear differential algebraic group whose identity component has a quotient isomorphic to the additive group $\mathbb{G}_a$ or the multiplicative group $\mathbb{G}_m$ cannot be a PPV-group over $k_0(x)$. It would be of interest to give an algebraic proof of all of Theorem 1.3.

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2. Linear algebraic groups with finitely generated Kolchin-dense subgroups

The proof of Proposition 1.2 depends on the following four lemmas.

Lemma 2.1. *Let G be a linear algebraic group defined over k_0 and G^0 be its identity component. $G(k_0)$ contains a Kolchin-dense finitely generated group if and only if $G^0(k_0)$ contains a Kolchin-dense finitely generated group.*

Proof. Assume that $G^0(k_0)$ contains a Kolchin-dense group generated by $g_1, \dots, g_s$. Let $\{h_1, \dots, h_t\}$ be a subset of $G(k_0)$ mapping surjectively onto $G(k_0)/G^0(k_0)$. The set $\{g_1, \dots, g_s, h_1, \dots, h_t\}$ generates a group that is Kolchin-dense in $G(k_0)$.

Assume that $G(k_0)$ contains elements $g_1, \dots, g_s$ that generate a Kolchin-dense subgroup. From [28, p. 142] or [1, lemme 5.11, p. 152], one knows that any linear algebraic group $G(k_0)$, k_0 algebraically closed, is of the form $HG^0(k_0)$ where H is a finite subgroup of $G(k_0)$. Therefore we may write each g_i as a product of an element of H and an element of $G^0(k_0)$ and so we may assume that there is a finite set $S = \{\tilde{g}_1, \dots, \tilde{g}_t\} \subset G^0(k_0)$ such that the group generated by S and H is Kolchin-dense in $G(k_0)$. Extending S if necessary, we may assume that S is stable under conjugation by elements of H and therefore that the group generated by S is stable under conjugation by the elements of H. An elementary topological argument shows that the Kolchin-closure G' of the group generated by S is also stable under conjugation by H. Therefore $H \cdot G'$ forms a group. It is a finite union of Kolchin-closed sets, so it is also Kolchin-closed. It contains H and S so it must be all of $G(k_0)$. Finally G' is normal and of finite index in $G(k_0)$ so it must contain $G^0(k_0)$. Clearly $G' \subset G^0(k_0)$ so $G^0(k_0) = G'$ and this shows that $G^0(k_0)$ is finitely generated. $\square$

Lemma 2.2. *Let $P \subset \mathrm{GL}_n$ be a connected semisimple linear algebraic group defined over k_0. Then $P(k_0)$ contains a finitely generated Kolchin-dense subgroup.*

Proof. From Proposition 1 of [25] or Lemma 5.13 of [24], we know that a linear algebraic group contains a Zariski-dense finitely generated subgroup H. We also know that P contains a maximal torus T of positive dimension. After conjugation, we may assume that T is diagonal and that the projection onto the first diagonal entry is a homomorphism of T onto $k_0^* = k_0 \backslash \{0\}$. Since k_0 is differentially closed, the derivations $\partial_1, \dots, \partial_m$ are linearly independent so there exist nonzero elements $x_1, \dots, x_m \in k_0$ such that $\det(\partial_i x_j)_{1 \leqslant i,j \leqslant m} \neq 0$ [12, Theorem 2, p. 96]. For each $i = 1, \dots, m$, let $g_i \in T$ be an element whose first diagonal entry is x_i. Let P' be the Kolchin-closure of the group generated by H and $\{g_1, \dots, g_m\}$. We claim $P' = P$.

To see this note that since P' contains H, P' is Zariski-dense in P. If $P' \neq P$, then results of [3] imply that there exist a nonempty subset $\Sigma \subset k_0\Pi$, the k_0 span of Π, such that P' is conjugate to a group of the form $P''(C)$ where P'' is a semisimple algebraic group defined over $\mathbb{Q}$ and $C = \{c \in k_0 \mid \partial c = 0 \text{ for all } \partial \in \Sigma\}$. This implies that each element of G has eigenvalues in C and so, for each x_i, $\partial(x_i) = 0$ for all $\partial \in \Sigma$. Yet, if $\partial = \sum_{j=1}^m a_j \partial_j$, not all a_j zero and $\partial(x_i) = 0$ for $i = 1, \dots, m$, then $(a_1, \dots, a_m)X = (0, \dots, 0)$ where $X = (\partial_i x_j)_{1 \leqslant i,j \leqslant m}$. This contradicts the fact that $\det X \neq 0$. Therefore $P' = P$. $\square$

Lemma 2.3. *Let $G(k) = P(k_0) \ltimes U(k_0)$ be a connected linear algebraic group where $P(k_0)$ is a semisimple linear algebraic group and $U(k_0)$ is a commutative unipotent group, both defined over k_0. If $G(k_0)$ has no quotient isomorphic to $\mathbb{G}_a(k_0)$, then $G(k_0)$ contains a Kolchin-dense finitely generated subgroup.*

Proof. Note that $U(k_0)$ is isomorphic to k_0^m for some m. Since P acts on U by conjugation, we may write $U = \bigoplus_{i=1}^m U_i$ where each U_i is an irreducible P-module. Furthermore, if the action of P on some U_j is trivial, then this U_j would be of the form $\mathbb{G}_a(k_0)$ and we could write $P \ltimes U = (P \ltimes \oplus_{i\neq j} U_i) \times \mathbb{G}_a(k_0)$. This would imply that there is an algebraic morphism of $G(k_0)$ onto $\mathbb{G}_a(k_0)$, a contradiction. Therefore we may assume the action of P on each U_i is nontrivial. Let B be a Borel subgroup of P. From the representation theory of semisimple algebraic groups [11, Ch. 13.3], we know that each U_i contains a unique B-stable one-dimensional subspace corresponding to a weight $\lambda_i : B \to \mathbb{G}_m(k_0)$ (the highest weight of U_i). For each i, let u_i span this one-dimensional space. We claim that the $P(k_0)$-orbit of u_i generates a group that equals $U_i(k_0)$. Note that since $B(k_0)$ is connected and λ_i is not trivial, we have the $P(k_0)$-orbit of u_i contains $\mathbb{G}_m(k_0)u_i$. Since U_i is an irreducible $P(k_0)$-module, there exist $g_1, \dots, g_s \in P(k)$, such that $g_1 u_i g_1^{-1}, \dots g_s u_i g_s^{-1}$ span U_i. Since $g_j(\mathbb{G}_m(k_0)u_i)g_j^{-1} = \mathbb{G}_m(k_0)(g_j u_i g_j^{-1})$ for $j = 1, \dots, s$, we have that the $P(k_0)$-orbit of u_i generates all of U_i.

Now Lemma 2.2 asserts that there exists a finite set $S \subset P(k_0)$ that generates a Kolchin-dense subgroup of $P(k_0)$. We then have that $S \cup \{u_i\}_{i=1}^m$ generates a Kolchin-dense subgroup of $G(k_0)$. □

Lemma 2.4. *The homomorphism $l\partial_1 : \mathbb{G}_m(k_0) \to \mathbb{G}_a(k_0)$ where $l\partial_1(u) = \partial_1(u)/u$ maps $\mathbb{G}_m(k_0)$ onto $\mathbb{G}_a(k_0)$.*

Proof. Since k_0 is differentially closed, we need only show that for any $u \in k_0$, there is a Π-differential extension F of k_0 such that $\partial_1 y = uy$ has a solution $y \neq 0$ in F. Let $\Pi_1 = \{\partial_2, \dots, \partial_m\}$ and let F be the Π_1-field $k_0\langle v\rangle$, where v is a Π_1-differentially transcendental element. We extend the derivation ∂_1 from k_0 to F by setting $\partial_1 v = uv$, and $\partial_1(\partial_2^{i_2} \dots \partial_m^{i_m} v) = \partial_2^{i_2} \dots \partial_m^{i_m}(\partial_1 v) = \partial_2^{i_2} \dots \partial_m^{i_m}(uv)$. With these definitions, F becomes a Π-differential extension of k_0 and $y = v$ satisfies $\partial_1 y = uy$. □

Proof of Proposition 1.2. Assume that $G(k_0)$ contains a Kolchin-dense finitely generated subgroup. Lemma 2.1 implies that $G^0(k_0)$ also contains a Kolchin-dense finitely generated subgroup. If there is an algebraic morphism of $G^0(k_0)$ onto $\mathbb{G}_m(k_0)$ then Lemma 2.4 implies that there is a differential algebraic morphism of $G^0(k_0)$ onto $\mathbb{G}_a(k_0)$. Therefore we may assume that we have a differential homomorphism of $G^0(k_0)$ onto $\mathbb{G}_a(k_0)$. This implies that $\mathbb{G}_a(k_0)$ would contain a Kolchin-dense subgroup generated by a finite set of elements $\{\alpha_i\}_{i=1}^m$. We will show that any finite set of elements of $\mathbb{G}_a(k_0)$ satisfy a linear differential equation over k_0 and so could not generate a Kolchin-dense subgroup of $\mathbb{G}_a(k_0)$. Let C be the ∂_1-constants of k_0 and $\beta_1, \dots, \beta_s$ a C-basis of the C-span of the α_i's. Let $R(Y) = wr(Y, \beta_1, \dots, \beta_s)$ where wr denotes the Wronskian determinant. $R(Y)$ is a linear differential polynomial yielding the desired $R \in k_0[\partial_1]$. Therefore there is no algebraic morphism of $G^0(k_0)$ onto $\mathbb{G}_m(k_0)$ or $\mathbb{G}_a(k_0)$.

Assume that there is no algebraic morphism of $G^0(k_0)$ onto $\mathbb{G}_m(k_0)$ or $\mathbb{G}_a(k_0)$. Lemma 2.1 implies that it is enough to show that $G^0(k_0)$ contains a Kolchin-dense finitely generated group. We may write $G^0 = P \ltimes R_u$ where P is a Levi subgroup and R_u is the unipotent radical of G [11, Ch. 30.2].

We first claim that P must be semisimple. We may write $P = (P, P)Z(P)$ where (P, P) is the derived subgroup of P and $Z(P)$ is the center of P. Furthermore, $Z(P)^0$ is a torus [11, Ch. 27.5]. We therefore have a composition of surjective morphisms

$$G^0 \to G^0/R_u \simeq P \to P/(P, P) \simeq Z(P)/\big(Z(P) \cap (P, P)\big).$$

Since G^0 is connected, its image lies in the image of $Z^0(P)$ in $Z(P)/(Z(P) \cap (P, P))$ and therefore is a torus. This torus, if not trivial, has a quotient isomorphic to $\mathbb{G}_m$. This would yield a homomorphism of $G^0(k_0)$ onto $\mathbb{G}_m(k_0)$ and, by assumption, this is not possible. Therefore $Z^0(P)$ is trivial. Since G^0 is connected we must have $Z(P) \subset (P, P)$. Therefore $P = (P, P)$ and is therefore semisimple.

We shall now show that it suffices to prove that $G^0(k_0)$ contains a Kolchin-dense finitely generated subgroup under the assumption that R_u is commutative. In [14], Kovacic shows [14, Lemma 2]: *Let G be an abstract group, H a subgroup and N a nilpotent normal subgroup of G. Suppose $H \cdot (N, N) = G$. Then $H = G$.* Therefore, if we can find a Kolchin-dense finitely generated subgroup of the k_0-points of $G^0/(R_u, R_u) \simeq P \ltimes (R_u/(R_u, R_u))$, then the preimage of this group under the homomorphism $G^0 \to G^0/(R_u, R_u)$ generates a Kolchin-dense subgroup of $G(k_0)$.

Therefore, we need only consider connected groups satisfying the hypotheses of Proposition 1.2 and of the form $P(k_0) \ltimes U(k_0)$, where P is semisimple and U is a commutative unipotent group. Lemma 2.3 guarantees that such a group has a finitely generated Kolchin-dense subgroup. □

We note that Theorem 1.1 allows one to show other classes of linear differential algebraic groups are PPV-groups of PPV-extensions of $k_0(x)$. As noted above, if G is a linear algebraic group defined over an algebraically closed field C, then $G(C)$ contains a finitely generated Zariski-dense subgroup. If C is the field of Π-constants of k_0, then the Zariski and Kolchin topologies are the same on $G(C)$. Therefore $G(C)$ will be a PPV-group over $k_0(x)$.

3. Linear differential algebraic groups a PPV-groups

As mentioned above, it would be of interest to give an algebraic proof of Theorem 1.3 and give an equally simple characterization of all linear differential algebraic groups that are PPV-groups over $k_0(x)$. In this section we show, by example, that the necessary condition of Theorem 1.3 insuring that a linear algebraic group is a PPV-group can be proved algebraically and extended to linear *differential* algebraic groups. We will also show that this condition is not sufficient to insure that a linear differential algebraic group is a PPV-group over $k_0(x)$.

Lemma 3.1. *If $G^0(k_0)$ has $\mathbb{G}_m(k_0)$ or $\mathbb{G}_a(k_0)$ as a homomorphic image (under a differential algebraic homomorphism) and $G(k_0)$ is a PPV-group of a PPV-extension of $k_0(x)$, then $\mathbb{G}_a(k_0)$ is a PPV-group of a PPV-extension of a finite algebraic extension E of $k_0(x)$.*

Proof. We will show that this result follows from the Galois theory of parameterized linear differential equations [17,4]. Let K be a PPV-extension of $k_0(x)$ having G as its PPV-group. The fixed field E of G^0 is a finite algebraic extension of $k_0(x)$. If G^0 has $\mathbb{G}_m(k_0)$ as a homomorphic image under a differential homomorphism then composing this homomorphism with $l\partial_1 : \mathbb{G}_m(k_0) \to \mathbb{G}_a(k_0)$ where $l\partial_1(u) = \partial_1(u)/u$, Lemma 2.4 implies that $\mathbb{G}_a(k_0)$ would also be a homomorphic image of $G^0(k_0)$ under a differential homomorphism. Therefore we shall only deal with this latter case. Let $\phi : G^0(k_0) \to \mathbb{G}_a(k_0)$ be a surjective differential algebraic homomorphism and let H be its kernel. The Galois theory [4, Theorem 9.5] implies that the fixed field of H is a PPV-extension F of E whose PPV-group over E is differentially isomorphic to $\mathbb{G}_a(k_0)$. □

The following lemma is the key to showing that $\mathbb{G}_a(k_0)$ is not a PPV-group over a finite algebraic extension of $k_0(x)$.

Lemma 3.2. *Let E be a finite algebraic extension of $k_0(x)$ and $f \in E$. Let K be the PPV-extension of $k_0(x)$ corresponding to the equation*

$$\partial_x y = f.$$

Let $z \in K$ satisfy $\partial_x z = f$. Then there exists a nonzero linear differential operator $L \in k_0[\partial_1]$ and an element $g \in E$ such that

$$L(z) = g.$$

Proof. The proof of this lemma is a slight modification of Manin's construction of the Picard–Fuchs equations (see Section 3, pp. 64–65 of the English translation of [19]). We shall use (as does Manin) ideas and results that appear in [5]. In Ch. VI, §7 of [5], Chevalley shows that ∂_1 can be used to define a map D on differentials of E satisfying $D(y\,dx) = (\partial_1 y)\,dx$. Furthermore, Theorem 13, Ch. VI, §7 of [5] states that for any differential ω and any place P of E, we have $\mathrm{res}_P\, D(\omega) = \partial_1(\mathrm{res}_P\, \omega)$ (where res_P denotes the residue at P). Let $\alpha_1, \dots, \alpha_m$ be the nonzero residues of $f\,dx$. As in the proof of Proposition 1.2, one can show that there is a nonzero linear differential operator $R \in k[\partial_1]$ such that $R(\alpha_i) = 0$, $i = 1, \dots, m$. We then have that for any place P, $\mathrm{res}_P(R(f)\,dx) = R(\mathrm{res}_P(f\,dx)) = 0$. Therefore $R(f)\,dx$ has residue 0 at all places, that is, it is a differential of the second kind. Note that $\partial_1^i(R(f))\,dx$ is also a differential of the second kind for any $i \geqslant 1$. The factor space of differentials of the second kind by the space of exact differentials has dimension $2g$ over k, where g is the genus of E [5, Corollary 1, Ch. VI, §8]. Therefore there exist $v_{2g}, \dots, v_0 \in k_0$ such that

$$v_{2g}\partial_1^{2g}\big(R(f)\big)\,dx + \cdots + v_0 R(f)\,dx = d\tilde{g} = \partial_x \tilde{g}\,dx$$

for some $\tilde{g} \in E$. This implies that there exists a linear differential operator $L \in k_0[\partial_1]$ such that

$$L(f) = \partial_x \tilde{g}.$$

Furthermore, $\partial_x(L(z)) = L(\partial_x z) = L(f) = \partial_x \tilde{g}$. Therefore $L(z) = g$ where $g = \tilde{g} + c$ for some $c \in k_0$. □

Proposition 3.3. *If G is a linear differential algebraic group defined over k_0 such that $G^0(k_0)$ has $\mathbb{G}_m(k_0)$ or $\mathbb{G}_a(k_0)$ as a quotient (as a linear differential group), then $G(k_0)$ cannot be a PPV-group of a PPV-extension of $k_0(x)$.*

Proof. Assume that $G(k_0)$ is a PPV-group of a PPV-extension of $k_0(x)$. Lemma 3.1 implies that, in this case, $\mathbb{G}_a(k_0)$ is a PPV-group of a PPV-extension K of E, where E is a finite algebraic extension of $k_0(x)$. From Proposition 9.12 of [4], K is the function field of a $\mathbb{G}_a(k_0)$-principal homogeneous space. The corollary to Theorem 4 of [13, Ch. VII, §3] implies that this principal homogeneous space is the trivial principal homogeneous space and so $K = E\langle z\rangle$ where for any $\sigma \in \mathbb{G}_a(k_0)$ there exists a $c_\sigma \in k_0$ such that $\sigma(z) = z + c_\sigma$. In particular, $\sigma(\partial_x z) = \partial_x z$ for all $\sigma \in \mathbb{G}_a(k_0)$ and so $\partial_x z = f \in E$. Lemma 3.2 implies that there exists a linear differential operator $L \in k_0[\partial_1]$ and an element $g \in E$ such that $L(z) = g$. For any $\sigma \in \mathbb{G}_a(k_0)$, we have $g = \sigma(g) = \sigma(L(z)) = L(\sigma(z)) = L(z + c_\sigma) = g + L(c_\sigma)$ so $L(c_\sigma) = 0$. This implies that the PPV-group of K over E is a proper subgroup of $\mathbb{G}_a(k_0)$, a contradiction. □

We shall now show that the necessary conditions of Proposition 3.3 are not sufficient, in general, for guaranteeing that a linear *differential* algebraic group is a PPV-group of a PPV-extension of $k_0(x)$.

Let k_0 be an ordinary differentially closed field with derivation ∂_1 and let

$$G = \left\{ \begin{pmatrix} 1 & 0 \\ a & b \end{pmatrix} \,\middle|\, a, b \in k_0,\ b \neq 0,\ \partial_1 b = 0 \right\} \simeq G_1 \rtimes G_2$$

where

$$G_1 = \left\{ \begin{pmatrix} 1 & 0 \\ a & 1 \end{pmatrix} \,\middle|\, a \in k_0 \right\} \simeq \mathbb{G}_a(k_0)$$

$$G_2 = \left\{ \begin{pmatrix} 1 & 0 \\ 0 & b \end{pmatrix} \,\middle|\, b \in k_0,\ b \neq 0,\ \partial_1 b = 0 \right\} \simeq \mathbb{G}_m(C)$$

where $C = \{c \in k_0 \mid \partial_1 c = 0\}$. Let $k = k_0(x)$ be a $\Delta = \{\partial_x, \partial_1\}$-field as in the introduction. We shall show that $G(k_0)$ contains no Kolchin-dense finitely generated subgroup, G is Kolchin-connected, and there is no surjective differential algebraic homomorphism of $G(k_0)$ onto $\mathbb{G}_a(k_0)$ or $\mathbb{G}_m(k_0)$. From the first property, Theorem 1.1 implies that $G(k_0)$ cannot be a PPV-group of a PPV-extension of $k_0(x)$.

To see that $G(k_0)$ contains no Kolchin-dense finitely generated subgroup, note that any element of $G(k_0)$ can be written as a product of an element of $\mathbb{G}_a(k_0)$ and $\mathbb{G}_m(C)$. Therefore it is enough to show that any set of elements of the form

$$\begin{pmatrix} 1 & 0 \\ a_1 & 1 \end{pmatrix}, \dots, \begin{pmatrix} 1 & 0 \\ a_n & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & b_1 \end{pmatrix}, \dots, \begin{pmatrix} 1 & 0 \\ 0 & b_m \end{pmatrix}$$

with the $a_i \in k_0$ and the $b_i \in C$ do not generate a Kolchin-dense subgroup of G. Let H be the group generated by these elements and let $L \in k[\partial_1]$ be a nonzero differential operator such that $L(a_i) = 0$ for all $i = 1, \dots, n$. A calculation shows that any element of H is of the form

$$\begin{pmatrix} 1 & 0 \\ c_1 a_1 + \dots + c_n a_n & b \end{pmatrix}$$

with b and the c_i in C. Therefore H is a subgroup of

$$\left\{ \begin{pmatrix} 1 & 0 \\ a & b \end{pmatrix} \,\middle|\, L(a) = 0,\ \partial_1 b = 0,\ b \neq 0 \right\}$$

which is a proper Kolchin-closed subgroup of G.

To see that G is Kolchin-connected, note that G is the product of Kolchin-irreducible Kolchin-closed sets and so must be irreducible.

We now show the last claimed property of G. Since $\mathbb{G}_a(k_0)$ is a differential homomorphic image of $\mathbb{G}_m(k_0)$, it suffices to show that there is no surjective differential algebraic homomorphism of $G(k_0)$ onto $\mathbb{G}_a(k_0)$. Assume not and let $\phi : G(k_0) \to \mathbb{G}_a(k_0)$. Restricting ϕ to G_2 yields an *algebraic* homomorphism of $\mathbb{G}_m(C)$ into $\mathbb{G}_a(k_0)$. Since algebraic homomorphisms preserve the property of being semisimple, we must have that $G_2 \subset \ker\phi$. Therefore for any $a \in k$ and any $b \in C^*$, we have

$$\begin{pmatrix} 1 & 0 \\ a & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & b \end{pmatrix} \begin{pmatrix} 1 & 0 \\ -a & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ a - ba & b \end{pmatrix} \in \ker\phi.$$

For any $\tilde{a} \in k_0$ and $1 \neq b \in C$ there exists an $a \in k_0$ such that $a - ba = \tilde{a}$, so $\ker\phi$ contains all elements of the form

$$\begin{pmatrix} 1 & 0 \\ \tilde{a} & b \end{pmatrix}$$

$a \in k_0$, $1 \neq b \in C$. Since $G_2 \subset \ker\phi$ as well, we have that $G \subset \ker\phi$, a contradiction.

References

[1] A. Borel, J.-P. Serre, Théorèmes de finitude en cohomologie galoisienne, Comment. Math. Helv. 39 (1964–1965) 111–164.
[2] P.J. Cassidy, Differential algebraic groups, Amer. J. Math. 94 (1972) 891–954.
[3] P.J. Cassidy, The classification of the semisimple differential algebraic groups and the linear semisimple differential algebraic Lie algebras, J. Algebra 121 (1) (1989) 169–238.
[4] P.J. Cassidy, M.F. Singer, Galois theory of parameterized differential equations and linear differential algebraic groups, in: D. Bertrand, et al. (Eds.), Differential Equations and Quantum Groups, in: IRMA Lect. Math. Theor. Phys., vol. 9, 2006, pp. 113–157.
[5] C. Chevalley, Introduction to the Theory of Algebraic Functions of One Variable, Math. Surveys, vol. 6, American Mathematical Society, New York, 1951.
[6] T. Dreyfus, A density theorem for parameterized differential Galois theory, preprint, arXiv:1203.2904.
[7] H. Gillet, S. Gorchinskiy, A. Ovchinnikov, Parameterized Picard–Vessiot extensions and Atiyah extensions, preprint, arXiv:1110.3526.

[8] C. Hardouin, M.F. Singer, Differential Galois theory of linear difference equations, Math. Ann. 342 (2) (2008) 333–377.
[9] J. Hartmann, On the inverse problem in differential Galois theory, J. Reine Angew. Math. 586 (2005) 21–44.
[10] J. Hartmann, Patching and differential Galois groups (joint work with David Harbater), in: Arithmetic and Differential Galois Groups, Oberwolfach Rep. 4 (2) (2007) 1490–1493.
[11] J.E. Humphreys, Linear Algebraic Groups, Grad. Texts in Math., Springer-Verlag, New York, 1975.
[12] E.R. Kolchin, Differential Algebra and Algebraic Groups, Academic Press, New York, 1976.
[13] E.R. Kolchin, Differential Algebraic Groups, Academic Press, Orlando, 1985.
[14] J. Kovacic, The inverse problem in the Galois theory of differential fields, Ann. of Math. 89 (1969) 583–608.
[15] J. Kovacic, On the inverse problem in the Galois theory of differential fields, Ann. of Math. 93 (1971) 269–284.
[16] J. Kovacic, The differential Galois theory of strongly normal extensions, Trans. Amer. Math. Soc. 355 (11) (2003) 4475–4522.
[17] P. Landesman, Generalized differential Galois theory, Trans. Amer. Math. Soc. 360 (8) (2008) 4441–4495.
[18] B. Malgrange, On nonlinear differential Galois theory, in: Frontiers in Mathematical Analysis and Numerical Methods, World Sci. Publ., River Edge, NJ, 2004, pp. 185–196.
[19] Ju.I. Manin, Algebraic curves over fields with differentiation, Izv. Akad. Nauk SSSR Ser. Mat. 22 (1958) 737–756, an English translation appears in Transl. Amer. Math. Soc. Ser. 2 37 (1964) 59–78.
[20] D. Marker, A. Pillay, Differential Galois theory III. Some inverse problems, Illinois J. Math. 41 (3) (1997) 453–561.
[21] C. Mitschi, M.F. Singer, Monodromy groups of parameterized linear differential equations with regular singular points, arXiv:1106.2664v1 [math.CA], June 2011.
[22] A. Pillay, Differential Galois theory II, Ann. Pure Appl. Logic 88 (2–3) (1997) 181–191.
[23] A. Pillay, Differential Galois theory I, Illinois J. Math. 42 (4) (1998) 678–699.
[24] M. van der Put, M.F. Singer, Galois Theory of Linear Differential Equations, Grundlehren Math. Wiss., vol. 328, Springer-Verlag, Heidelberg, 2003.
[25] C. Tretkoff, M. Tretkoff, Solution of the inverse problem in differential Galois theory in the classical case, Amer. J. Math. 101 (1979) 1327–1332.
[26] H. Umemura, Invitation to Galois theory, in: D. Bertrand, et al. (Eds.), Differential Equations and Quantum Groups, in: IRMA Lect. Math. Theor. Phys., vol. 9, 2007, pp. 269–289.
[27] H. Volklein, Groups as Galois Groups, Cambridge Stud. Adv. Math., vol. 53, Cambridge University Press, Cambridge, 1996.
[28] B.A.F. Wehrfritz, Infinite Linear Groups, Ergeb. Math., Springer-Verlag, Berlin, 1973.
[29] M. Wibmer, Existence of ∂-parameterized Picard–Vessiot extensions over fields with algebraically closed constants, arXiv:1104.3514v1 [math.AC], April 2011.