Consistent systems of linear differential and difference equations

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Abstract

We consider systems of linear differential and difference equations

$$\delta Y(x) = A(x)Y(x), \quad \sigma Y(x) = B(x)Y(x)$$

with $\delta = \frac{d}{dx}$, σ a shift operator $\sigma(x) = x + a$, q-dilation operator $\sigma(x) = qx$ or Mahler operator $\sigma(x) = x^p$ and systems of two linear difference equations

$$\sigma_1 Y(x) = A(x)Y(x), \quad \sigma_2 Y(x) = B(x)Y(x)$$

with (σ_1, σ_2) a sufficiently independent pair of shift operators, pair of q-dilation operators or pair of Mahler operators. Here $A(x)$ and $B(x)$ are $n \times n$ matrices with rational function entries. Assuming a consistency hypothesis, we show that such systems can be reduced to a system of a very simple form. Using this we characterize functions satisfying two linear scalar differential or difference equations with respect to these operators. We also indicate how these results have consequences both in the theory of automatic sets, leading to a new proof of Cobham's Theorem, and in the Galois theories of linear difference and differential equations, leading to hypertranscendence results.

Keywords. linear differential equations, linear difference equations, consistent systems, shift operator, q-difference equation, Mahler operator

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1 Introduction

In [37], J.-P. Ramis showed that if a formal power series $f(x)$ is a solution of a linear differential equation and a linear q-difference equation[2], $q \neq 0$, q transcendental if $|q| = 1$, both with polynomial coefficients, then f is the expansion at the origin of a rational function. Rationality has also been shown for a formal power series satisfying

- a linear differential equation and a linear σ-difference equation with polynomial coefficients, where σ is the Mahler operator $\sigma(f)(x) = f(x^k), k$ an integer ≥ 2 [12],
- a linear q_1-difference equation and a linear q_2-difference equation, both with polynomial coefficients and with q_1 and q_2 multiplicatively independent (*i.e.,* no integer power of q_1 is equal to an integer power of q_2) [15] [3],
- a linear σ_1-difference equation and σ_2-difference equation with polynomial coefficients and with Mahler operators σ_1, σ_2 having multiplicatively independent exponents [1].

Other results characterizing entire solutions of a linear differential equation and a linear σ-equation with polynomial coefficients where σ is the operator $\sigma(x) = x + \alpha$, $\sigma(x) = qx$, or $\sigma(x) = x^k$ and entire solutions of two linear σ-difference equations involving these operators can be found in [13], [14],[17], [28], [30], [31], and [38].

These results have been proved with a variety of ideas such as the structure of ideals of entire functions, Gevrey-type estimates, p-adic behavior and mod p reductions. In our work we present a unified approach to all these results, reproving and generalizing them to also characterize meromorphic solutions on the plane and certain Riemann surfaces.

Our results spring from two fundamental ideas. The first is that questions concerning the form of solutions of two scalar linear differential/difference equations can be reduced to showing that consistent pairs of first order systems are equivalent to very simple systems. The second is that the hypothesis of consistency allows us to show that the singular points are of a very simple nature, to describe the interaction of local solutions at different singular points and to continue local solutions meromorphically. These conclusions, in turn, allow us to prove that the systems are equivalent to systems of a very simple form. For our approach, it is crucial that δ and σ or σ_1 and σ_2, respectively, commute except for some constant factor. The commutativity is closely related to the consistency condition.

[2] A linear difference equation involving the operator $\sigma(x) = qx$

[3] The result of [15] needs some restrictions, see below Corollary 15.

Our approach is best explained with the example of Ramis's result in the case $|q| \neq 0, 1$. Let $f(x)$ be a power series satisfying both a linear differential equation and a linear q-difference equation with coefficients in $\mathbb{C}(x)$. Using these equations, one shows that the $\mathbb{C}(x)$-vector space V spanned by $\{(x\frac{d}{dx})^i f(q^j x)\}$ with $0 \leq i < \infty, -\infty < j < \infty$ is finite dimensional (see Corollary 3). This space consists of Laurent series and is invariant under the map σ that sends x to qx and the derivation $\delta = x\frac{d}{dx}$. If $y_1(x), \ldots, y_n(x)$ is a $\mathbb{C}(x)$-basis of V and $y(x) = (y_1(x), \ldots, y_n(x))^T$, then

$$\begin{aligned} x\frac{dy(x)}{dx} &= \quad A(x)y(x),\ \ A(x) \in \mathrm{gl}_n(\mathbb{C}(x)) \\ y(qx) &= \quad B(x)y(x),\ \ B(x) \in \mathrm{GL}_n(\mathbb{C}(x)). \end{aligned} \tag{1}$$

$B(x)$ is invertible because σ is an automorphism. Calculating $\sigma(\delta(y(x))) = \delta(\sigma(y(x)))$ in two ways and using that the components of y are linearly independent over $\mathbb{C}(x)$, we obtain that $A(x)$ and $B(x)$ satisfy the consistency condition

$$x\frac{dB(x)}{dx} + B(x)A(x) = A(qx)B(x). \tag{2}$$

The first principal result of our work (Theorem 2) states in this case that there exists a matrix $G(x) \in \mathrm{GL}_n(\mathbb{C}(x))$ such that the gauge transformation $y(x) = G(x)z(x)$ results in a new simpler system

$$\begin{aligned} x\frac{dz(x)}{dx} &= \quad \tilde{A}z(x), \\ z(qx) &= \quad \tilde{B}z(x) \end{aligned} \tag{3}$$

with $\tilde{A} \in \mathrm{gl}_n(\mathbb{C})$ and $\tilde{B} \in \mathrm{GL}_n(\mathbb{C})$.

This implies that there is a new basis $z(x) = (z_1(x), \ldots, z_n(x))^T$ of V given by $z(x) = G(x)^{-1}y(x)$ such that $x\frac{dz(x)}{dx} = \tilde{A}z(x)$ and $z(qx) = \tilde{B}z(x)$ with $\tilde{A}$ and $\tilde{B}$ constant matrices. It is not hard to show that the entries of $z(x)$ must be Laurent polynomials and therefore rational. We then conclude that $y(x)$ is also rational and hence also the given $f(x)$ is rational.

We now give an idea of the proof of Theorem 2 in the context of the present case. A calculation shows that the consistency condition implies: *if* $Y(x)$ *is a solution of* $x\frac{dY(x)}{dx} = A(x)Y(x)$ *then* $Z(x) = B(x)Y(x)$ *is a solution of* $x\frac{dZ(x)}{dx} = A(qx)Z(x)$. Repeating this observation we have that for any m, there is a gauge transformation $Y(x) = D_m(x)Z(x)$ taking solutions of $x\frac{dY(x)}{dx} = A(x)Y(x)$ to solutions of $x\frac{dZ(x)}{dx} = A(q^m x)Z(x)$. This gauge transformation only introduces apparent singularities, that is, those at which one has a meromorphic fundamental solution matrix. Since the singular points in $\mathbb{C}\backslash\{0, \infty\}$ of these two equations are disjoint for sufficiently large m, we can conclude that all the singular points, other than $0, \infty$, of $\frac{dY(x)}{dx} = A(x)Y(x)$ are apparent (see Lemma 8).

If $Y(x)$ is a formal fundamental solution of $x\frac{dY(x)}{dx} = A(x)Y(x)$, then as seen above $Z(x) = B(x)Y(x)$ is a formal fundamental solution of $x\frac{dZ(x)}{dx} = A(qx)Z(x)$. Comparing it with the formal fundamental solution $\tilde{Z}(x) = Y(qx)$ of this equation, it follows that 0 is a regular singular point (see Lemma 9). A similar statement holds for ∞. We then have that if $Y(x)$ is a fundamental solution analytic in a neighborhood of an ordinary point, then $Y(x)$ can be analytically continued to a meromorphic function on the universal cover $\hat{\mathbb{C}}$ of $\mathbb{C}\backslash\{0\}$. If $Y(xe^{2\pi i})$ is the solution matrix obtained by analytically continuing $Y(x)$ once around 0, we have that $Y(xe^{2\pi i}) = Y(x)H$ for some $H \in \mathrm{GL}_n(\mathbb{C})$. Writing $H = e^{2\pi i\tilde{A}}$ with a non-resonant $\tilde{A}$, a calculation shows that $G(x) = Y(x)x^{-\tilde{A}}$ is a matrix valued meromorphic function on $\mathbb{C}\backslash\{0\}$. Using the fact that 0 and ∞ are regular singular points, one deduces that $G(x)$ has moderate growth at these points and so must have rational function entries. Therefore the gauge transformation $y(x) = G(x)z(x)$ transforms $x\frac{dy(x)}{dx} = A(x)y(x)$ to $x\frac{dz(x)}{dx} = \tilde{A}z(x)$. One then shows that the consistency condition implies that this transformation also results in a constant q-difference equation for $z(x)$ and so (1) is transformed into (3) (see Lemma 10 for details).

The rest of our work is organized as follows. In Section 2 we consider systems (1) where $\sigma(x) = x + 1, qx$ ($q \neq 0$ not a root of unity) or x^q, (q an integer ≥ 2) and $A(x) \in \mathrm{gl}_n(C(x)), B(x) \in \mathrm{GL}_n(C(x)),$ C an algebraically closed field of characteristic zero. Assuming a consistency condition analogous to (2) we show in Theorem 2 that there is a transformation $Y(x) = D(x)Z(x)$ with $D(x) \in GL_n(C(x))$ taking system (1) to a much simpler system. When $\sigma(x) = qx$ or x^q, we characterize those $y(x) \in C[[x]][x^{-1}]$ and those $y(x)$ meromorphic on the Riemann surface of $\log x$ (when $C = \mathbb{C}$) that simultaneously satisfy a linear differential equation and a linear σ-difference equation over $C(x)$ (Corollary 3). When $\sigma(x) = x + 1$, we characterize those $y(x) \in C[[x^{-1}]][x]$ and those $y(x)$ meromorphic on $\mathbb{C}$ that simultaneously satisfy a linear differential equation and a linear σ-difference equation over $C(x)$ (Corollary 5). Theorem 2 allows us to also characterize in Corollary 6 when the time-1-operator of a linear differential system has rational entries.

In Section 3 we consider systems of the form

$$\sigma_j(Y) = \quad B_j Y,\ j = 1, 2 \tag{4}$$

with $B_j \in \mathrm{GL}_n(C(x))$ satisfying a suitable consistency condition and (σ_1, σ_2) defined by $(\sigma_1(x) = x + 1, \sigma_2(x) = x + \alpha), \alpha \in \mathbb{C}\backslash\mathbb{Q}$ or $(\sigma_1(x) = q_1x, \sigma_2(x) = q_2x)$ or $(\sigma_1(x) = x^{q_1}, \sigma_2(x) = x^{q_2})$ with q_1 and q_2 multiplicatively independent. Theorem 13 states that there is a gauge transformation $Y(x) = D(x)Z(x), D(x) \in \mathrm{GL}_n(C(x))$ in the first two cases and $D(x) \in \mathrm{GL}_n(K), K = C(\{x^{1/s} \mid s \in \mathbb{N}^*\})$ in the last case, transforming such a system into a system with constant coefficients. Once again the proofs depend on showing that the singular points and the connection relations are particularly simple. We again have corollaries characterizing formal solutions and solutions on various domains of two linear σ equations in each of these three cases (Corollaries 14, 15 and 16).

We end this introduction with a discussion of two applications of our results. The first concerns properties of *automatic sets* (See [3] for a general introduction to these sets and [10] and [38] for connections to Mahler equations). A subset $\mathcal{N} \subset \mathbb{N}$ of integers is called *k-automatic* if there is a finite-state machine that accepts as input the base-k representation of an integer and outputs 1 if the integer is in $\mathcal{N}$ and 0 if it is not in $\mathcal{N}$. Many sets can be k-automatic for fixed k (for example the set of powers of 2 is 2-automatic) but only very simple sets can be k- and ℓ-automatic for multiplicatively independent integers k and ℓ. This fact is formalized in Cobham's Theorem [19], [23].

Theorem (Cobham). *Let k and ℓ be two multiplicatively independent integers. Then a set $\mathcal{N} \subset \mathbb{N}$ is both k- and ℓ-automatic if and only if it is the union of a finite set and a finite number of arithmetic progressions.*

Linear difference equations involving the Mahler operator and k-automatic sets are related by the following fact: If $\mathcal{N}$ is a k-automatic set then $F(x) = \sum_{n\in\mathcal{N}} x^n$ satisfies a scalar linear difference equation over $\mathbb{Q}(x)$ with respect to the Mahler operator $\sigma(x) = x^k$, that is, a k-Mahler equation. In ([1], Theorem 1.1), Adamczewski and Bell show: a power series $f(x) \in C[[x]][x^{-1}]$ satisfies both a k- and ℓ-Mahler equation if and only if it is a rational function, proving a conjecture of Loxton and van der Poorten [34]. Their proof relies on Cobham's Theorem. On the other hand, it is known that their Theorem 1.1 implies Cobham's Theorem (see, for example, Section 2, [1] or Chapitre 7, [38]). In our work we prove and generalize the Adamczewski-Bell result (Corollary 16) without using Cobham's Theorem, therefore yielding a new proof of this latter result. Our proof of the Adamczewski-Bell result follows the general philosophy of our work. We show that proving that a power series of two such Mahler equations is rational can be reduced to showing that consistent pairs of first order Mahler systems must be of a very simple nature. In fact, although we deduce the Adamczewski-Bell result from Theorem 13 mentioned above, we do not need its full strength and can also prove this result from the weaker statement contained in Proposition 22.

The second application concerns the Galois theory of difference equations. In [25] a differential Galois theory of linear difference equations was developed as a tool to understand the differential properties of solutions of linear difference equations. This theory associates to a system of linear difference equations $Y(\sigma(x)) = B(x)Y(x)$ a group called the *differential Galois group*. This is a linear differential algebraic group, that is a group of matrices whose entries are functions satisfying a fixed set of (not necessarily linear) differential equations. Differential properties of solutions of the linear difference equation are measured by group theoretic properties of the associated group. For example, a group theoretic proof is given in [25] of Hölder's Theorem that the Gamma Function satisfies no polynomial differential equation, that is, the Gamma Function is hypertranscendental. In general, one can measure the amount of differential dependence among the entries of a fundamental solution matrix of $Y(\sigma(x)) = B(x)Y(x)$ by the size of its associated group; the larger the group the fewer differential relations hold among these entries. This theme has been taken up in [21] where the authors develop criteria to show that the generating series $F(x) = \sum_{n\in\mathcal{N}} x^n$ of certain k-automatic sets $\mathcal{N}$ are hypertranscendental. As we mentioned above, these generating series satisfy Mahler equations and Dreyfus, Hardouin and Roques in [21] develop criteria to insure that a given Mahler equation has SL_n or GL_n as its associated group. The proofs of the validity of their criteria depend on Bézivin's result [12] that a power series that simultaneously satisfies a Mahler equation and a linear differential equation must be a rational function. In [22], the authors develop similar criteria (using the result of Ramis mentioned in the Introduction) for linear q-difference equations. Both Ramis's result and Bézivin's result appear in Corollary 3 as a consequence of Theorem 2 in the present work. Using Theorem 2 directly, the authors of [6] classify the differential Galois groups that can occur for the equations considered in this latter theorem and, in particular, rule out certain groups from occurring. Using this classification, it is shown in [6] how the criteria of [21] and [22] can be extended and given simple proofs. The results of [6] can also be used in designing algorithms to compute the differential Galois group of linear difference equations (c.f., [5]). Using the Galois theory presented in [33], the authors of [22] also develop criteria to determine when a solution of a linear q-difference equation satisfies no q'-difference relation (even nonlinear) with respect to a multiplicatively independent q'. This is done, in a manner analogous to the results of [21], by developing criteria to insure that the Galois groups in this context are large. Their result depends on the results of Bézivin and Boutabaa [15]. The results of Section 3 can also be used to sharpen the criteria in [22].

2 Reduction of systems of differential and difference equations

Let C be an algebraically closed field[4] and $k = C(x)$. Let δ be a derivation on k with constants C and σ be a C-algebra endomorphism of k. We suppose that there is a constant

[4] All fields considered in this work are of characteristic zero.

$\mu \in C$ such that $\delta\sigma = \mu\,\sigma\delta$. This commutativity except for a constant factor is crucial for our approach.

We consider three cases of couples (δ, σ) below.

case S: The derivation is $\delta = d/dx$ and σ is the shift operator defined by $\sigma(x) = x + 1$. Here $\mu = 1$.

case Q: The derivation is $\delta = x\,d/dx$ and σ is the q-dilation operator defined by $\sigma(x) = q\,x$ with some $q \in C$, $q \neq 0$ and not a root of unity. Note that $\mu = 1$ here as well.

case M: The derivation is $\delta = x\,d/dx$ and σ is the Mahler operator defined by $\sigma(x) = x^q$ with an integer $q \geq 2$. Here we have $\mu = q$.

Observe that σ is bijective in cases S and Q, but not in case M.

We will consider systems

$$\begin{aligned} \delta(Y) = &\quad AY \\ \sigma(Y) = &\quad BY \end{aligned} \tag{5}$$

with $A \in \mathrm{gl}_n(k)$, $B \in \mathrm{GL}_n(k)$ that are *consistent*, that is A and B satisfy the consistency condition given by

$$\delta(B) = \quad \mu\,\sigma(A)B - BA. \tag{6}$$

The consistency condition is closely related to the almost-commutativity of δ, σ. Note that it guarantees that $\delta(\sigma(Z)) = \mu\sigma(\delta(Z))$ holds for any solution Z of the system (5) in any extension of $C(x)$. It is satisfied if there exists a fundamental solution of $\delta(Y) = AY$ that is also a solution of $\sigma(Y) = BY$ in some extension of $C(x)$ in which δ and σ commute or if there exists a solution vector of the system in such an extension such that its components are linearly independent over $C(x)$. The consistency condition is satisfied for the systems (5) constructed from the applications to common solutions of pairs of linear scalar equations, again because δ and σ commute except for a constant factor.

We say that (5) is *equivalent* (over k) to a system

$$\begin{aligned} \delta(Z) = &\quad \tilde{A}Z \\ \sigma(Z) = &\quad \tilde{B}Z \end{aligned} \tag{7}$$

with $\tilde{A} \in \mathrm{gl}_n(k)$, $\tilde{B} \in \mathrm{GL}_n(k)$ if for some $G \in \mathrm{GL}_n(k)$,

$$\begin{aligned} \tilde{A} &= \delta(G)G^{-1} + GAG^{-1} \\ \tilde{B} &= \sigma(G)BG^{-1} \end{aligned} \tag{8}$$

that is, if (7) comes from (5) via the *gauge transformation* $Z = GY$. Note that the property of consistency is preserved under equivalence.

A simple, but crucial observation is the fact that the consistency condition can be expressed as an equivalence.

Lemma 1. *Consider the system (5). It satisfies the consistency condition (6) if and only if it is equivalent to the system*

$$\delta(Z) = \mu\sigma(A)Z, \;\; \sigma(Z) = \sigma(B)Z \tag{9}$$

by the gauge transformation $Z = BY$. For $N \in \mathbb{N}^*$, it is equivalent to the systems

$$\delta(Z) = \mu^N\sigma^N(A)Z, \;\; \sigma(Z) = \sigma^N(B)Z. \tag{10}$$

Proof. Rewriting (6) yields the first part, iteration using the fact that all these equivalent systems are again consistent yields the second. ∎

Observe that (10) is also obtained by applying σ^N to (5).

The main result of this section expresses that the consistency condition is very restrictive.

Theorem 2. *The system (5) satisfying the consistency condition (6) is equivalent over k to a system (7) with $\tilde{A} \in \mathrm{gl}_n(C)$, $\tilde{B} \in \mathrm{GL}_n(k)$. Moreover:*

case S: *$\tilde{A}$ is diagonal, $\tilde{B} \in \mathrm{GL}_n(C)$ is constant and upper triangular and commutes with $\tilde{A}$.*

case Q: *If λ_1, λ_2 are eigenvalues of $\tilde{A}$, then $\lambda_1 - \lambda_2 \notin \mathbb{Z} \setminus \{0\}$. $\tilde{B} \in \mathrm{GL}_n(C)$ is constant and commutes with $\tilde{A}$.*

case M: *The eigenvalues of $\tilde{A}$ are rational and in the interval $[0, 1[$ and there exists a diagonalisable matrix D with integer eigenvalues commuting with $\tilde{A}$ such that $\tilde{A} + D$ is conjugate to $q\,\tilde{A}$. We have $\tilde{B} \in \mathrm{GL}_n(C[x, x^{-1}])$, such that the exponents m appearing with nonzero coefficient in $\tilde{B}$ are integer differences of the form $q\lambda_1 - \lambda_2$ of eigenvalues λ_1, λ_2 of $\tilde{A}$.*

Remark: Simple counter-examples show that the statement in case Q no longer holds if q is a root of unity. Consider for instance the natural consistent system satisfied by $y_j(x) = \exp(q^j x)$, $j = 0, ..., n-1$, if $q^n = 1$.

Before giving a proof of this result in sections 2.1 and 2.2, we deduce several corollaries concerning common solutions of the linear differential and σ-difference equations

$$\begin{aligned} L(f(x)) &= \delta^n(f(x)) + a_{n-1}(x)\delta^{n-1}(f(x)) + \ldots + a_0(x)f(x) = 0 \text{ and} \\ S(f(x)) &= \sigma^m(f(x)) + b_{m-1}(x)\sigma^{m-1}(f(x)) + \ldots + b_0(x)f(x) = 0, \end{aligned} \tag{11}$$

with $a_i(x), b_i(x) \in \mathbb{C}(x)$.

Corollary 3. *Consider δ, σ as in case Q or M. Let E be the field of meromorphic functions on $\hat{\mathbb{C}}$, where $\hat{\mathbb{C}}$ denotes the Riemann surface of the logarithm over $\mathbb{C} \setminus \{0\}$. $\mathbb{C}(x)$ is considered as a subfield of E. If $f \in E$ satisfies the linear differential and σ-difference equations (11) then*

$$f(x) = \sum_{i,j=1}^{t} r_{ij}(x) x^{\alpha_i} \log(x)^j \tag{12}$$

where $\alpha_i \in \mathbb{C}$ and $r_{ij} \in \mathbb{C}(x)$. In case M, we obtain moreover that the α_i are rational. Conversely, any such function satisfies a pair of linear differential and σ-difference equations with coefficients in $\mathbb{C}(x)$.

Assume $f(x) \in \mathbb{C}[[x]][x^{-1}]$ with $\mathbb{C}(x)$ considered as a subfield of $\mathbb{C}[[x]][x^{-1}]$. If $f(x)$ satisfies the linear differential and σ-difference equations (11) then f is rational, i.e. $f \in \mathbb{C}(x)$.

Remark: 1. The Corollary can be extended to functions satisfying non-homogeneous systems

$$L(f(x)) = h_1(x), \quad S(f(x)) = h_2(x),$$

where $h_1(x)$ and $h_2(x)$ are of the form (12). Indeed, as the latter satisfy homogeneous systems of the form (4), it is straightforward to eliminate them from the non-homogeneous system at the expense of increasing the orders n and m of the equations. A similar remark applies to the corresponding corollaries in case S and in cases 2S and 2Q in section 3.

2. Functions of the form (12) are elementary function. Algorithms for finding such solutions of linear difference equations are known (eg. [20], [36], Ch. 4) and can be modified to find solutions of this special type. Possible values of α and relevant powers of $\log x$ can be calculated using effective procedures to determine canonical forms of such equations at singular points ([36], Ch. 3.1). Again a similar remark applies to the corresponding corollaries in case S and in cases 2S and 2Q of section 3 ([35]).

3. The final assertion of the above corollary corresponds to results of Ramis [37] in case Q and of Bézivin [12] in case M. Their proofs proceed by examining the asymptotic behavior of solutions of the two scalar linear differential equations rather than our approach. In [37], Ramis assumed that $q \neq 0$ and q is transcendental if $|q| = 1$. This condition can be reduced to $|q| \neq 0, 1$. The latter was needed to ensure that $q^n x \to 0$ or $q^n x \to \infty$ when $n \to \infty$ and allowed asymptotic results for q-difference equations to be applied. As mentioned above, our approach only requires that $\sigma(x) = qx, q \neq 0$ has no periodic points other than the fixed points.

4. Although we use Theorem 2 to prove the final statement of this corollary, one can prove this directly from Lemmas 8 and 9 below as noted in the remark following Lemma 9.

Proof. We begin by proving the assertion concerning a function $f(x) \in E$. Let W_0 be the $\mathbb{C}(x)$-subspace of E spanned by

$$\{\sigma^j \delta^i (f(x))\}$$

where $0 \leq i \leq n-1$ and $0 \leq j \leq m-1$. Using the fact that $\delta\sigma = \mu\sigma\delta$ and the equation $L(f(x)) = 0$, one sees that W_0 is left invariant under δ. Similarly, using $S(f(x)) = 0$, one sees that W_0 is invariant under σ. Unfortunately, σ does not preserve linear independence over $\mathbb{C}(x)$ in case M – just consider 1 and $x^{1/q}$. Therefore we consider now the vector spaces W_ℓ generated by the elements of $\sigma^\ell(W_0)$, $\ell = 0, 1, ...$ These are again invariant under δ and σ. As they form a descending chain of finite dimensional $\mathbb{C}(x)$-vector spaces there must be a first index s such $W_s = W_{s+1}$. It is easy to see that we then have that $W_\ell = W_{\ell+1}$ for all $\ell \geq s$. In the sequel, we consider W_s and omit the index s.

Let $w_1, \ldots, w_t$ be a $\mathbb{C}(x)$-basis of W and let $w = (w_1, \ldots, w_t)^T$. We have that

$$\begin{aligned} \delta(w) = &\quad Aw \\ \sigma(w) = &\quad Bw \end{aligned} \tag{13}$$

for some $A \in \mathrm{gl}_t(k), B \in \mathrm{GL}_t(k)$ because $\sigma(w_1), ..., \sigma(w_t)$ again generate W. We claim that A and B satisfy (6). To see this, note that

$$0 = \mu\sigma\delta(w) - \delta\sigma(w) = (\mu\sigma(A)B - BA - \delta(B))w.$$

Since the entries of w are linearly independent over k, we have (6).

Before we continue with the proof of this corollary, we remark that the above argument does not use special properties of δ and σ and will again be useful later. We note it as

Lemma 4. *Let C be an algebraically closed field, $k = C(x)$, E a k-algebra, δ a derivation on E annihilating C satisfying $\delta(x) \in k$ and σ a C-algebra endomorphism on E satisfying $\sigma(k) \subset k$ and $\delta\sigma = \mu\sigma\delta$ for some $\mu \in C^*$. Suppose that there exists an $f \in E$ satisfying a system (11) of equations. Then there exist $s, t \in \mathbb{N}$ [5], a solution vector $w = (w_1, ..., w_t)^T \in E^t$ of a system (13) satisfying the consistency condition (6) and $r_i \in k$, $i = 1, .., t$, such that $\sigma^s f = \sum_{i=1}^t r_i w_i$.*

We now apply Theorem 2 to the equations $\delta(Y) = AY, \sigma(Y) = BY$. We conclude that there is a gauge transformation $Z = GY$ that transforms this system to a system $\delta(Z) = \tilde{A}Z, \sigma(Z) = \tilde{B}(x)Z$ where $\tilde{A}$ is a constant matrix (with rational eigenvalues in case M). Letting $z = (z_1, \ldots, z_t)^T = Gw$ we see that $z \in E^t$ satisfies $\delta z = \tilde{A}z$ and hence $z = x^{\tilde{A}}C$ for some constant matrix C. Therefore the z_i are of the desired form.

[5] $\mathbb{N}$ denotes the set of non-negative integers in the present text.

Since the z_i are again a $\mathbb{C}(x)$-basis of W, there exists a non-negative integer s such that $\sigma^s(f)$ is a $\mathbb{C}(x)$-linear combination of the z_i. In case Q, it follows immediately that f is also a $\mathbb{C}(x)$-linear combination of terms of the wanted form. In case M, we find that f is a $\mathbb{C}(x^{q^{-s}})$-linear combination of terms of the desired form, that is, it is itself of that form. Thus we have proved the first statement of the corollary.

To prove the second part of the corollary, first assume that $f = x^\alpha \log(x)^j$ with $\alpha \in \mathbb{C}$, $j \in \mathbb{N}$. A simple calculation shows that the operator $L = (\delta - \alpha)^{j+1}$ annihilates f.

In case Q, we obtain that $S = (\sigma - q^\alpha)^{j+1}$ also annihilates f.

In case M, it is required that α be rational. As there are only finitely many rationals in $[0, 1[$ with the same denominator as α, there must exist positive integers $m \neq n$ such that $(q^m - q^n)\alpha =: r \in \mathbb{Z}$. Then $\sigma^m(f) = q^{j(m-n)} x^r \sigma^n(f)$ and we have found a σ-difference equation for f.

The general case follows from the fact that sums and products with rational functions of solutions of differential or σ-difference equations, respectively, again satisfy such equations.

We now turn to the assertion concerning a function $f(x) \in \mathbb{C}[[x]][x^{-1}]$. We can again use Lemma 4 and apply Theorem 2. Following the proof of the first part of this corollary we will find the solutions of $\delta z = \tilde{A} z$ in $\mathbb{C}[[x]][x^{-1}]^t$. Clearly the coefficient $z^{(\ell)}$ of z in front of x^ℓ must satisfy $\tilde{A}\, z^{(\ell)} = \ell\, z^{(\ell)}$. Hence $z^{(\ell)}$ can only be different from 0 for finitely many ℓ and so $z \in \mathbb{C}[x, x^{-1}]^t \subset \mathbb{C}(x)^t$. Since the entries in the vector z form a $\mathbb{C}(x)$-basis of W, we must have $t = 1$. Therefore z is a scalar in $\mathbb{C}(x)$ and so $\sigma^s(f)$ is also rational, $\sigma^s(f)(x) = g(x) \in \mathbb{C}(x)$. In case Q, the proof is finished since we can immediately conclude that $f(x) \in \mathbb{C}(x)$. In case M, we have that $\sigma^s(f)$ is in $\mathbb{C}\{x^{q^s}\}[x^{-q^s}]$ and hence invariant under the substitutions $x \to x\exp(2\pi i\ell/q^s)$, $\ell = 0, ..., q^s - 1$. Therefore we also have that

$$\sigma^s(f)(x) = q^{-s} \sum_{\ell=0}^{q^s-1} g(x \exp(2\pi i \ell/q^s)) \in \mathbb{C}(x^{q^s}).$$

Hence $f \in \mathbb{C}(x)$. ∎

In the shift case, the corollaries obtained are slightly different as we cannot consider the difference equation on $\hat{\mathbb{C}}$ or $\mathbb{C}[[x]][x^{-1}]$. It is possible here to consider the difference equation for meromorphic functions of $\mathbb{C}$ and for formal Laurent series in $1/x$, that is in $\mathbb{C}[[x^{-1}]][x]$. Again $\mathbb{C}(x)$ is considered as a subset of $\mathbb{C}[[x^{-1}]][x]$, where the inclusion is given by series expansion at $x = \infty$.

Corollary 5. *Consider $\delta = d/dx$ and σ induced by $\sigma(x) = x + 1$. If $f(x)$ is a meromorphic function on some horizontal strip $\{x \in \mathbb{C} \mid m < \operatorname{Im} x < M\}$ satisfying the system*

(11) of a linear differential and difference equations with coefficients in $\mathbb{C}(x)$ then

$$f(x) = \sum_{i=1}^{t} r_i(x) e^{\alpha_i x}$$

where $\alpha_i \in \mathbb{C}$ and $r_i(x) \in \mathbb{C}(x)$. Conversely, any such function satisfies a pair of linear differential and difference equations with rational coefficients.

If $f(x) \in \mathbb{C}[[x^{-1}]][x]$ satisfies the system (11) of a linear differential and difference equations with coefficients in $\mathbb{C}(x)$ then $f(x)$ is rational.

Remark: In [13], Bézivin and Gramain consider *entire* solutions of linear differential/difference equations of the type considered in Corollary 5 and show that such solutions must be of the form described. Their proof ultimately depends on an analytic result of Kelleher and Taylor describing the growth properties of a set of entire functions equivalent to the property that they generate an ideal equal to the whole ring of entire functions. A crucial part of their proof also involves reduction to equations with constant coefficients but in a different manner than the proof presented here. They also give conditions on the equations that guarantee that the $r_i(x)$ appearing in the expression for $f(x)$ are polynomials.

Such a condition can also be given here. Let $\ell(x)$ be the least common denominator of the coefficients $a_j(x)$, $j = 0, ..., n-1$ of L in (11), $r(x)$ the least common denominator of the coefficients $b_j(x)$, $j = 0, ..., m-1$ of S. By the $\mathbb{C}(x)$-linear independence of $e^{\alpha_i x}$, $i = 1, ..., t$, also the individual terms $r_i(x)e^{\alpha_i x}$ are solutions of the scalar equations (11). Therefore the poles of some $r_i(x)$ must be among the zeroes of $\ell(x)$. Now let α be a pole of some $r_i(x)$ with minimal real part. Then it is a zero of $\ell(x)$ as seen above, but in view of $\sigma^{-m}S(r_i(x)e^{\alpha_i x}) = 0$, α must also be a zero of $\sigma^{-m}r$. As a consequence, all r_i must be polynomials, if ℓ and $\sigma^{-m}r$ have no common zero.

Proof. Using Lemma 4 and Theorem 2, we obtain that $f(x)$ is a $\mathbb{C}(x)$-linear combination of solutions of $\delta z = \tilde{A}z$ with diagonal constant $\tilde{A}$. This yields the first part of the corollary.

To prove the second part of the corollary, first assume that the r_i are polynomials with $\deg(p_i) = n_i$. A simple calculation shows that the operators $L = \prod_{i=1}^{t}(\delta - \alpha_i)^{n_i+1}$ and $S = \prod_{i=1}^{t}(\sigma - e^{\alpha_i})^{n_i+1}$ annihilate f. In general, let $p(x)$ be a common denominator of the r_i and let $\tilde{L}$ and $\tilde{S}$ be the operators annihilating $p(x)f$. Then clearly $L = \frac{1}{p(x)}\tilde{L} \circ M_{p(x)}$ and $S = \frac{1}{\sigma^n p(x)}\tilde{S} \circ M_{p(x)}$, $n = t + \sum n_i$, $M_{p(x)}$ the multiplication operator, satisfy the conditions of the theorem.

For the last part of the corollary we use again Lemma 4 and Theorem 2 and obtain again that $f(x)$ is a $\mathbb{C}(x)$-linear combination of solutions of $\delta z = \tilde{A}z$, with diagonal constant $\tilde{A}$, now in $\mathbb{C}[[x^{-1}]][x]^t$. Hence z is also constant [6]. As a consequence, $f(x)$ is in $\mathbb{C}(x)$. ∎

Another consequence of Theorem 2 in case S concerns the *time-1-operator* associated to a system $\frac{dy}{dx} = A(x)y$, $A \in \mathbb{C}(x)$. It is defined by $T_A(x) = Y(x+1, x)$, where $Y(t,x)$ denotes the solution matrix of $\frac{d}{dt}Y(t,x) = A(t)Y(t,x)$ satisfying $Y(x,x) = I$. It is holomorphic on $\mathbb{C} \setminus \cup_{\ell=1}^{L}[x_\ell - 1, x_\ell + 1]$ if x_ℓ, $\ell = 1, ..., L$ are the singularities of $A(x)$. By the theorem on the dependence of solutions upon initial conditions, it is readily verified that $W(t,x) = \frac{\partial Y}{\partial x}(t,x)$ satisfies $\frac{d}{dt}W(t,x) = A(t)W(t,x)$ and $W(x,x) = -A(x)$ and hence $\frac{\partial Y}{\partial x}(t,x) = -Y(t,x)A(x)$. Thus we find that

$$\frac{d}{dx}T_A(x) = A(x+1)T_A(x) - T_A(x)A(x).$$

The system

$$\begin{array}{rcl} \frac{d}{dx}y & = & A(x)y \\ y(x+1) & = & T_A(x)y(x) \end{array} \tag{14}$$

therefore satisfies the consistency condition (6). This implies that $T_A(x)$ may be continued analytically to the universal covering of $\mathbb{C} \setminus \{\xi, \xi+1 \mid \xi \text{ a singularity of } A(x)\}$. It also allows us to show

Corollary 6. *The time-1-operator defined for a system of linear differential equations $\frac{dy}{dx} = A(x)y$, $A(x) \in \mathrm{gl}_n(\mathbb{C}(x))$, has rational functions as entries if and only if the system is equivalent over $\mathbb{C}(x)$ to a system with a constant diagonal coefficient matrix.*

Proof. It suffices to put $B(x) = T_A(x)$ and to apply Theorem 1. ∎

An analogous corollary can be stated for the "q-multiplication-operator" that can be defined for a system $x\frac{dy}{dx}y = A(x)y$. Details are left to the reader.

In case M, the statement of Theorem 1 is not entirely satisfactory, because the matrix $B(x)$ obtained is not constant. This can be repaired by changing the base field. Consider the field $K = C(\{x^{1/\ell} \mid \ell \in \mathbb{N}\})$ (see [21]) containing all fractional powers of x. This field also has the advantage that σ, now defined by mapping x^α to $x^{q\alpha}$ for all rational α, is an automorphism[7]. The derivation $\tilde{\delta} = x\log(x)\frac{d}{dx}$ and the automorphism σ mapping x^α to $x^{q\alpha}$, $\log(x)$ to $q\log(x)$ commute on the base field $K(\log x)$ (*i.e.* $\mu = 1$). Furthermore, as the consistency conditions for the couple (δ, σ) and the couple $(\tilde{\delta}, \sigma)$ are equivalent we do not consider $\tilde{\delta}$ here.

[6] By the way, as the entries of z form a $\mathbb{C}(x)$-basis of the space W of the proof of Lemma 4, we have $t = 1$.

[7] For this property it would not be necessary to have all fractional powers of x in the field. The field $K_0 = C(\{x^{1/q^\ell} \mid \ell \in \mathbb{N}\})$ would suffice.

Corollary 7. *Consider the system (5) in case M with* $A \in \mathrm{gl}_n(K), B \in \mathrm{GL}_n(K)$ *satisfying the consistency condition (6). It is equivalent over* K *to a system (7) with nilpotent constant* $\tilde{A} \in \mathrm{gl}_n(C)$ *and constant* $\tilde{B} \in \mathrm{GL}_n(C)$ *satisfying* $q\tilde{A}\tilde{B} = \tilde{B}\tilde{A}$*. It is equivalent over* $K(\log(x))$ *to a system (7) with* $\tilde{A} = 0$ *and constant* $\tilde{B} \in \mathrm{GL}_n(C)$.

Proof. Making a change of variables $x = t^N$ with a suitable positive integer N, if necessary, we can assume that A, B have entries in $C(x)$. By Theorem 1, there is a gauge transformation changing (5) into $\delta U = \bar{A}U$, $\sigma U = \bar{B}(x)U$ with constant $\bar{A}$ having rational eigenvalues. By a constant transformation, we can assume that $\bar{A}$ is in Jordan canonical form. If $D = \mathrm{diag}(r_1, ..., r_n)$ is the diagonal of $\bar{A}$, then $U = \mathrm{diag}(x^{r_1}, ..., x^{r_n})Z$ is a gauge transformation in $\mathrm{GL}_n(K)$ changing the system into (7) where now $\tilde{A} = \bar{A} - D$ is nilpotent. The matrix $\tilde{B}(x)$ has entries in $C[x^{1/N}, x^{-1/N}]$ for some positive integer N and satisfies the consistency condition. Writing $\tilde{B}(x) = \sum_{m=-t}^{t} B_m x^{m/N}$ yields $q\tilde{A}\, B_m - B_m\tilde{A} = \frac{m}{N}B_m$ for any $m \in \{-t, \dots, t\}$. Since $\tilde{A}$ is nilpotent, so is the operator mapping X to $q\tilde{A}\, X - X\tilde{A}$. Hence $B_m = 0$ unless $m = 0$. This proves the first statement.

In order to prove the second statement it suffices to use the first and to make the gauge transformation $V = \exp(-\log(x)\tilde{A})Z$. Observe that the consistency condition for $\tilde{A}, \tilde{B}$ is $q\tilde{A}\tilde{B} = \tilde{B}\tilde{A}$ and that is implies $\exp(q\log(x)\tilde{A})\tilde{B} = \tilde{B}\exp(\log(x)\tilde{A})$. ∎

We now turn to a proof of Theorem 2. In a first step, the consistency condition will be used in the form of Lemma 1 to characterize the singularities of the equation $\delta(Y) = A(x)Y$. We say that a singular point x_1 of the equation $\delta(Y) = A(x)Y$ is an *apparent singular point* if there is a fundamental solution matrix whose entries are in $C\{x - x_1\}[(x - x_1)^{-1}]$ [8]. Note that this condition is the same as saying that there is an equivalent system for which x_1 is a regular point. To see this note that truncating the entries of such a fundamental solution matrix at a sufficiently high power, we obtain a matrix G such the gauge transformation $Y = GZ$ leads to an equivalent system for which x_1 is a regular point. In Lemma 8 we show that the finite singular points must be apparent singular points (with the exception of 0 in cases Q and M). In cases Q and M, Lemma 9 shows the effect of consistency, that is Lemma 1, on the structure of local solutions at these points. Finally, monodromy theory is used to show that the differential equation is equivalent to $\delta z = Mz$ with some constant matrix M and that the spectrum of M has the desired properties. In case S, Lemma 11 states the first consequence for the solutions at infinity. Then we have to work considerably harder to arrive at the wanted reduced form of the differential equation (c.f., Proposition 12.) The rest of the conclusions of Theorem 2 will follow easily.

[8] Although some authors use this term to mean that the equation has a fundamental solution matrix *holomorphic* at x_1, we will use the above extended meaning throughout this work.

The entries of A and B have coefficients that lie in a countable algebraically closed field so we may replace C, if necessary, with a countable algebraically closed field, again denoted by C. Furthermore, we may assume that C is a subfield of the complex numbers $\mathbb{C}$. In the rest of the proof it can be verified that all equivalent systems can be chosen to have entries in this $C(x)$.

Lemma 8. *Consider δ, σ as in cases S,Q or M and a system (5) satisfying the consistency condition (6). Then each finite singular point ξ of the differential equation $\delta Y = AY$, except maybe 0 in cases Q and M, is an apparent singular point.*

Remark: Using the Lemma, it can be shown (see e.g. [9]) that the system (5) is equivalent to a system (7) where $\tilde{A}$ and $\tilde{B}$ have entries in $C[x, x^{-1}]$ in cases Q and M and $\tilde{A}, \tilde{B} \in C[x]$ in case S. We do not need this statement in our proof.

Proof. Consider the differential equation

$$\delta(Y) \;=\; AY. \tag{15}$$

By Lemma 1, it is equivalent to

$$\delta Z \;=\; \mu^N \sigma^N(A) Z. \tag{16}$$

for any positive integer N. Let $s(x) = \sigma(x) \in C[x]$. Let $\mathcal{S}$ be the set of finite singularities of (15), except 0 in cases Q and M and let $\mathcal{S}_N$ be the analogous set for (16). Note that $x_1 \in \mathcal{S}_N$ if and only if $\sigma^N(x_1) = s(s(\ldots s(x_1)\ldots))$ is in $\mathcal{S}$. It can be verified in each of the cases S,Q and M that for any finite singularity $\xi \in \mathcal{S}$ and large enough N, there exists $x_1 \in \mathcal{S}_N$ not in $\mathcal{S}$ satisfying $\sigma^N(x_1) = \xi$. Since (15) and (16) are equivalent, there exists a basis of solutions of (16) in $C\{x - x_1\}[(x - x_1)^{-1}]^n$. Applying σ^{-N} yields a basis of solutions of (15) in $C\{x - \xi\}[(x - \xi)^{-1}]^n$ in cases S and Q. Substituting the branch $x = x_1 \left(1 + \frac{t-\xi}{\xi}\right)^{1/q^N}$ of t^{1/q^N}, we see that $\delta y(t) = A(t) y$ has a basis of solutions in $C\{t - \xi\}[(t - \xi)^{-1}]^n$ in case M as well.

Hence every finite singular point ξ of (15,) except 0 in cases Q and M, is an apparent singular point. ∎

We now consider the behavior of solutions of equations (5) at infinity. In general, if one has a linear differential equation $\delta Y = A(x) Y$ with $A(x) \in \mathrm{gl}_n(C((x^{-1})))$, there exists a formal fundamental solution matrix of the form

$$Y(x) \;=\; \Phi(x) x^L e^{Q(x)} \tag{17}$$

where $\Phi(x)$ is a formal power series in $x^{-1/r}$ for some integer r, L is a constant matrix and $Q(x) = \sum_{j=1}^{h} Q_j x^{r_j}$ where the Q_j are diagonal matrices with entries in C and the r_j are positive rational numbers with $r_h > r_{h-1} > \ldots > r_1 > 0$ (or $Q(x) \equiv 0$); furthermore L and $Q(x)$ commute (c.f., [7]).

In the rest of the proof of Theorem 2, we will treat cases Q and M together and then treat case S.

2.1 Proof of Theorem 2: Cases Q and M.

We begin by showing that under our hypotheses $Q(x) = 0$ in these cases.

Lemma 9. *Let $Y(x)$ be a formal fundamental solution matrix of $\delta(Y) = AY$ as in (17). In cases Q and M we then have that $Q(x) = 0$, i.e.∞ is a regular singular point.*

In the same way, it is shown that 0 also must be a regular singular point in those two cases.

Remark. We note that Lemmas 8 and 9 are sufficient to prove, in cases Q and M, that any $f(x) \in \mathbb{C}[[x]][x^{-1}]$ satisfying (11) must be in $\mathbb{C}(x)$. Using the argument of Corollary 3, we may assume that $f(x)$ is a component of a vector $y(x)$ of power series such that $y(x)$ is a solution of a consistent system (13). Lemma 9 implies that $f(x)$ converges in a neighborhood of 0 since this point is a regular singular point and Lemma 8 implies that $y(x)$ can be continued analytically to a meromorphic function on $\mathbb{C}$. Finally, $y(x)$ has at most polynomial growth since ∞ is also a regular singular point. Altogether, we obtain that $y(x)$ has rational components and so $f(x)$ is rational.

Proof. Let $Q(x) = \mathrm{diag}(p_1(x), \ldots, p_n(x))$. By Lemma 1, we have that $B(x)Y(x)$ is a solution of $\delta(Y) = \mu\sigma(A(x))Y(x)$ and $\sigma(Y(x))$ as well. Let $p(x)$ be a nonzero diagonal entry of $Q(x)$. By the uniqueness of the p_i (Theorem 2,[7]), we must have that for some j, $\sigma(p(x)) = p_j(x)$. This implies that the map $p(x) \mapsto \sigma(p(x))$ permutes the p_i . From this we conclude that for some m, that $\sigma^m(p(x)) = p(x)$, a contradiction in cases Q and M. Therefore we have that $Q(x) = 0$. ∎

Lemma 10. *In cases* Q *and* M *there exists a matrix $T \in \mathrm{gl}_n(C)$ such that $\lambda_1 - \lambda_2 \notin \mathbb{Z}^*$ for eigenvalues λ_1, λ_2 of T and a gauge transformation $Z = FY$, $F \in \mathrm{GL}_n(C(x))$, that transforms $\delta Y = A(x)Y$ into $\delta(Z) = TZ$.*

Moreover in case M, the matrix T has the following properties:

1. *Its eigenvalues λ are rational with smallest denominators prime to q, $0 \leq \lambda < 1$.*

2. *There exists a diagonalisable matrix D with integer eigenvalues commuting with T such that $T + D$ is conjugate to $q\,T$.*

Proof. Consider any local fundamental matrix $Y(x)$ of (15). Because of Lemma 8 it can be continued analytically to a meromorphic function on $\hat{\mathbb{C}}$. Let $Y(x\,e^{2\pi i})$ be the fundamental matrix obtained by going once around 0. There exists a constant invertible matrix $H \in \mathrm{GL}_n(\mathbb{C})$ such that

$$Y(x\,e^{2\pi i}) = Y(x)H,$$

the so-called monodromy matrix associated to Y and (15). It is well known that there exists a matrix $T \in \mathrm{gl}_n(\mathbb{C})$ such that $H = \exp(2\pi i T)$. Consider now $G(x) = Y(x)x^{-T} = Y(x)\exp(-\log(x)\,T)$. By construction, $G(x)$ is invertible for every x that is not a pole and $G(x\,e^{2\pi i}) = G(x)$, *i.e.* G is single valued. By the previous lemma, 0 and ∞ are regular singular points of (15) and therefore the growth of $G(x)$ as $x \to 0$ or $x \to \infty$ is at most polynomial (*i.e.* there exists $K, L > 0$ such that $|G(x)| \le L(|x| + |x|^{-1})^K$). Hence the meromorphic function G on $\mathbb{C}^*$ must be a rational function,*i.e.* $G \in \mathrm{GL}_n(\mathbb{C}(x))$. The transformation $y = G(x)z$ changes $\delta y = A(x)y$ into a system of differential equations having x^T as a fundamental solution, *i.e.* $\delta z = T\,z$. The matrices T and $G(x)$ satisfy $A = \delta(G)G^{-1} + GTG^{-1},\ \det(G) \neq 0$. Therefore the entries of T and the coefficients of the entries of G satisfy a finite number of equations and inequations with coefficients in C. The Hilbert Nullstellensatz can be applied and yields that there also exists a solution in C. Therefore we may assume that $T \in \mathrm{GL}_n(C)$ and $G(x) \in \mathrm{GL}_n(C(x))$. A further constant gauge transformation allows us to assume that T is in triangular form. To insure that the eigenvalues of T do not differ by nonzero integers, one can make a gauge transformation by a diagonal matrix S whose diagonal entries are powers of x (c.f., Lemma 3.11, [36]).

In case M, it also remains to show the stated properties. Observe that $B(x)Y(x)$ and $Y(x^q)$ are both fundamental solutions of $\delta Y = q\,A(x^q)\,Y$ – this was used before. Hence there exists a constant invertible matrix D such that

$$B(x)Y(x) = Y(x^q)\,D.$$

For the corresponding monodromy matrices this implies the relation $H = D^{-1}\,H^q\,D$ because x^q goes q times around 0 when x goes once. Therefore H and H^q are conjugate. As a first consequence, the mapping $\lambda \to \lambda^q$ must induce a permutation of the eigenvalues of H. Hence, for every eigenvalue λ of H there exists a positive integer ℓ such that $\lambda^{q^\ell} = \lambda$. The eigenvalues of H are therefore roots of unity and the smallest positive integer m satisfying $\lambda^m = 1$ must be prime to q. This shows that the matrix T with $H = \exp(2\pi i T)$ can be chosen having the first property.

We can assume that T is in Jordan canonical form $T = \operatorname{diag}(T_1, ..., T_r)$ with Jordan blocks T_i of size n_i, say. The mapping $\lambda \bmod \mathbb{Z} \to q\lambda \bmod \mathbb{Z}$ induces a permutation of the equivalence classes $\lambda \bmod \mathbb{Z}$ of the eigenvalues λ of T. Hence there is a permutation matrix P such that the diagonal blocks of $P^{-1}\, qT\, P$ have the same size and modulo $\mathbb{Z}$ the same eigenvalues as those of T. Therefore there is a diagonal matrix $D = \operatorname{diag}(d_1 I_{n_1}, ... d_r I_{n_r})$ with integer d_i such that the blocks of $P^{-1}\, qT\, P$ and $T + D$ have same size and same eigenvalues and thus the two matrices are conjugate. This proves the second property of the matrix T. ∎

We can now complete the proof of Theorem 2 in cases Q and M. From the above lemmas we know that there is a gauge transformation that transforms $\delta(Y) = AY$ into a new equation $\delta(Y) = TY$ where T has the stated properties. Apply the same gauge transformation to $\sigma(Y) = BY$ to yield $\sigma(Y) = \tilde{B}Y$. The consistency condition (6) implies that

$$\delta \tilde{B}(x) = \mu T \tilde{B}(x) - \tilde{B}(x) T.$$

Comparing the orders of the poles, we see that $\tilde{B}(x)$ can have no finite poles other than 0 and we have $\tilde{B}(x) \in C[x, x^{-1}]$. Now let $\tilde{B}(x) = \sum_{m=-m_0}^{m_0} x^m B_m$.

In case Q we obtain that $TB_m - B_m T = mB_m$ for all $m = -m_0, \ldots, m_0$. As the eigenvalues of the operator $X \mapsto TX - XT$ are exactly the differences $\lambda_1 - \lambda_2$ where λ_1, λ_2 are eigenvalues of T the condition for T shows that $B_m = 0$ unless $m = 0$. This proves the theorem in case Q.

In case M we then have $qTB_m - B_m T = mB_m$ for all $m = -m_0, ..., m_0$. Hence $B_m \neq 0$ can only hold for integers m such that there exist eigenvalues λ_1, λ_2 of T such that $q\lambda_1 - \lambda_2 = m$. This shows the remaining statement of the theorem in case M. ∎

2.2 Proof of Theorem 1: Case S

In the remaining case S, we consider again the formal fundamental solution matrix at infinity of the form (17). We will first show that under our hypotheses $r_h \leq 1$ in case S.

Lemma 11. *Let $Y(x)$ be a formal fundamental solution matrix of $\delta(Y) = AY$ as in (17). We then have that $r_h \leq 1$ in case S.*

Proof. Let $Q(x) = \operatorname{diag}(q_1(x), \ldots, q_n(x))$. Let $q(x) = a_h x^{r_h} + a_{h-1} x^{r_{h-1}} + \ldots + a_1 x^{r_1}$ be one of the q_i and assume $r_h > 1$. Note that for any integer m we have $q(x+m) = q(x) + m r_h x^{r_h - 1} +$ other terms of order less than $r_h - 1$. We have that $B(x)Y(x)$ is a solution of $\delta(Y) = A(x+1)Y(x)$. Therefore, by the uniqueness of the q_i (Theorem 2,[7]), we must have that for some j, $q(x+1) = q_j(x)$ modulo terms of order ≤ 0. This implies that the map $q(x) \mapsto q(x+1)$ permutes the q_i modulo non-positive terms. From this we conclude that for some m, that $q(x+m) = q(x)$ modulo terms of order less than $r_h - 1$, a contradiction. Therefore we have that $r_h \leq 1$. ∎

The following proposition concerns only linear differential equations whose formal solutions satisfy the conclusions of the Lemmas 11 and 8, with no mention of difference equations.

Proposition 12. *Let $Y(x)$ be a formal fundamental solution matrix as in (17) of the differential equation $\delta(Y) = AY, A \in \mathrm{gl}_n(C(x))$ with $r_h \leq 1$. Assume that all finite singular points of the differential equation are apparent in the sense of Lemma 8. Then there exists a diagonal matrix $\tilde{A}$ with constant entries and a gauge transformation $Z = FY$ such that Z satisfies $\delta(Z) = \tilde{A}Z$.*

Proof. For almost all directions θ there exist

1. sectors S and T of openings greater than π bisected by θ and $\theta + \pi$, respectively,
2. a number $R > 0$ and functions $\Phi_S(x)$ and $\Phi_T(x)$ analytic for $|x| > R$ in S and T, respectively, such that
   (a) $\Phi_S(x)$ and $\Phi_T(x)$ are asymptotic to $\Phi(x)$ as $x \to \infty$ in their respective sectors, and
   (b) $Y^S(x) = \Phi_S(x)x^L e^{Q(x)}$ and $Y^T(x) = \Phi_T(x)x^L e^{Q(x)}$ are solutions of $\delta Y = AY$.

This can be proved using multisummation (c.f., Section 7.8, [36]) but can also be obtained independently (c.f., the proposition of [29], p.85).

We write $Q(x) = \mathrm{diag}(q_1(x)I_1, \ldots, q_s(x)I_s)$ with distinct $q_i(x)$ and I_j identity matrices of an appropriate size. We can furthermore order the $q_i(x)$ so that in the direction $\arg(x) = \psi = \theta + \pi/2$ we have $\mathrm{Re}(q_j(x)) < \mathrm{Re}(q_k(x))$ for large $|x|$ if $j < k$. Let us first assume that there are no monomials of the form $x^\alpha, 0 < \alpha < 1$ in any of the q_i (we shall show below that this is indeed the case). We may then write $Q(x) = \Lambda x$ where $\Lambda = \mathrm{diag}(\lambda_1 I_1, \ldots, \lambda_s I_s)$ with $\mathrm{Re}(\lambda_j x) < \mathrm{Re}(\lambda_k x)$ for $j < k$ and $\arg(x) = \psi$. Let C^+ be the Stokes matrix defined by $Y^S(x) = Y^T(x)C^+$ in the component of the intersection of S and T that contains the line $\arg(x) = \psi$. Because of the ordering of the λ_i we have that C^+ is upper triangular with 1 on the diagonal (§3,[8]; c.f., Theorem 8.13, [36]).

We include a short proof of this fact for the convenience of the reader. Write $Y^S = (Y_1^S|...|Y_s^S)$ and $Y^T = (Y_1^T|...|Y_s^T)$ in block columns according to the subdivision of

Q and do the analogous subdivision for Φ. Split $L = \mathrm{diag}(L_1, ..., L_s)$ and write $C^+ = (C^+_{i,j})_{i,j=1,...,s}$ in corresponding blocks. First suppose that there is a non-vanishing block C^+_{ij},$i > j$, below the diagonal. Fix one such j. Then

$$Y_j^S(x) = \sum_{i=1}^{m} Y_i^T C_{ij}^+ \tag{18}$$

where $m \leq s$ is chosen as the last index i such that $C^+_{i,j} \neq 0$. By assumption we have $m > j$. Hence $Y_j^S \exp(-q_m(x))$ has a non-vanishing asymptotic expansion $\Phi_m(x)x^{L_m}C^+_{mj}$ as $x \to \infty$ on the line $\arg x = \psi$ contradicting the fact that it also has an expansion $\Phi_j(x)x^{L_j}\exp(q_j(x) - q_m(x))$ and hence vanishes faster than any power of x as $x \to \infty$ on this line because of $m > j$. Therefore there is no non-vanishing block C^+_{ij} below the diagonal. In the same way one shows that the diagonal blocks C_{jj} must equal I_j. This completes the proof that C^+ is upper triangular with 1 on the diagonal.

Similarly to C^+, let C^- be the Stokes matrix defined by $Y^S(x) = Y^T(x)C^-$ in the component of the intersection of S and T that contains the line $\arg(x) = \psi - \pi$. Note that $\mathrm{Re}(\lambda_j x) > \mathrm{Re}(\lambda_k x)$ on the line $\arg(x) = \psi - \pi$ if $j < k$. We therefore have that C^- is lower triangular with 1 on the diagonal. As all finite singular points are apparent, both $Y^S(x)$ and $Y^T(x)$ can be extended as meromorphic functions on $\mathbb{C}$ with finitely many poles. We therefore have $C^+ = C^- = I$. The matrix

$$F(x) = Y^S(x)e^{-\Lambda x} = Y^T(x)e^{-\Lambda x}$$

has entries that are meromorphic with finitely many poles and of polynomial growth at infinity, that is, rational entries. Therefore the transformation $Z = F^{-1}(x)Y$ yields the system $\delta(Y) = \Lambda Y$.

We now show that monomials of the form $x^\alpha, 0 < \alpha < 1$ cannot appear in any of the q_i. Assuming that this is not the case we will argue to a contradiction. We write each $q_i(x)$ as

$$q_i(x) = \lambda_i x + \text{ terms involving } x^\alpha \text{ with } 0 < \alpha < 1.$$

To fix notation, we assume that the same branches of $\log x$ are used to define $Y^S(x)$ and $Y^T(x)$ on the component of $S \cap T$ containing ψ but that on the other component we use $\arg x$ near $\psi - \pi$ in S and $\arg x$ near $\psi + \pi$ in T.

We define the Stokes matrix C^+ as above. Again we have that C^+ is upper triangular with 1 on the diagonal. We divide this matrix again $C^+ = (C^+_{ij})$ according to the diagonal blocks of Q. We do not claim that C^+ is the identity matrix because different determinations of the powers are used in the definitions of $Y^S(x)$ and $Y^T(x)$ in the components of the intersection of S and T that contain the lines $\arg(x) = \psi \pm \pi$. Nevertheless $Y^S(x)$ and

$Y^T(x)$ are meromorphic on $\mathbb{C}$ with finitely many poles by the assumption of the Proposition so $Y^S(x) = Y^T(s)C^+$ also holds in this component. This implies that $C^+_{ij} = 0$ if $i < j$ and $\lambda_i \neq \lambda_j$ because otherwise $\mathrm{Re}(\lambda_i x) > \mathrm{Re}(\lambda_j x)$ on the line $\arg(x) = \psi + \pi$, thus $\mathrm{Re}(q_i(x)) > \mathrm{Re}(q_j(xe^{-2\pi i}))$ for large x on this line and therefore by (18) the right hand side would grow faster as $|x| \to \infty$ on that line than the left hand side if $C^+_{ij} \neq 0$, a contradiction.

To complete the argument that no $x^\alpha, 0 < \alpha < 1$ appear in the $q_i(x)$, note the following:

> If some $q_i(x)$ appears in $Q(x)$ we must have, for all integers m, that the conjugate $q_i(xe^{2m\pi i})$ also appears ([7], §1).

Continuing, we assume that some $q_i(x)$ contains a monomial $x^\alpha, 0 < \alpha < 1,$ and let j be the smallest index for which this is true. We claim that $\lambda_i \neq \lambda_j$ for $i < j$. If not then, for some i, $q_i(x) = \lambda x$ and $q_j(x) = \lambda x + cx^\alpha +$ lower order terms. Among the conjugates $q_j(e^{2m\pi i}x)$, m integer, we can find one such that $\mathrm{Re}(ce^{2m\pi i\alpha}x^\alpha)$ tends to $-\infty$ as x approaches infinity along the line $\arg(x) = \psi$. By the minimality of j, this is also the case for $m = 0$. We then have $\mathrm{Re}(q_i(x)) > \mathrm{Re}(q_j(x)))$ eventually along this line as well, contradicting $i < j$. Hence the blocks C^+_{ij} do not only vanish if $i > j$, but as seen above also if $i < j$ (since $\lambda_i \neq \lambda_j$ in this case).

Therefore $Y^S_j = Y^T_j$ for the block column corresponding to $q_j(x)$. This is impossible since near $\arg x = \psi + \pi$, $Y^T_j(x)$ is asymptotic to something times $e^{q_j(x)}$ while $Y^S_j(x)$ is asymptotic to something times $e^{q_j(xe^{-2\pi i})}$. This contradiction allows us to conclude that no fractional powers appear in the $q_i(x)$ and so completes the proof. ∎

We can now complete the proof of Theorem 2 in case S. From Proposition 12 we know that there is a gauge transformation that transforms $\delta(Y) = AY$ into a new equation $\delta(Y) = \tilde{A}Y$ where $\tilde{A}$ is the diagonal matrix $\mathrm{diag}(a_1, \ldots, a_n)$ with constant entries. Apply the same gauge transformation to $\sigma(Y) = BY$ to yield $\sigma(Y) = \tilde{B}Y$, $\tilde{B} = (b_{i,j})$. Since the a_i are constant, the consistency condition (6) implies

$$\delta(b_{i,j}) \quad = \quad (a_i - a_j)b_{i,j} \tag{19}$$

If $a_i \neq a_j$ then $b_{i,j} = 0$ since (19) then has no nonzero solution in $C(x)$. If $a_i = a_j$, then $\delta(b_{i,j}) = 0$. Therefore $\tilde{B}$ has constant entries. Equation (6) now implies that $\tilde{A}$ and $\tilde{B}$ commute. This implies that there is a matrix $D \in \mathrm{GL}_n(C)$ which commutes with $\tilde{A}$ such that $D\tilde{B}D^{-1}$ is upper diagonal. ∎

3 Reduction of systems of difference equations

Here we present results analogous to Section 2 for systems of two difference equations with shifts having irrational quotient, for systems of two q-difference equations with "independent" q and for systems of two Mahler equations with independent q.

We consider two commuting C-algebra endomorphisms σ_1, σ_2 on k extending the trivial automorphism on C, more specifically, the three cases of couples (σ_1, σ_2) below.

case 2S: Two shift operators σ_j defined by $\sigma_1(x) = x+1$ and $\sigma_2(x) = x+\alpha$ where $\alpha \in C\backslash\mathbb{Q}$.

case 2Q: Two q-dilation operators σ_j, $j = 1, 2$, defined by $\sigma_j(x) = q_j\, x$ with multiplicatively independent[9] $q_j \in C$, $|q_j| \neq 0$, q_j not a root of unity. We also assume that at least one of the q_j does not have modulus 1, without loss of generality $|q_1| \neq 1$.[10] When considering σ_j on the Riemann surface $\hat{\mathbb{C}}$ of the logarithm, we fix logarithms of q_j used to determine $q_j x$ in $\hat{\mathbb{C}}$ for given $x \in \hat{\mathbb{C}}$.

case 2M: Two Mahler operators σ_j, $j = 1, 2$, defined by $\sigma_j(x) = x^{q_j}$ with some multiplicatively independent positive integers q_j.

More precisely, we will consider systems

$$\sigma_j(Y) = \quad B_j\, Y,\ j = 1, 2 \tag{20}$$

with $B_j \in \mathrm{GL}_n(k)$ that are *consistent*, that is B_1 and B_2 satisfy the consistency condition given by

$$\sigma_1(B_2)B_1 = \sigma_2(B_1)B_2. \tag{21}$$

As in Section 2, the consistency condition is closely related to the commutativity of σ_1, σ_2. Both are fundamental for our approach. As (6) did in Section 2, the consistency condition guarantees that $\sigma_1(\sigma_2(Z)) = \sigma_2(\sigma_1(Z))$ holds for any solution Z of the system (20) in any extension of $C(x)$. The other remarks following (6) apply analogously.

We say that (20) is *equivalent* (over k) to a system

$$\sigma_j(Z) = \quad \tilde{B}_j Z,\ j = 1, 2 \tag{22}$$

with $\tilde{B}_j \in \mathrm{GL}_n(k)$ if for some $G \in \mathrm{GL}_n(k)$,

$$\tilde{B}_j = \sigma_j(G)B_jG^{-1}, j = 1, 2 \tag{23}$$

[9] *i.e.* there are no nonzero integers n_j such that $q_2^{n_2} = q_1^{n_1}$.

[10] Alternatively to $|q_1| \neq 1$, one can assume that $|q_1| = |q_2| = 1$, q_2 not a root of unity, q_1 transcendental over $\mathbb{Q}$ or q_1 algebraic over $\mathbb{Q}$ such that its minimal polynomial has a root in $\mathbb{C}$ of absolute value not equal to 1. See Remark 21.

that is, if (22) comes from (20) via the *gauge transformation* $Z = GY$. Note that the property of consistency is preserved under equivalence. In the present context we can prove

Theorem 13. *In cases 2S and 2Q, the system (20) satisfying the consistency condition (21) is equivalent over k to a system (22) with constant invertible commuting $\tilde{B}_j$, $j = 1, 2$, that is $\tilde{B}_j \in \mathrm{GL}_n(C)$, $j = 1, 2$, and $\tilde{B}_2\, \tilde{B}_1 = \tilde{B}_1\, \tilde{B}_2$.*

In case 2M, the system (20) satisfying the consistency condition (21) is equivalent over $K = C(\{x^{1/s} \mid s \in \mathbb{N}^*\})$ to a system (22) with constant invertible commuting $\tilde{B}_1$, $\tilde{B}_2$.

Remark: 1. When $n = 1$ in case 2S, the result of this theorem is a reformulation of Lemma 3.1 in [17] where the authors give a purely algebraic proof of this special case.

2. In case 2S, one of the equations can be made diagonal by a (polynomial) transformation $Z = HU$, $H = \exp(Nx)$ with a certain nilpotent matrix N commuting with its coefficient matrix.

3. In case 2M, the statement of the theorem also holds for $B_j \in \mathrm{GL}_n(K)$, $j = 1, 2$, because fractional powers of x can be removed by some change of variables $x = t^N$ with a suitable positive integer N.

The proof of this theorem will be given for each of the cases separately in Sections 3.1, 3.2 and 3.3. As in Section 2, we present a few consequences of the theorem before presenting these proofs. These concern a system

$$S_j(f(x)) = \sigma_j^{m_j}(f(x)) + b_{j,m_j-1}(x)\sigma_j^{m_j-1}(f(x)) + \ldots + b_{j,0}(x)f(x) = 0,\ j = 1, 2 \quad (24)$$

with $b_{j,i}(x) \in \mathbb{C}(x)$.

In view of the simplest nontrivial system of this form, $y(x+1) = y(x)$, $y(x+\alpha) = y(x)$, in case 2S with non-real α, it is necessary to consider elliptic functions if we are interested in solving (24) using meromorphic functions. We recall the functions needed here (see [32], section 23.2). The Weierstrass $\wp$-function is the unique 1- and α-periodic meromorphic function that has exactly one double pole in the basic parallelogram with vertices $\pm 1/2 \pm \alpha/2$ and satisfies $\wp(x) = \frac{1}{x^2} + \mathcal{O}(x)$ as $x \to 0$. All elliptic (*i.e.* meromorphic 1- and α-periodic) functions can be expressed as rational functions of $\wp$ and $\wp'$. The Weierstrass ζ-function is the odd antiderivative of $\wp$. It satisfies $\zeta(x+1) = \zeta(x) + 2\eta_1, \zeta(x+\alpha) = \zeta(x) + 2\eta_2$ for all x, where η_j are certain constants such that the vectors (η_1, η_2) and $(1, \alpha)$ are linearly independent. The Weierstrass σ-function is the solution of $\sigma'/\sigma = -\zeta$ with $\sigma'(0) = 1$. It is an entire function vanishing at the origin and satisfies $\sigma(x+1) = e^{2\eta_1 x + \eta_1}\sigma(x)$, $\sigma(x+\alpha) = e^{2\eta_2 x + \eta_2 \alpha}\sigma(x)$ for $x \in \mathbb{C}$. We introduce an additional function ρ by $\rho(\delta, x) = \sigma(x + \delta)/\sigma(x)$.[11] It is a meromorphic function with a simple pole at the origin that satisfies

$$\rho(\delta, x + 1) = e^{2\eta_1\delta}\rho(\delta, x) \text{ and } \rho(\delta, x + \alpha) = e^{2\eta_2\delta}\rho(\delta, x).$$

Corollary 14. *Consider σ_1 and σ_2 as in case 2S.*

If $f(x)$ is a meromorphic function in $\mathbb{C}$ that solves a system (24) and α is nonreal, then

$$f(x) = \sum_{i=1}^{I}\sum_{k=0}^{K} r_{i,k}(x)g_{i,k}(x)\zeta(x)^k e^{\alpha_i x}\rho(\delta_i, x) \tag{25}$$

where $\alpha_i, \delta_i \in \mathbb{C}$, $r_{i,k}(x) \in \mathbb{C}(x)$ and $g_{i,k}(x)$ are elliptic functions. The latter and the functions ζ and ρ are taken with respect to the periods 1 and α. Conversely, any such function satisfies a pair of linear difference equations with rational coefficients.

If $f(x)$ is a meromorphic function in $\mathbb{C}$ that solves a system (24) and α is real or $f(x)$ has only finitely many poles, then

$$f(x) = \sum_{i=1}^{I} r_i(x)e^{\alpha_i x} \tag{26}$$

where $\alpha_i \in \mathbb{C}$ and $r_i(x) \in \mathbb{C}(x)$.

If $f(x) \in \mathbb{C}[[x^{-1}]][x]$ satisfies a system (24) then $f(x)$ is rational.

Remark: 1. In the case of real irrational α, a slight extensions of the proof shows that the statement also holds if $f(x)$ is a function on the real line continuous in all but finitely many points solving a system (24). Indeed, it suffices to use the vector space E of all functions on the real line continuous in all but finitely many points and to apply Fejér's Theorem instead of the simple Fourier series. This also implies that the given $f(x)$ is analytic at the points of continuity and can be continued analytically to a meromorphic function with finitely many poles on the whole complex plane. Similar extensions can be obtained in the second part of the subsequent corollary and in Corollary 16. A similar reasoning is also crucial in the proof of the Theorem in case 2M.

2. In [13], Bézivin and Gramain consider *entire* solutions of (24) for case 2S under the assumption that $\alpha \in \mathbb{C}\backslash\mathbb{R}$. They show that such solutions must be of the form given in (26). They generalize this result to entire functions of s variables satisfying $2s$ difference equations with respect to suitably independent multi-shifts. The techniques are similar to those mentioned in the Remark following Corollary 5. In [17], Brisebarre and Habsieger replace the condition that $\alpha \in \mathbb{C}\backslash\mathbb{R}$ with $\alpha \in \mathbb{C}\backslash\mathbb{Q}$ but need several other

[11] This function appears already in the classical works of [4, 26], but seems to be not so well known.

nontrivial technical conditions on the coefficients of (24) to show that entire solutions are of the form given in (26). The techniques are essentially algebraic, reducing this problem to a similar problem for equations with constant coefficients in a manner different from our approach. In [14], Bézivin shows essentially the statement in remark 1, *i.e.* that for $\alpha \in \mathbb{R}\backslash\mathbb{Q}$, a solution $f(x)$, continuous on $\mathbb{R}$, of (24) is of the form given in (26). Using properties of skew polynomial rings, Bézivin reduces the problem to the case of constant coefficient equations. In [31], Marteau considers systems of scalar equations of the form $\sum_{i=0}^{N} a_i(x) f(x+\alpha_i) = 0$ where the $a_i(x)$ are polynomials and the $\alpha_i \in \mathbb{C}$. He shows that, under certain restrictions on the α_i, real valued continuous solutions and entire solutions of such systems must also be of the form given in (26). Using essentially algebraic techniques, constant coefficients systems are considered in [28, 31].

Corollary 15. *Consider σ_1 and σ_2 as in case 2Q. If $\alpha := \log(q_2)/\log(q_1)$ is nonreal and $f(x)$ is a meromorphic function on the Riemann surface $\hat{\mathbb{C}}$ of the logarithm and if $f(x)$ is a solution of a system (24) then*

$$f(x) = \sum_{i=1}^{I}\sum_{j=0}^{J}\sum_{k=0}^{K} r_{i,j,k}(x)\log(x)^j x^{\alpha_i} g_{i,k}(t)\zeta(t)^k \rho(\delta_i, t),\ t = \log(x)/\log(q_1), \tag{27}$$

where $\alpha_i, \delta_i \in \mathbb{C}$, $r_{i,j,k}(x) \in \mathbb{C}(x)$ and $g_{i,k}(t)$ are elliptic functions. The latter and the functions ζ and ρ are taken with respect to the periods 1 and α. Conversely, any such function satisfies a pair of linear q-difference equations with rational coefficients.

If $f(x)$ is a meromorphic function on $\hat{\mathbb{C}}$ solving a system (24) and α is real or $f(x)$ has only finitely many poles then

$$f(x) = \sum_{i,j=0}^{I} r_{ij}(x) x^{\alpha_i} \log(x)^j \tag{28}$$

where $\alpha_i \in \mathbb{C}$ and $r_{ij} \in \mathbb{C}(x)$.

If $f(x) \in \mathbb{C}[[x]][x^{-1}]$ satisfies a system (24) then $f(x)$ is rational.

Remark: In [15] Bézivin and Boutabaa use p-adic techniques to show that if $f(x) \in F[[x]][x^{-1}]$, where F *is the field of algebraic numbers*, satisfies a system (24) for case 2Q with $q_1, q_2 \in K$ multiplicatively independent, then $f(x)$ is rational. They prove a similar result for F any characteristic 0 field assuming q_1 and q_2 are algebraically independent over $\mathbb{Q}$. In [14], Bézivin shows that for q_1, q_2 multiplicatively independent positive real numbers, a solution of (24) that is continuous on the $]0, \infty[$ is of the form (28). The proof again uses properties of skew polynomial rings and constant coefficient equations.

In case 2M, equations like $\log(\log(x^q)) = \log(\log(x)) + \log(q)$ yield interesting solutions. This suggests to consider the Riemann surface $\check{\mathbb{C}}$ of $\log(\log(x))$. It is obtained by deleting the point $1e^{i0}$, *i.e.* the point with logarithm 0, from the Riemann surface $\hat{\mathbb{C}}$ of the logarithm and taking the universal covering of the remaining manifold. It is biholomorphically mapped to $\hat{\mathbb{C}}$ by $t = \log(x)$, biholomorphically to $\mathbb{C}$ by $s = \log(\log(x))$.

Corollary 16. *Consider σ_1 and σ_2 as in case 2M.*

If $f(x)$ is a meromorphic function on the universal cover of the open punctured unit disk $D(0,1) \setminus \{0\}$ (or on the universal cover of the annulus $\{x \in \mathbb{C} \mid |x| > 1\}$) solving a system (24), then $f(x)$ can be continued analytically to a meromorphic function on $\check{\mathbb{C}}$ and

$$f(x) = \sum_{i,j=0}^{I} r_{ij}(x)(\log(x))^{\alpha_i} \log(\log(x))^j \tag{29}$$

where $\alpha_i \in \mathbb{C}$ and $r_{ij} \in \mathbb{C}(\{x^{1/r} \mid r \in \mathbb{N}^*\})$. Conversely, any such function satisfies a pair of linear Mahler equations with rational coefficients.

If $f(x)$ is a meromorphic function on $\hat{\mathbb{C}}$ solving a system (24) then

$$f(x) = \sum_{j=-I}^{I} r_j(x)(\log(x))^j \tag{30}$$

where $r_j \in \mathbb{C}(\{x^{1/r} \mid r \in \mathbb{N}^*\})$.

If $f(x) \in \mathbb{C}[[x]][x^{-1}]$ satisfies a system (24) then $f(x)$ is rational.

Remark: As noted in the Introduction, the last statement of the above corollary was recently proved by Adamczewski and Bell in [1]. Their tools include a local-global principle to reduce the problem to a similar problem over finite fields, Chebotarev's Density Theorem, Cobham's Theorem and some asymptotics - all very different from the techniques used in the present work.

If one is only interested in proving the last statement, the application of Theorem 13 in the proof of this corollary can be replaced with the weaker Proposition 22. This is discussed in the remarks following Proposition 22.

We now turn to the proof of the three corollaries.

Proof. We will prove these three corollaries in parallel, diverging from this plan only when the cases force us to. We can assume without loss in generality that $b_{j,0}(x) \neq 0$, $j = 1, 2$. Otherwise in cases 2S and 2Q, we can simply apply the inverses of σ_1 or σ_2. In case 2M, we rewrite the system as a system for a new function $\tilde{f}(x) = \sigma_1^a \sigma_2^b(f(x))$ with suitable positive integers a, b, applying some powers of σ_1 or σ_2 to the equations. We then first obtain that $\tilde{f}(x)$ is as stated in Corollary 16. In the first two cases of this corollary, it follows immediately that $f(x)$ also has the wanted form. In the last case, we obtain a series $f(x) \in \mathbb{C}[[x]][x^{-1}]$ such that $\sigma_1^a\sigma_2^b(f(x)) = f(x^{q_1^a q_2^b}) \in \mathbb{C}(x)$. As shown at the end of the proof of Corollary 3, this implies that $f(x) \in \mathbb{C}(x)$.

We proceed as for the corollaries concerning differential and difference equations. The first part is the same for all the cases. Let E be the vector space of all meromorphic functions f on $\mathbb{C}$ for the first two cases of Corollary 14, the vector space of all meromorphic functions g on $\hat{\mathbb{C}}$ for the first two cases of Corollary 15 and the second case of Corollary 16, and the vector space of all meromorphic functions on $D(0,1) \setminus \{1\}$ in the first case of Corollary 16. Let $E = \mathbb{C}[[x^{-1}]][x]$ or $E = \mathbb{C}[[x]][x^{-1}]$ or $E = \cup_{s\in\mathbb{N}^*}\mathbb{C}[[x^{1/s}]][x^{-1/s}]$ in the remaining cases concerning formal power series, respectively. Let $L = \mathbb{C}(x)$ for Corollaries 14, 15 and $L = K = \mathbb{C}(\{x^{1/r} \mid r \in \mathbb{N}^*\})$ for Corollary 16. In all cases, σ_j, $j = 1, 2$, are extended in the canonical way to automorphisms of E and the extensions commute.

Consider the L-subspace W of E generated by $\sigma_1^m\sigma_2^n(f)$, $m = 0, ..., m_1 - 1,\ n = 0, ...m_2 - 1$. By (24), W is invariant under σ_j and σ_j^{-1}; here the facts that σ_j commute and that $b_{j,0}(x) \neq 0$ are used.

Let $w_1, ..., w_s$ be an L-basis of W and let $w = (w_1, ..., w_s)^T$. Then we have that

$$\sigma_j(w) = B_j(x)w,\ j = 1, 2 \tag{31}$$

with $B_j \in \mathrm{GL}_s(L)$ because the components of $\sigma_j(w)$ are again a basis of W. The coefficient matrices of (31) satisfy the consistency condition (21). Indeed,

$$0 = \sigma_1(\sigma_2(w)) - \sigma_2(\sigma_1(w)) = (\sigma_1(B_2)B_1 - \sigma_2(B_1)B_2)w$$

and as the components of w form a basis we obtain (31).

Now we apply Theorem 13 to the system $\sigma_j(Y) = B_j(x)Y,\ j = 1, 2$. It yields a gauge transformation $Z = GY$, $G \in \mathrm{GL}_n(L)$, that transforms the system to $\sigma_j(Z) = \tilde{B}_jZ,\ j = 1, 2$ where $\tilde{B}_j$ are constant commuting matrices. The vector $z = Gw \in W^s$ satisfies $\sigma_j(z) = \tilde{B}_jz,\ j = 1, 2$. Once we have proved that its components $z_1, ..., z_s$ have the form desired in each of the cases of the corollaries, the same will be true for $f(x)$ because the z_i again form a basis of W – only for the last part of Corollary 16 this has to be modified somewhat.

It remains to solve the system

$$\sigma_j(z) = \tilde{B}_jz,\ j = 1, 2 \tag{32}$$

with constant invertible commuting $\tilde{B}_j$ in each of the spaces E.

We now proceed as follows. The three corollaries contain nine cases in total. In each of these cases we will consider functions meromorphic on a given domain or formal Laurent series and show that they are of the desired form. Once we have finished this we will prove the converse statements.

We first consider case 2S on the space of meromorphic functions on $\mathbb{C}$. If α is nonreal, then (32) can be solved using the function ρ and exponential functions. Consider two commuting logarithms of $\tilde{B}_j$, *i.e.* commuting matrices L_j such that $\tilde{B}_j = \exp(L_j)$, $j = 1, 2$. Then there are uniquely determined (commuting) matrices Δ and C such that

$$\begin{array}{rcl} 2\eta_1\,\Delta + C &=& L_1 \\ 2\eta_2\,\Delta + \alpha C &=& L_2 \end{array}$$

where (η_1, η_2) are as in the above definition of the ζ-function. The matrix-valued function[12] $Z(x) = e^{Cx}\rho(\Delta, x) = \frac{1}{\sigma(x)}e^{Cx}\sigma(x+\Delta)$ has meromorphic entries and satisfies $Z(x+1) = e^C e^{2\eta_1\Delta}Z(x) = \tilde{B}_1 Z(x)$ and similarly $Z(x+\alpha) = \tilde{B}_2 Z(x)$. If z is a vectorial solution of (32), then $c(x) = Z(x)^{-1}z(x)$ is 1- and α-periodic and meromorphic. Hence its components are elliptic functions.

In order to express $\sigma(x+\Delta)$ using scalar functions we proceed as it is well known for the exponential. Let T be invertible such that $J = T^{-1}\Delta T$ is in Jordan canonical form and let $J = D + N$ with diagonal D and nilpotent N, $D = \mathrm{diag}(d_1, ..., d_s)$. Then

$$\sigma(x+\Delta) = T\sigma(x+D)\left(I + \sum_{k=1}^{s-1}\frac{1}{k!}\sigma(x+D)^{-1}\sigma^{(k)}(x+D)N^k\right)T^{-1}\ ;$$

here $\sigma(x+D) = \mathrm{diag}(\sigma(x+d_1), ..., \sigma(x+d_s))$ and similarly $\sigma(x+D)^{-1}\sigma^{(k)}(x+D)$ is the diagonal matrix of the quotients $\frac{\sigma^{(k)}(x+d_j)}{\sigma(x+d_j)}$ which in turn can be expressed using powers of $\zeta(x+d_j)$ and elliptic functions. Finally as $\zeta(x+d_j) - \zeta(x)$ are elliptic, $\frac{1}{\sigma(x)}\sigma(x+\Delta)$ can be expressed using the functions $\rho(d_j, x)$, powers of $\zeta(x)$ and elliptic functions.

If α is real and irrational or the number of poles of f is finite, then we consider again commuting logarithms L_j of the coefficient matrices $\tilde{B}_j$. Put $c = \exp(-L_1 x)z$. Then c satisfies $\sigma_1(c) = c$ and $\sigma_2(c) = \tilde{B}c$, $\tilde{B} = \tilde{B}_2 e^{-\alpha L_1}$. Then c cannot have a pole at all, because otherwise by the two equations all points of the lattice $\{k + m\alpha | k, m \in \mathbb{Z}\}$ would be poles and hence the set of poles would be dense for real irrational α or infinite otherwise.

As c is 1-periodic and entire, it can be expanded in a Fourier series on $\mathbb{C}$. On some horizontal strip S of finite width, we have the uniformly convergent series

$$c(x) = \sum_{k=-\infty}^{\infty} c_k e^{2\pi i k x},\ x \in S.$$

[12]See [24], chapter V, for the extension of entire functions to matrices.

The second equation satisfied by c, *i.e.*

$$c(x+\alpha) = \tilde{B}c(x) \text{ for } x \in S$$

implies that $\tilde{B}c_k = \exp(2\pi i k\alpha)c_k$ for $k \in \mathbb{Z}$. If $c_k \neq 0$ for some k then $\exp(2\pi i k\alpha)$ is an eigenvalue of $\tilde{B}$. As there are only finitely many such eigenvalues, the above Fourier series can contain only a finite number of terms. This proves that $c(x) = \sum_{k=-k_0}^{k_0} c_k e^{2\pi i k x},\ x \in \mathbb{C}$ with some positive integer k_0. As a consequence $z = \exp(L_1 x)c$ is of the desired form.

On $E = \mathbb{C}[[x^{-1}]][x]$, we can assume that $\tilde{B}_1$ is diagonal according to a remark following the theorem. Hence it suffices to show that the solution of $\sigma_1(z(x)) = d\, z(x)$ in E is a constant for every complex number d. Write such a solution as $z(x) = \sum_{k=k_0}^{\infty} c_k\, x^{-k}$ with some $k_0 \in \mathbb{Z}$ and $c_k \in \mathbb{C}$, $k \geq k_0$. Unless $z = 0$, we can assume that $c_{k_0} \neq 0$. Comparing the coefficients of x^{-k_0}, we find that $d\, c_{k_0} = c_{k_0}$. Hence $d = 1$. Comparing the coefficients of x^{-k_0-1} we find that $c_{k_0+1} = c_{k_0+1} - c_{k_0} k_0$. Hence $k_0 = 0$. The new series $\tilde{z}(x) = z(x) - c_0$ is again a solution of $\sigma_1(\tilde{z}(x)) = \tilde{z}(x)$ but unless $\tilde{z}(x) = 0$, the corresponding series $\tilde{z}(x) = \sum_{k=k_1}^{\infty} c_k\, x^{-k}$ has a positive k_1 which is impossible as seen above. Hence $\tilde{z}(x) = 0$ and z is a constant.

In case 2Q and in the space of meromorphic functions on $\hat{\mathbb{C}}$, we simply put $t = \log(x)/\log(q_1)$ and consider z as a vectorial function of t now. The system $\sigma_j(z) = \tilde{B}_j z,\ j = 1,2$ is then transformed into a system $z(t+1) = \tilde{B}_1 z(t)$, $z(t+\alpha) = \tilde{B}_2 z(t)$, for a function z meromorphic on $\mathbb{C}$. The above considerations then prove that the entries of z have the desired form in the cases of nonreal and real α. The only difference is that in the expansion of $x^C = e^{C\log(q_1)t}$ using scalar functions, logarithms of x may occur.

In case 2Q, $E = \mathbb{C}[[x]][x^{-1}]$, we write a solution of $\sigma_j(z(x)) = \tilde{B}_j\, z(x)$, $j = 1,2$, as a series $z(x) = \sum_{k=k_0}^{\infty} c_k\, x^k$ with some $k_0 \in \mathbb{Z}$ and $c_k \in \mathbb{C}^s$, $k \geq k_0$. We obtain that each c_k satisfies $\tilde{B}_j c_k = q_j^k c_k$, $j = 1,2$. Therefore unless $c_k = 0$ for some k, the numbers q_j^k must be eigenvalues of $\tilde{B}_j$ for $j = 1,2$. As there are only finitely many eigenvalues we obtain that $z(x) \in (\mathbb{C}[x,x^{-1}])^s \subset (\mathbb{C}(x))^s$.

In case 2M and in the space of meromorphic functions on $D(0,1) \setminus \{1\}$, we put $s = \frac{\log(\log(x))}{\log(q_1)}$ and consider z as a function of s now. Proceeding as above [13] with the resulting system of difference equations with irrational $\alpha = \frac{\log(q_2)}{\log(q_1)}$, the result follows readily. The result concerning meromorphic functions on the annulus $\{x \in \mathbb{C} \mid |x| > 1\}$ is reduced to the one on the punctured unit disk by the change of variables $x \to 1/x$. Observe that in this case, the reduction to a constant system uses a gauge transformation with coefficients in $\mathbb{C}(\{x^{1/r} \mid r \in \mathbb{N}^*\})$.

In case 2M and in the space of meromorphic functions on $\hat{\mathbb{C}}$, we have to find solutions of the system $\sigma_j(z(x)) = \tilde{B}_j z(x),\ j = 1,2$, that are meromorphic in $\hat{\mathbb{C}}$. Letting $t = \log x$

[13] The restriction of the domain causes no additional problem.

we can reduce this problem to the search for solutions, meromorphic in $\mathbb{C}$, of a system with constant coefficients in case 2Q. One then considers the series expansion at the origin of such solutions and concludes, as in case 2Q, that these vectors have entries in $\mathbb{C}[t, t^{-1}]$. Therefore the entries of $z(x)$ involve powers of $\log x$.

In case 2M, $f(x) \in \mathbb{C}[[x]][x^{-1}]$, we have to solve $\sigma_j(z(x)) = \tilde{B}_j\, z(x)$, $j = 1, 2$, in $E = \cup_{s\in\mathbb{N}^*}\mathbb{C}[[x^{1/s}]][x^{-1/s}]$. Expanding a solution in a series immediately proves that $z(x)$ is constant. Therefore $f(x)$ is an element of K. We write $f(x) = g(x^{1/s})$ with some rational function $g \in \mathbb{C}(x)$ and some positive integer s. As in the proof of Corollary 3, this yields $f(t^s) = g(t)$ and therefore also $f(t^s) = g(\xi t)$ for any s-th root of unity ξ. Hence $f(t^s) = \frac{1}{s}\sum_{\xi^s=1} g(\xi t) \in \mathbb{C}(t^s)$. This yields that $f(x)$ is rational.

For the converse concerning the shift operators, consider first $f(x) = g(x)\zeta(x)^j e^{\beta x}\rho(\delta, x)$ where $\beta, \delta \in \mathbb{C}$, j is some positive integer and g is an elliptic function. By the properties of ρ and the exponential, we obtain with $a_1 = e^{\beta+2\eta_1\delta}$ that $(\sigma_1 - a_1)^{j+1}(f(x)) = 0$. Again products of solutions of σ_1-difference equations with rational functions also satisfy σ_1-difference equations and sums of solutions of σ_1-difference equations also satisfy some σ_1-difference equation. Therefore $f(x)$ given by (25) also does. We obtain the σ_2-difference equation for $f(x)$ in the same way.

For the proof of the converse concerning two q-difference equations, we proceed analogously. A slight change is that we have to consider

$$f(x) = \log(x)^i x^\beta g(t)\zeta(t)^j \rho(\delta, t),\ t = \log(x)/\log(q_1).$$

Here $(\sigma_1 - a_1)^{i+j+1}(f(x)) = 0$ for $a_1 = q_1^\beta e^{2\eta_1\delta}$ and $(\sigma_2 - a_2)^{i+j+1}(f(x)) = 0$ for $a_2 = q_2^\beta e^{2\eta_2\delta}$.

For the proof of the converse concerning two Mahler systems, consider

$$f(x) = x^r(\log(x))^\alpha(\log(\log(x)))^j$$

with $r \in \mathbb{Q}$, $\alpha \in \mathbb{C}$ and $j \in \mathbb{N}$. There are $k \in \mathbb{N}^*$, $m \in \mathbb{N}$ such that $(q_1^k - 1)q_1^m r =: \ell$ is an integer. Then

$$(q_1^{-k\alpha}x^{-\ell}\sigma_1^k - id)^{j+1}\sigma_1^m(f(x)) = 0.$$

It follows as before that $f(x)$ given by (29) satisfies a σ_1-Mahler equation. For the second equation concerning σ_2, we proceed analogously. ∎

We now turn to the proof of Theorem 13. The consistency condition (21) can be interpreted as follows.

Remark 17. The gauge transformation $Z = B_2 Y$ transforms

$$\sigma_1(Y) = B_1\, Y \tag{33}$$

into the equivalent system $\sigma_1(Z) = \sigma_2(B_1)\, Z$. Iterating this procedure, we find that the system (33) is equivalent to the systems

$$\sigma_1(U) = \sigma_2^N(B_1)\, U \tag{34}$$

for any positive integer N by the gauge transformation $Z = G_N Y, G_N = \sigma_2^{N-1}(B_2)...\sigma_2(B_2)B_2$. Thus, if W is a fundamental solution matrix of (33) then both $\sigma_1^N(W)$ and $G_N W$ are fundamental solution matrices of (34).

The rest of the proof of Theorem 13 will be given separately for the three cases.

3.1 Proof of Theorem 13: Case 2S

In a first step, we consider analytic continuation of solutions of (33) under the hypotheses of the Theorem.

Lemma 18. *In case 2S, consider a strip $S = \{x \in \mathbb{C} | a_1 < \operatorname{Im} x < a_2\}$, $-\infty \le a_1 < a_2 \le +\infty$. If $g(x)$ is a holomorphic solution of (33) for $x \in S$ with sufficiently large positive real part or with sufficiently large negative real part then $g(x)$ can be continued analytically to a meromorphic function in S with finitely many poles.*

Proof. Let $\mathcal{M}$ denote the set of $x_1 \in \mathbb{C}$ such that $B_1(x)$ or $B_1(x)^{-1}$ has a pole at $x = x_1$. Consider a solution g holomorphic for $x \in S$ with large positive real part. By the difference equation (33), it can be continued analytically to S except for possible poles in $(\mathcal{M} - \mathbb{N}) \cap S$. By Remark 17, for $N \in \mathbb{N}$, the function $g_N = G_N g$ is a solution of $\sigma_1(U) = \sigma_2^N(B_1)U$ holomorphic for $x \in S$ with large real part. By its difference equation, it can be continued analytically to S except for possible poles in $(\sigma_2^{-N}(\mathcal{M}) - \mathbb{N}) \cap S = (\mathcal{M} - \mathbb{N} - N\alpha) \cap S$, because x_1 is a pole of $\sigma_2^N(B_1^{\pm 1})$ if and only if $\sigma_2^N(x_1)$ is a pole of $B_1^{\pm 1}$. This implies that $g = G_N^{-1} g_N$ can also be analytically continued to S except possible poles in $(\sigma_2^{-N}(\mathcal{M}) - \mathbb{N}) \cap S$ and in $\mathcal{N}_N \cap S$, where $\mathcal{N}_N$ is the (finite) set of poles of G_N^{-1}. Therefore g can be continued analytically to S with the exception of $(\mathcal{M} - \mathbb{N} - N\alpha) \cap (\mathcal{M} - \mathbb{N}) \cap S$ and $\mathcal{N}_N \cap S$.

We claim that the former intersection is empty for appropriate N. The set $\{d - c + \mathbb{Z} \mid c, d \in \mathcal{M}\}$ is finite since $\mathcal{M}$ is finite. As α is irrational, the set $\{N\alpha + \mathbb{Z} \mid N \in \mathbb{N}\}$ is infinite. Hence we can select $N \in \mathbb{N}$ such that for all $c, d \in \mathcal{M}$, the difference $d - c \not\equiv N\alpha \mod \mathbb{Z}$. If the first intersection is nonempty, there exists $h \in S$ such that $h = d - N\alpha - m$ and $h = c - n$, where $c, d \in \mathcal{M}$ and $m, n \in \mathbb{N}$. This implies that $d - c - N\alpha = m - n \in \mathbb{Z}$, a contradiction. Thus the intersection is indeed empty. This means that g can be continued analytically to $S \setminus \mathcal{N}_N$, where $\mathcal{N}_N$ is finite.

The proof in the case of g analytic for $x \in S$ with large negative real part is analogous. ∎

Concerning the behavior at infinity, it is known that $\sigma_1(Y) = B_1 Y$ has a formal fundamental solution

$$Y(x) = \Phi(x) x^L \, e^{Q(x)} x^{Dx}, \tag{35}$$

where $\Phi(x)$ is a formal power series in $x^{-1/r}$ for some integer r, L is a constant matrix with eigenvalues γ_j satisfying $0 \le \mathrm{Re}(\gamma_j) < \frac{1}{r}$, $Q(x) = \sum_{j=1}^{h} Q_j x^{r_j}$ where the Q_j are diagonal matrices with entries in C and the r_j are positive rational numbers with $1 = r_h > r_{h-1} > \ldots > r_1 > 0$ (or $Q(x) \equiv 0$) and D is diagonal with entries in $\frac{1}{r}\mathbb{Z}$; furthermore L, $Q(x)$ and D commute (*c.f.,* [27], chapter I, and [35], section 6.1). The leading term Q_h of $Q(x)$ is chosen such that the imaginary parts of its entries are between 0 and 2π[14], 2π excluded. We can write $D = \mathrm{diag}(d_\ell I_\ell, \ell = 1, ..., m)$ and $Q(x) = \mathrm{diag}(q_\ell(x) I_\ell, \ell = 1, ..., m)$ with identity matrices of appropriate size $n_\ell \times n_\ell$ and distinct couples $(d_\ell, q_\ell(x))$, $\ell = 1, ..., m$. Then also $L = \mathrm{diag}(L_1, ..., L_m)$ with diagonal blocks of corresponding size. The formal fundamental solution is essentially unique, *i.e.* except for a permutation of the diagonal blocks and passage from some L_ℓ to a conjugate matrix.

By Remark 17, both $B_2(x)Y(x)$ and $\sigma_2(Y(x)) = Y(x+\alpha)$ are formal fundamental solutions of $\sigma_1(Z) = \sigma_2(B_1)\, Z$. Re-expanding we find that $Y(x+\alpha) = \tilde{\Phi}(x) x^{\tilde{L}} e^{Q(x)} x^{Dx}$, where $\tilde{L} \equiv L + \alpha D \mod \frac{1}{r}\mathbb{Z}$. Also writing $\tilde{L} = \mathrm{diag}(\tilde{L}_\ell, \ell = 1, ..., m)$ we obtain from the essential uniqueness of the formal fundamental solution that for each ℓ, the matrices $\tilde{L}_\ell \equiv L_\ell + \alpha d_\ell \mod \frac{1}{r}\mathbb{Z}$ and L_ℓ are conjugate.[15] Now if $a_k + \frac{1}{r}\mathbb{Z}$, $k = 1, ..., r_\ell$ are the equivalence classes of the eigenvalues of L_ℓ modulo $\frac{1}{r}\mathbb{Z}$, then $a_k + \alpha d_\ell + \frac{1}{r}\mathbb{Z}$ are those of $\tilde{L}_\ell$. Hence the mapping $x + \frac{1}{r}\mathbb{Z} \mapsto x + \alpha d_\ell + \frac{1}{r}\mathbb{Z}$ induces a permutation of the equivalence classes of the eigenvalues of L_ℓ. Applying it several times, if necessary, to some eigenvalue a_k, we obtain the existence of some positive integer N such that $a_k + \frac{1}{r}\mathbb{Z} = a_k + N\alpha d_\ell + \frac{1}{r}\mathbb{Z}$. Since α is not rational, but d_ℓ is, this is impossible unless $d_\ell = 0$ and we obtain that $D = 0$. The difference equation (33) and by symmetry also $\sigma_2 Y = B_2 Y$ are hence *mild* in the sense of [35], section 7.1.

Next we show the statement analogous to Proposition 12.

Lemma 19. *In the present context, there exists a diagonal matrix $\tilde{B}_1$ with constant entries and a gauge transformation $Z = FY$, $F \in \mathrm{GL}_n(k)$, such that Z satisfies $\sigma_1(Z) = \tilde{B}_1 Z$.*

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[14]Recall that solutions of σ_1-difference equations remain solutions when multiplied by 1-periodic functions.

[15]Observe that there can be no permutaion of diagonal blocks because the couples $(d_\ell, q_\ell(x))$, $\ell = 1, ..., m$ are distinct.

Proof. This is an adaptation of the proof of Proposition 12. For $\theta \neq 0$ sufficiently close to 0, there exist

1. sectors S and T of openings greater than π bisected by θ and $\theta + \pi$, respectively,
2. a number $R > 0$ and functions $\Phi_S(x)$ and $\Phi_T(x)$ analytic for $|x| > R$ in S and T, respectively, such that
   (a) $\Phi_S(x)$ and $\Phi_T(x)$ are asymptotic to $\Phi(x)$ as $x \to \infty$ in their respective sectors, and
   (b) $Y^S(x) = \Phi_S(x)x^L e^{Q(x)}$ and $Y^T(x) = \Phi_T(x)x^L e^{Q(x)}$ are solutions of $\sigma_1 Y = AY$.

This can be proved using multisummation (c.f., [16], [35], section 9.1).

We write $Q(x) = \mathrm{diag}(q_1(x)I_1, \ldots, q_s(x)I_s)$ with distinct $q_j(x)$ and I_j identity matrices of an appropriate size. We split

$$q_i(x) = \lambda_i x + \text{ terms involving } x^\alpha \text{ with } 0 < \alpha < 1.$$

As $0 \leq \mathrm{Im}\,\lambda_i < 2\pi$, we may assume that θ is so close to 0 that

$$\max_{j,k}(\mathrm{Re}(\lambda_j x) - \mathrm{Re}(\lambda_k x)) < -\,\mathrm{Re}(2\pi i x) \text{ if } \arg(x) = \psi = \theta + \pi/2. \qquad (36)$$

Indeed, this is equivalent to $\max_{j,k} \mathrm{Im}\,\left((\lambda_j - \lambda_k)e^{i\theta}\right) < 2\pi\cos(\theta)$ which is true for sufficiently small $|\theta|$.

According to Lemma 18 both $Y^S(x)$ and $Y^T(x)$ can be continued analytically to meromorphic functions on $\mathbb{C}$ with finitely many poles. The same is true for their inverses, because their transposed inverses also satisfy equations to which Lemma 18 applies. We call these extensions again $Y^S(x)$, $Y^T(x)$.

Let $D(x)$ be the connection matrix defined by $Y^S(x) = Y^T(x)D(x)$. It is meromorphic on $\mathbb{C}$ with finitely many poles and 1-periodic. Hence it is entire. Moreover, as $Y^S(x)$, $Y^T(x)$ and $(Y^T)^{-1}(x)$, it has at most exponential growth as $|\mathrm{Im}\,x| \to \infty$ in the intersections of S and T. Hence the Fourier series of $D(x)$ has at most finitely many terms. Let us write $D(x) = \sum_{k=-k_0}^{k_0} D^{(k)}e^{2\pi ikx}$. The blocks of $D(x)$ corresponding to the subdivision of $Q(x)$ are denoted by $D_{i,j}(x)$, those of $D^{(k)}$ by $D_{i,j}^{(k)}$. Write $Y^S = (Y_1^S|...|Y_s^S)$ and $Y^T = (Y_1^T|...|Y_s^T)$ in corresponding block columns.

We claim that $D(x)$ is constant. To show this, fix some j. Then

$$Y_j^S(x) = \sum_{m=1}^{s}\sum_{k=-k_0}^{k_0} Y_m^T(x)e^{2\pi ikx}D_{m,j}^{(k)}.$$

Because of (36) we obtain that $D_{m,j}^{(k)} = 0$ if $k < 0$. Otherwise the right hand side would grow more rapidly as $|x| \to \infty$, $\arg(x) = \psi$ than the left hand side. For the line $\arg(x) = \psi + \pi$, the exponentials $e^{2\pi ikx}$ are ordered inversely with respect to growth as $|x| \to \infty$. Here we find that $D_{m,j}^{(k)} = 0$ if $k > 0$. Altogether we obtain that $D(x)$ is constant.

The rest of the proof is identical to the one of Proposition 12 with $D(x)$ replacing the matrix C^+ in the latter argument. ∎

We can now complete the proof of Theorem 13 in case 2S. Apply the gauge transformation $Z = FY$ of Lemma 19 to $\sigma_2 Y = B_2 Y$ to yield $\sigma_2 Z = \tilde{B}_2 Z$, $\tilde{B}_2 = (b_{i,j})$. Since $\tilde{B}_1 = \mathrm{diag}(a_1, ..., a_n)$ is constant diagonal, the consistency condition (21) implies that

$$\sigma_1(b_{i,j}) = \frac{a_i}{a_j} b_{i,j}.$$

If $a_i \neq a_j$ we obtain that $b_{i,j} = 0$ since the above equation has no nonzero solution in $C(x)$ then. If $a_i = a_j$ we have $\sigma_1(b_{i,j}) = b_{i,j}$ and hence $b_{i,j}$ is a constant. Therefore $\tilde{B}_2$ is also constant. This completes the proof in case 2S.

3.2 Proof of Theorem 13: Case 2Q

In case 2Q, we omit the index 1 of q_1 and assume $|q| > 1$ for simplicity. Fix a logarithm $2\pi i\tau$ of q. Concerning analytic continuation we prove here

Lemma 20. *In case 2Q, consider a spiraling strip $S = \{e^{2\pi i\tau t} \in \hat{\mathbb{C}} | a_1 < \mathrm{Im}\, t < a_2\}$, $-\infty \leq a_1 < a_2 \leq +\infty$. If $g(x)$ is a holomorphic solution of (33) for $x \in S$ with sufficiently large modulus or with sufficiently small modulus then $g(x)$ can be continued analytically to a meromorphic function in S such that the projections of its poles to $\mathbb{C}^*$ form a finite set.*

Proof. The proof is analogous to that of Lemma 18. The only sets of possible poles are now $(\mathcal{M} \cdot q^{-\mathbb{N}}) \cap S$ or $(\mathcal{M} \cdot q_2^{-N} \cdot q^{-\mathbb{N}}) \cap S$, respectively, and their intersection is empty for convenient N because of the condition imposed on q and q_2 in case 2Q. We leave it to the reader to fill in the details. ∎

Concerning the behavior at 0 (and similarly at ∞), it is known that there exists a formal gauge transformation $Z = GY$, $G \in \mathrm{GL}_n(\mathbb{C}[[x^{1/s}]][x^{-1/s}])$, $s \in \mathbb{N}^*$, that reduces (33) to a system $\sigma(Z) = x^D A_0 Z$, where D is a diagonal matrix with entries in $\frac{1}{s}\mathbb{Z}$ and $A_0 \in \mathrm{GL}_n(\mathbb{C})$ such that any eigenvalue λ of A_0 satisfies $1 \leq |\lambda| < |q|^{1/s}$, moreover D and A_0 commute. If we write $D = \mathrm{diag}(d_1 I_1, ..., d_r I_r)$ with distinct d_j and I_j identity matrices of an appropriate size, then $A_0 = \mathrm{diag}(A_0^1, ..., A_0^r)$ with diagonal blocks A_0^j of corresponding size. D and A_0 are essentially unique, *i.e.* except for a permutation of the diagonal blocks and passage from some A_0^j to a conjugate matrix. If D happens to be 0, then s can be chosen to be 1 and G is convergent (see [35], ch. 12, [2], [18]).

According to Remark 17, our system (33) is equivalent to $\sigma(U) = \sigma_2(B_1)\,U$. The gauge transformation $V = \sigma_2(G)U$ now transforms this system to $\sigma(V) = \sigma_2(x^D A_0)V$. Now $\sigma_2(x^D A_0) = x^D\, q_2^D A_0$ and there is a diagonal matrix F with entries in $\frac{1}{s}\mathbb{Z}$ commuting with D and A_0 such that the gauge transformation $W = x^F V$ reduces the latter system to $\sigma(V) = x^D \tilde{A}_0 V$, where $\tilde{A}_0 = q^{-F} q_2^D A_0$ has again eigenvalues with modulus in $[1, |q|^{1/s}[$. Now we write $\tilde{A}_0 = \mathrm{diag}(\tilde{A}_0^1, ..., \tilde{A}_0^r)$ and fix some $j \in \{1, ..., r\}$. If $a_1^j, ..., a_{r_j}^j$ are the eigenvalues of A_0^j then $q^{-f_j} q_2^{d_j} a_\ell^j$, $\ell = 1, ..., r_j$, are those of $\tilde{A}_0^j$. By the uniqueness of the reduced form, the mapping $x \mapsto q^{-f_j} q_2^{d_j} x$ induces a permutation of the eigenvalues of A_0^j. If we apply it several times, if necessary, we obtain the existence of some $\ell \in \{1, ..., r_j\}$ and of some positive integer k such that $q^{-kf_j} q_2^{kd_j} a_\ell^j = a_\ell^j$. Due to our condition on q and q_2 this is only possible if $d_j = 0$. Thus we have proved that $D = 0$.

Hence 0 and ∞ are regular singular points of (33). There is a gauge transformation $Z = G_0(x)Y$, $G_0(x) \in \mathrm{GL}_n(\mathbb{C}\{x\}[x^{-1}])$ reducing the system to $\sigma(Z) = A_0 Z$ and a gauge transformation $V = G_\infty(x)Y$, $G_\infty(x) \in \mathrm{GL}_n(\mathbb{C}\{x^{-1}\}[x])$ reducing the system to $\sigma(V) = A_\infty V$, where A_0, A_∞ are constant invertible matrices with eigenvalues in the annulus $1 \le |\lambda| < |q|[$.

Now we fix a matrix L_0 such that $A_0 = q^{L_0} (= \exp(2\pi i \tau L_0)))$ and thus $Y_0(x) = G_0(x)^{-1} x^{L_0}$ is a solution of (33) in some neighborhood of 0 in $\hat{\mathbb{C}}$. By Lemma 20, this solution can be continued analytically to a meromorphic function on $\hat{\mathbb{C}}$ such that the projections of its poles to $\mathbb{C}^*$ form a finite set. This implies that $G_0(x)^{-1}$ can be continued to a meromorphic function on $\mathbb{C}$ with finitely many poles. We use the same name for this extension. In some annulus $K < |x| < \infty$, K sufficiently large, it can be expanded in a convergent Laurent series

$$G_0(x)^{-1} = \sum_{m=-\infty}^{\infty} G_m x^m$$

with some constant matrices G_m and therefore also

$$G_\infty(x) G_0(x)^{-1} = \sum_{m=-\infty}^{\infty} P_m x^m$$

with constant matrices P_m.

Now by construction, $G_\infty(x)G_0(x)^{-1}x^{L_0}$ is a solution of the equation $\sigma(V) = A_\infty V$. Using $q^{L_0} = A_0$, this implies the equations

$$P_m\, q^m A_0 = A_\infty P_m,\ m \in \mathbb{Z}.$$

Now if $m \neq 0$, then $q^m A_0$ and A_∞ cannot have a common eigenvalue as those of A_0, A_∞ have a modulus in $[1, |q|[$. Hence $P_m = 0$ for $m \neq 0$ and there is a constant invertible matrix P_0 such that $G_\infty(x) = P_0 G_0(x)$ for $|x| > K$. In particular, the above Laurent series of $G_0(x)^{-1}$ has only finitely many terms corresponding to positive powers of x since this is the case for $G_\infty(x)^{-1}$. This means that ∞ is only a pole of the $G_0(x)^{-1}$, a meromorphic function on $\mathbb{C}$ with finitely many poles. We obtain that $G_0(x)^{-1}$ and hence also $G_0(x)$ have entries that are rational functions.

Thus we have shown that (33) is equivalent over $\mathbb{C}(x)$ to $\sigma(Z) = A_0 Z$ with some constant invertible matrix A_0. To complete the proof of Theorem 13 for case 2Q, apply the same gauge transformation to $\sigma_2(Y) = B_2(x)Y$ to obtain $\sigma_2(Z) = \tilde{B}_2(x)Z$. The consistency condition implies here that

$$\sigma_1(\tilde{B}_2(x))A_0 = A_0\tilde{B}_2(x).$$

Expanding $\tilde{B}_2(x)$ in a convergent Laurent series in a punctured disk centered at the origin

$$\tilde{B}_2(x) = \sum_{m=-M}^{\infty} C_m x^m$$

we find that the coefficients satisfy the equations $C_m\, q^m A_0 = A_0 C_m, m = -M, \ldots$. Again $q^m A_0$ and A_0 have no common eigenvalue and hence $C_m = 0$ if $m \neq 0$. Thus $\tilde{B}_2(x)$ is also a constant and the theorem is proved in case 2Q as well.

Remark 21. Following ideas of Bézivin ([37, Page 90]) we can establish Theorem 13, case 2Q under the assumptions that $|q_1| = |q_2| = 1$, q_1, q_2 multiplicatively independent, q_2 not a root of unity, q_1 transcendental over $\mathbb{Q}$ or q_1 algebraic over $\mathbb{Q}$ such that its minimal polynomial has a root in $\mathbb{C}$ of absolute value not equal to 1. For the proof, let $F \subset \mathbb{C}$ be the field generated by q_1 and q_2 and the coefficients of the entries of B_1 and B_2. This is a finitely generated extension of $\mathbb{Q}$. Hence by the assumption on q_1, there is an embedding $\psi : F \to \mathbb{C}$ such that such that $|\psi(q_1)| \neq 1$. We can extend ψ to an automorphism of $\mathbb{C}$ which we denote again by ψ. After applying ψ to the system (20), we get a new system that satisfies the hypotheses of Theorem 12. Therefore there exists a gauge transformation $Z = GY$ transforming this new system to a system with constant invertible commuting matrices $\tilde{B}_1, \tilde{B}_2$. One sees that the system with matrices $\psi^{-1}(\tilde{B}_1), \psi^{-1}(\tilde{B}_2)$ satisfies the conclusion of Theorem 13 with respect to our original system.

3.3 Proof of Theorem 13: Case 2M

In case 2M, it is more convenient to use different notation. We consider

$$y(x^p) = A(x)y(x), \quad y(x^q) = B(x)y(x) \tag{37}$$

with multiplicatively independent positive integers p and q and $A(x), B(x) \in \mathrm{GL}_n(C(x))$ satisfying the consistency condition

$$A(x^q)B(x) = B(x^p)A(x). \tag{38}$$

We want to show that there exist $G(x) \in \mathrm{GL}_n(K)$ and commuting $A_0, B_0 \in \mathrm{GL}_n(C)$ such that the gauge transformation $y = G(x)z$ reduces (37) to

$$z(x^p) = A_0 z(x), \quad z(x^q) = B_0 z(x). \tag{39}$$

The proof is more involved than in cases 2S and 2Q because there are several fixed points of the mapping $x \mapsto x^p$, namely $0, \infty$ and the $(p-1)$-th roots of unity, because solutions holomorphic in some neighborhood of the origin or ∞ can only be extended using (37) to the unit disk or the annulus $|x| > 1$, respectively, and because the behavior of the solutions of Mahler systems near 0 and ∞ is not well understood. The consistency condition (38) is crucial and will be used many times.

The plan is as follows. In a first step we prove that any formal vectorial solution of a system (37) satisfying (38) is rational and deduce the statement of the theorem under the additional hypothesis that $x = 0$ is a *regular singular point* of $y(x^p) = A(x)y(x)$, *i.e.* there is a formal series $G_0(x) \in \mathrm{GL}_n(\hat{K}),\ \hat{K} = \bigcup_{r\in\mathbb{N}^*} C[[x^{1/r}]][x^{-1/r}]$, such that the gauge transformation $y = G_0(x)z$ reduces the equation to one with a constant coefficient matrix. In a second step, we prove that $x = 0$ is always regular singular for a consistent system (37), (38) thus completing the proof.

Proposition 22. *In case 2M, consider the system (37) satisfying the consistency condition (38) and suppose that $g(x) \in (C[[x]][x^{-1}])^n$ is a formal vectorial solution. Then $g(x) \in C(x)^n$.*

Corollary 23. *In case 2M, consider the system (37) satisfying the consistency condition (38) and suppose that the point $x = 0$ for the first equation of (37) is regular singular. Then (37) is equivalent over $K = C(\{x^{1/s} \mid s \in \mathbb{N}^*\})$ to a system (39) with constant invertible commuting A_0 and B_0.*

Remark: The Proposition could be deduced from the last part of Corollary 16, *i.e.* from Theorem 1.1 of [1]. Indeed, consider the $C(x)$-subspace space of $C[[x]][x^{-1}]$ generated by the components of $g(x)$. By (37), it is invariant under σ_1, σ_2 and it follows as usual that each component of $g(x)$ satisfies a system of two scalar linear p- and q-Mahler equations and hence is rational by the Theorem 1.1 of [1].

Conversely, Theorem 1.1 of [1] can be deduced from Proposition 22: Given a formal solution $f(x)$ of a system (24), the first part of the proof of Corollary 16 constructs a system (37) satisfying the consistency condition (38) having a solution vector in $(\mathbb{C}[[x]][x^{-1}])^n$. As this solution vector is actually a basis of some vector space containing $f(x)$, we can assume that one of its components is $f(x)$. Proposition 22 then yields that $f(x)$ is rational.

Observe that Theorem 13 in case 2M (and hence also the first part of Corollary 16) is not an immediate consequence of the result of [1] because it is not clear a priori that the point 0 is regular singular under the hypotheses of the theorem. This statement is the contents of Proposition 26.

Proof of Corollary 23. We consider a formal series $G_0(x) \in \mathrm{GL}_n(\hat{K})$ reducing $y(x^p) = A(x)y(x)$ to $z(x^p) = A_0 z(x)$ with a constant invertible matrix A_0. This means

$$G_0(x^p) = A(x)G_0(x)A_0^{-1}. \tag{40}$$

By a change of variables $x = t^r$, if necessary, we can assume that $G_0(x) \in \mathrm{GL}_n(\hat{k})$, $\hat{k} = C[[x]][x^{-1}]$ (we do not use that $A(x), B(x)$ are in $C(x^r)$ then).

Applying the gauge transformation $y = G_0(x)z$ to the second equation $y(x^q) = B(x)y(x)$, we obtain $z(x^q) = \tilde{B}(x)z(x)$ with some $\tilde{B}(x) \in \mathrm{GL}_n(\hat{K})$ satisfying the consistency condition $A_0\,\tilde{B}(x) = \tilde{B}(x^p)A_0$. Using the series expansion of $\tilde{B}(x)$, it is readily shown that $\tilde{B}(x)$ must be constant and commutes with A_0. We write $\tilde{B}(x) =: B_0$. Thus $G_0(x)$ also satisfies

$$G_0(x^q) = B(x)G_0(x)B_0^{-1}. \tag{41}$$

The system (40), (41) can be considered as a vectorial system

$$Y(x^p) = \bar{A}(x)Y(x), \;\; Y(x^q) = \bar{B}(x)Y(x),$$

where $Y(x)$ has coefficients in $gl_n(\hat{k}) \sim \hat{k}^{n^2}$ and $\bar{A}(x)$, $\bar{B}(x)$ are the matrices of the linear operators mapping Z to $A(x)ZA_0^{-1}$ and $B(x)ZB_0^{-1}$, respectively. This system satisfies the consistency condition, because $A(x)$ and $B(x)$ do and A_0, B_0 commute. Now Proposition 22 can be applied to its formal solution $G_0(x)$ and we obtain that $G_0(x)$ has rational entries. ∎

Proof of Proposition 22. We first show that $g(x)$ is actually convergent. This could be deduced from [38], section 4. For the convenience of the reader, we provide a short proof.

To do that, we truncate $g(x)$ at a sufficiently high power of x to obtain $h(x) \in \mathrm{gl}_n(C[x][x^{-1}])$ and introduce $r(x) = h(x) - A(x)^{-1}h(x^p)$ and $\tilde{g}(x) = g(x) - h(x)$. Then we have

$$\tilde{g}(x) = A(x)^{-1}\tilde{g}(x^p) - r(x). \tag{42}$$

We denote the valuation of $A(x)^{-1}$ at the origin by $s \in \mathbb{Z}$ and introduce $\tilde{A}(x) = x^{-s}A(x)^{-1}$ which is holomorphic at the origin.

First choose $M \in \mathbb{N}$ such that $pM + s > M$ and $h(x)$ such that $g(x) - h(x)$ has at least valuation M. Then by (42), $r(x)$ also has at least valuation M. Now consider $R > 0$ such that $\tilde{A}(x)$ is holomorphic and bounded on $D(0, R)$. Then consider for positive $\rho < \min(R, 1)$ the vector space E_ρ of all series $F(x) = \sum_{m=M}^{\infty} F_m x^m$ such that $\sum_{m=M}^{\infty} |F_m|\rho^m$ converges and define the norm $|F(x)|_\rho$ as this sum. Then E_ρ equipped with $|\ |_\rho$ is a Banach space and the existence of a unique solution of (42) in E_ρ for sufficiently small $\rho > 0$ follows from the Banach fixed-point theorem using that $|x^s F(x^p)|_\rho \leq \rho^{Mp+s-M}|F(x)|_\rho$ for $F(x) \in E_\rho$. This proves the convergence of $\tilde{g}(x)$ and hence of $g(x)$.

By (37), rewritten $g(x) = A(x)^{-1}g(x^p)$, the function g can only be extended analytically to a meromorphic function on the unit disk. According to Theorem 4.2 of [38] (see also [11]), it is sufficient to show that $g(x)$ does not have the unit circle as a natural boundary and the rationality of $g(x)$ follows. We show how it follows naturally, in our context, that $g(x)$ can be continued analytically as a meromorphic function to all of $\mathbb{C}$ and, as well, that it has only finitely many poles. The rationality of $g(x)$ then follows as in [38] and [11] from a growth estimate.

As we want to extend $g(x)$ beyond the unit disk, we use the change of variables $x = e^t$, $u(t) = y(e^t)$ and obtain a system of q-difference equations

$$u(pt) = \bar{A}(t)u(t), \quad u(qt) = \bar{B}(t)u(t) \tag{43}$$

with $\bar{A}(t) = A(e^t)$, $\bar{B}(t) = B(e^t)$. It satisfies the consistency condition

$$\bar{A}(qt)\bar{B}(t) = \bar{B}(pt)\bar{A}(t). \tag{44}$$

We are not in case 2Q, however, because $\bar{A}(t), \bar{B}(t)$ are not rational in t, but rational in e^t.

Nevertheless, the local theory at the origin used in the proof of Lemma 20 applies to the present consistent system. In particular, we know that the origin is a regular singular point in case 2Q. We therefore obtain a matrix A_1 with eigenvalues λ in the annulus $1 \leq |\lambda| < p$ and $G_1(t) \in \mathrm{GL}_n(C\{t\}[t^{-1}])$ such that $u = G_1(t)v$ reduces the first equation of (43) to $v(pt) = A_1 v(t)$. This means

$$G_1(pt) = \bar{A}(t)G_1(t)A_1^{-1} \text{ for small } t. \tag{45}$$

Applying the same gauge transformation to the second equation of (43) yields an equation $v(qt) = \tilde{\bar{B}}(t)v(t)$ with some $\tilde{\bar{B}}(t) \in \mathrm{GL}_n(C\{t\}[t^{-1}])$. It satisfies the consistency condition $A_1\tilde{\bar{B}}(t) = \tilde{\bar{B}}(pt)A_1$. As at the end of the proof in case 2Q, we expand $\tilde{\bar{B}}(t) = \sum_{m=m_0}^{\infty} C_m t^m$. The coefficients satisfy $A_1 C_m = C_m(p^m A_1)$, $m \geq m_0$. As A_1 and $p^m A_1$ have no common eigenvalue unless $m = 0$, we obtain that $\tilde{\bar{B}}(t) =: B_1$ is constant and commutes with A_1. We note the second equation satisfied by G_1

$$G_1(qt) = \bar{B}(t)G_1(t)B_1^{-1} \text{ for small } t. \tag{46}$$

Lemma 24. *The functions $G_1(t)^{\pm 1}$ can be continued analytically to meromorphic functions on $\mathbb{C}$ and there exists $\delta > 0$ such that both can be continued analytically to the sectors $\{t \in \mathbb{C}^* \mid \delta < \arg(\pm t) < 2\delta\}$.*

Proof of the lemma. Let $\mathcal{M}$ be the set of poles of $\bar{A}(t)^{\pm 1}$, *i.e.* the set of t such that e^t is a pole of $A(x)$ or $A(x)^{-1}$. Note that $\mathcal{M}$ is $2\pi i$-periodic, has no finite accumulation point and is contained in some vertical strip $\{t \in \mathbb{C} \mid -D < \operatorname{Re} t < D\}$. By (45), $G_1(t)^{\pm 1}$ can be continued analytically to $\mathbb{C}^* \setminus (\mathcal{M} \cdot p^{\mathbb{N}})$ and thus to meromorphic functions on $\mathbb{C}$ which we denote by the same name. By construction, $G_1(t)^{\pm 1}$ are also analytic in some punctured neighborhood of the origin. By the properties of $\mathcal{M}$, the infimum of the $|\operatorname{Re} t_1|$ on the set of all $t_1 \in \mathcal{M}$ having nonzero real part is a positive number. As $\mathcal{M}$ is contained in some vertical strip there exist sectors $\{t \in \mathbb{C}^* \mid \delta < \arg(\pm t) < 2\delta\}$ disjoint to $\mathcal{M}$ and hence to $\mathcal{M} \cdot p^{\mathbb{N}}$. Therefore $G_1(t)^{\pm 1}$ can be analytically continued to these sectors and the lemma is proved. ∎

Consider now the function $d(t) = G_1(t)^{-1}g(e^t)$. By Lemma 24 and because $g(x)$ is holomorphic in some punctured neighborhood of $x = 0$, $d(t)$ is defined and holomorphic for some sector $S = \{t \in \mathbb{C} \mid |t| > K,\ \pi + \delta < \arg t < \pi + 2\delta\}$. By (37), (45), and (46) it satisfies

$$d(pt) = A_1 d(t), \;\; d(qt) = B_1 d(t) \text{ for } t \in S. \tag{47}$$

To solve (47), consider a matrix L_1 commuting with B_1 such that $p^{L_1} = A_1$. Put $F(t) = t^{-L_1}d(t)$. Then

$$F(pt) = F(t), \;\; F(qt) = \tilde{B}_1 F(t) \text{ for } t \in S \tag{48}$$

where $\tilde{B}_1 = B_1 q^{-L_1}$. Thus $H(s) = F(e^s)$ is $\log(p)$-periodic on the half-strip $B = \{s \in \mathbb{C} \mid \operatorname{Re} s > \log(K),\ \pi + \delta < \operatorname{Im} s < \pi + 2\delta\}$ and can be expanded in a Fourier series. This implies that

$$F(t) = \sum_{\ell=-\infty}^{\infty} F_\ell \, t^{\frac{2\pi i}{\log(p)}\ell} \text{ for } t \in S. \tag{49}$$

The second equation of (48) yields conditions on the Fourier coefficients

$$F_\ell \exp\left(2\pi i \frac{\log(q)}{\log(p)}\ell\right) = \tilde{B}_1 F_\ell \text{ for } \ell \in \mathbb{Z}.$$

Therefore $F_\ell = 0$ unless $\exp\left(2\pi i \frac{\log(q)}{\log(p)}\ell\right)$ is an eigenvalue of $\tilde{B}_1$. Since p and q are multiplicatively independent, the quotient $\frac{\log(q)}{\log(p)}$ is irrational and hence $\exp\left(2\pi i \frac{\log(q)}{\log(p)}\right)$ is not a root of unity. Therefore all the numbers $\exp\left(2\pi i \frac{\log(q)}{\log(p)}\ell\right)$, $\ell \in \mathbb{Z}$ are different and only finitely many of them can be eigenvalues of $\tilde{B}_1$. This shows that the Fourier series (49) has finitely many terms and thus $F(t)$ can be analytically continued to the whole Riemann surface $\hat{\mathbb{C}}$ of $\log(t)$. The same holds for $d(t) = t^{L_1}F(t)$.

Since $g(x)$ is a convergent Laurent series in $C[[x]][x^{-1}]$, the function $h(t) = g(e^t)$ is holomorphic for t with large negative real part and $2\pi i$-periodic. We conclude using Lemma 24 that $h(t) = G_1(t)d(t)$ can be analytically continued to a meromorphic function on $\hat{\mathbb{C}}$, in particular the point $t = 2\pi i$ is at most a pole of h. By its periodicity, this implies that $t = 0$ also is at most a pole of h and that it can be continued analytically to a $2\pi i$-periodic meromorphic function on $\mathbb{C}$ which we denote by the same name.

This periodicity allows one to define a meromorphic function $\tilde{g}(x)$ on $\mathbb{C} \setminus \{0\}$ by $\tilde{g}(e^t) = h(t)$. As $\tilde{g}(x) = g(x)$ for small $|x| \neq 0$ by the construction of h, we have shown that $g(x)$ can be continued analytically to a meromorphic function on $\mathbb{C}$ which we again will denote by the same name.

The formula $h(t) = G_1(t)d(t)$ and Lemma 24 also imply that $h(t)$ is analytic in some sector $\tilde{S} = \{t \in \mathbb{C}^* \mid \delta < \arg t < 2\delta\}$ with small positive δ. As this sector contains some half strip $\{t \in \mathbb{C} \mid \operatorname{Re} t > L, \mu \operatorname{Re} t < \operatorname{Im} t < \mu \operatorname{Re} t + 3\pi\}$ for some positive L, μ which has vertical width larger than 2π and h is $2\pi i$-periodic, its poles are contained in some vertical strip $\{t \in \mathbb{C} \mid -L < \operatorname{Re} t < L\}$. For the function $g(x)$ this means that it can be continued analytically to a meromorphic function on $\mathbb{C}$ with finitely many poles.

The proposition is proved once we have shown that $g(x)$ has polynomial growth as $|x| \to \infty$. This is done as in the proof of Theorem 4.2 in [38] (see also [11]). Consider $r_0 > 1$ such that $g(x)$ and $A(x)$ are holomorphic on the annulus $|x| > r_0/2$. There are positive numbers K, M such that $|A(x)| \leq K|x|^M$ for $|x| \geq r_0$. Consider now the annuli

$$\mathcal{A}_j = \{x \in \mathbb{C} \mid r_0^{p^j} \leq |x| < r_0^{p^{j+1}}\},\ j = 0, 1, ...$$

covering the annulus $|x| \geq r_0$. Any $x \in \mathcal{A}_j$ can be written $x = \xi^{p^j}$ with some $\xi \in \mathcal{A}_0$. Then we estimate using (37) and the inequality for $|A(x)|$

$$|g(x)| = |g(\xi^{p^j})| \leq K^j \left(|\xi|^{p^{j-1}} \cdots |\xi|^p |\xi|\right)^M \max_{r_0 \leq |\xi| \leq r_0^p} |g(\xi)|.$$

Hence there is a positive constant L such that $|g(x)| \leq L\, K^j\, |x|^{\frac{M}{p-1}}$ for $x \in \mathcal{A}_j$. Assuming $\log(r_0) \geq 1$ without loss in generality, we find that $j \leq \log(\log(|x|))/\log(p)$ for $x \in \mathcal{A}_j$. Hence there exists $d > 0$ such that

$$|g(x)| \leq L\,(\log(|x|))^d\, |x|^{\frac{M}{p-1}} \text{ for } |x| > r_0$$

and the proof of Proposition 22 is complete. ∎

For later use, we note another corollary of Proposition 22.

Corollary 25. *Consider multiplicatively independent positive integers p, q. Let $A(x)$ and $B(x)$ be two matrices with polynomial entries such that the consistency condition (38) holds and $A(0)$ and $B(0)$ are invertible. Then there exists a matrix $G(x)$ with polynomial entries, $G(0) = I$ and*

$$A(x) = G(x^p)\, A(0)\, G(x)^{-1}, \;\; B(x) = G(x^q)\, B(0)\, G(x)^{-1}.$$

As an application, consider two polynomials $a(x), b(x)$ without constant term satisfying

$$a(x^q) - a(x) = b(x^p) - b(x).$$

This is the consistency condition for the matrices $A(x) = \begin{pmatrix} 1 & 0 \\ a(x) & 1 \end{pmatrix}$ and $B(x) = \begin{pmatrix} 1 & 0 \\ b(x) & 1 \end{pmatrix}$. The above Corollary yields the existence of a polynomial matrix $G(x)$ with the stated properties. The conditions encoded in $A(x)G(x) = G(x^p)$ and $G(0) = I$ imply that $G(x) = \begin{pmatrix} 1 & 0 \\ g(x) & 1 \end{pmatrix}$ with some polynomial $g(x)$ without constant term. We obtain

$$a(x) = g(x^p) - g(x), \;\; b(x) = g(x^q) - g(x).$$

In the case of p, q without common divisor, the existence of such a polynomial $g(x)$ can be proved directly. Under the present hypothesis that p and q are multiplicatively independent, this is possible but less elementary. A similar reasoning will appear in the proof of Corollary 26.

Proof of Corollary 25. Putting $G(x) = I + H(x)$, the first equation of the statement is equivalent to

$$H(x) = A(x)^{-1}A(0) - I + A(x)^{-1}H(x^p)A(0).$$

Using the fixed point principle in $\mathcal{M} = \mathrm{gl}_n(x\,\mathbb{C}[[x]])$ equipped with the x-adic norm, it follows that the latter equation has a unique solution in $\mathcal{M}$. This also follows from Proposition 34 of [39]. Hence there is a unique formal series $G(x) = I + ...$ such that

$A(x) = G(x^p)\,A(0)\,G(x)^{-1}$. Now put $\tilde{B}(x) = G(x^q)^{-1}B(x)G(x)$. As $A(x), B(x)$ satisfy the consistency condition, so do $A_0 := A(0)$ and $\tilde{B}(x)$. Again a series expansion yields that $\tilde{B}(x) =: B_0$ is constant.

As in the proof of Corollary 23, the formal solution $G(x) \in I + \mathcal{M}$ of the system

$$G(x^p) = A(x)G(x)A_0^{-1}, \;\; G(x^q) = B(x)G(x)B_0^{-1} \tag{50}$$

can be considered as a formal (matricial) solution of a consistent system (37) and thus must have rational entries by Proposition 22.

It remains to prove that $G(x)$ actually has polynomial entries. As $A(x)$ has polynomial entries, the first equation of (50) implies the following statement: If $x = x_0$ is a finite regular point of all entries of $G(x)$ then so is $x = x_0^p$. The contrapositive – which is also true – says: If $x = x_0$ is a finite pole of some entry of $G(x)$ then so is any p-th root of x_0. Now there is no finite subset of $\mathbb{C}$ that is stable with respect to taking any p-th root of any element besides the empty set and $\{0\}$. As the entries of $G(x)$ have no poles at $x = 0$, the set of their finite poles must be empty. Hence $G(x)$ has polynomial entries. ∎

Unfortunately not every Mahler system is regular singular.

Example. The system $y(x^p) = A(x)y(x),\; A(x) = \begin{pmatrix} 1 & 0 \\ x^{-1} & 2 \end{pmatrix}$ is not regular singular at the origin. Otherwise, there exist an invertible matrix $T(x) \in \mathrm{GL}_2(E)$, $E = \mathbb{C}[[x^{1/r}]][x^{-1/r}]$ for some positive integer r and some constant triangular matrix $C = \begin{pmatrix} a & 0 \\ c & b \end{pmatrix}$ with nonzero a, b such that $A(x)T(x) = T(x^p)C$. For the determinants this means that $2\det(T(x)) = ab\det(T(x^p))$; hence $\det(T(x))$ is a constant and $ab = 2$. Considering the upper right elements of both sides implies $T_{12}(x) = T_{12}(x^p)b$.

If $T_{12}(x) \neq 0$, we obtain $b = 1$ and thus $a = 2$. This means that C has the distinct eigenvalues $1, 2$ and hence with an additional transformation, we can replace it by $\mathrm{diag}(1, 2)$ and as a consequence, $T(x)$ is replaced by some lower triangular matrix. If $T_{12}(x) = 0$, we immediately obtain that $a = 1$, $b = 2$ and hence we can replace C by $\mathrm{diag}(1, 2)$ also in this case.

Thus we can assume in both cases that $C = \mathrm{diag}(1, 2)$ and that $T(x)$ is lower triangular and hence its diagonal elements are constants.

Now put $T(x) = \begin{pmatrix} c & 0 \\ u(x) & d \end{pmatrix}$. Then $u(x) \in E$ satisfies $cx^{-1} + 2u(x) = u(x^p)$. We now consider the vector space F of all formal Laurent series $\sum_{j=-\infty}^{\infty} a_j x^{-j/r}$. Then $u(x) \in F$ and also $v(x) := -c\sum_{k=0}^{\infty} 2^{-k-1}x^{-p^k} \in F$. As $v(x)$ satisfies $cx^{-1} + 2v(x) = v(x^p)$, the difference $d(x) = u(x) - v(x) \in F$ satisfies $2d(x) = d(x^p)$. In view of the series expansions, this means that $d(x) =: d$ is a constant. We obtain that $u(x) = v(x) + d$ in contradiction to $u(x) \in E$.

Fortunately, our matrices $A(x)$ are very special. The proof of Theorem 13 in case 2M is complete, once we have shown

Proposition 26. *Consider $A(x), B(x) \in \mathrm{GL}_n(\hat{K})$, $\hat{K} = \bigcup_{r\in\mathbb{N}^*} C[[x^{1/r}]][x^{-1/r}]$ satisfying the consistency condition (38). Then $x = 0$ is a regular singular point of $y(x^p) = A(x)y(x)$.*

Proof. We will show later

Lemma 27. *Under the hypotheses of Proposition 26, there exists a gauge transformation $y = H(x)z$, $H(x) \in \mathrm{GL}_n(\hat{K})$, that changes (37) into*

$$z(x^p) = \tilde{A}(x)z(x), \;\; z(x^q) = \tilde{B}(x)z(x), \tag{51}$$

where $\tilde{A}(x) = d\,I$ with some $d \in C$ or $\tilde{A}(x), \tilde{B}(x)$ are both lower block triangular with blocks of the same size, i.e. there exist $m \in \{1, ..., n-1\}$ and $A_{11}(x), B_{11}(x) \in \mathrm{GL}_m(\hat{K})$, $A_{22}(x), B_{22}(x) \in \mathrm{GL}_{n-m}(\hat{K})$, $A_{21}(x), B_{21}(x) \in \hat{K}^{n-m,m}$ such that

$$\tilde{A}(x) = \begin{pmatrix} A_{11}(x) & 0 \\ A_{21}(x) & A_{22}(x) \end{pmatrix}, \;\; \tilde{B}(x) = \begin{pmatrix} B_{11}(x) & 0 \\ B_{21}(x) & B_{22}(x) \end{pmatrix}. \tag{52}$$

Recall from the beginning of section 3 that gauge transformations preserve the consistency condition. Hence it also holds for $\tilde{A}(x)$ and $\tilde{B}(x)$.

We now prove the Proposition by induction. In case $n = 1$, it has been shown in Proposition 23 of [39] that $y(x^p) = A(x)y(x)$ is always regular singular. Indeed, if $A(x) = ax^s b(x)$, where $a \in C^*$, $s \in \mathbb{Q}$ and $b(x) \in C[[x]]$ with $b(0) = 1$, then there is a formal series $c(x) \in C[[x]]$ with $c(0) = 1$ satisfying $c(x^p) = b(x)c(x)$ as a power series expansion readily shows. Thus $y(x^p) = A(x)y(x)$ is equivalent to $z(x^p) = ax^s z(x)$ and by $z = x^{\frac{s}{p-1}}v$, this is equivalent to $v(x^p) = a\,v(x)$.

So suppose that the statement has been proved for all dimensions smaller than n. Given $A(x), B(x)$ as in the hypothesis, we now invoke Lemma 27. When $\tilde{A}(x) = dI$, there is nothing to do. Otherwise, we observe that the couples $A_{11}(x), B_{11}(x)$ and $A_{22}(x), B_{22}(x)$ also satisfy the consistency condition and therefore by the induction hypothesis, $x = 0$ is a regular singular point of $u(x^p) = A_{jj}(x)u(x)$, $j = 1, 2$. Thus there exists gauge transformations $u = F_{jj}(x)v$ with entries in $\hat{K}$ that transform the systems to constant ones $v(x^p) = \tilde{A}_{jj}v(x)$, $j = 1, 2$. Performing the same gauge transformations on the systems $u(x^q) = B_{jj}(x)u(x)$, $j = 1, 2$ yields systems $v(x^q) = \tilde{B}_{jj}(x)v(x)$ with $\tilde{B}_{jj}(x)$ having coefficients in $\hat{K}$. As we still have the consistency condition, we have

$$\tilde{A}_{jj}\tilde{B}_{jj}(x) = \tilde{B}_{jj}(x^p)\tilde{A}_{jj}, \;\; j = 1, 2.$$

As used before, this implies that the $\tilde{B}_{jj}(x)$ are also constant and commute with $\tilde{A}_{jj}$. Hence using the block diagonal matrix $F(x) = \operatorname{diag}(F_{11}(x), F_{22}(x))$, the system (51) is reduced to one, where additionally $A_{jj},\ B_{jj}$ are constant and commute.

It remains to show that for a system (51), (52) with constant commuting $A_{jj},\ B_{jj}$ satisfying the consistency condition, the point $x = 0$ is a regular singular point of the first equation of (51). Observe that the consistency condition reduces to an equation for the lower left block

$$A_{21}(x^q)B_{11} + A_{22}B_{21}(x) = B_{21}(x^p)A_{11} + B_{22}A_{21}(x). \tag{53}$$

This suggests splitting $A_{21}(x), B_{21}(x)$ into their polar parts $A_{21}^-(x), B_{21}^-(x)$ and their regular parts $A_{21}^+(x), B_{21}^+(x)$ containing the terms with negative or non-negative exponents, respectively. Then (53) splits into two equations

$$A_{21}^\pm(x^q)B_{11} + A_{22}B_{21}^\pm(x) = B_{21}^\pm(x^p)A_{11} + B_{22}A_{21}^\pm(x). \tag{54}$$

We are interested in the polar parts that have to be removed. They are polynomial in $x^{-1/r}$ for some positive integer r. So we replace $x = \xi^{-r}$ and consider the matrices

$$A^-(\xi) = \begin{pmatrix} A_{11} & 0 \\ A_{21}^-(\xi^{-r}) & A_{22} \end{pmatrix},\ \ B^-(\xi) = \begin{pmatrix} B_{11} & 0 \\ B_{21}(\xi^{-r}) & B_{22} \end{pmatrix}. \tag{55}$$

These matrices are polynomial in ξ and satisfy the consistency condition because of (54). Here Corollary 25 applies and yields a polynomial matrix $G(\xi)$ with $G(0) = I$ such that

$$A^-(\xi)G(\xi) = G(\xi^p)A^-(0). \tag{56}$$

Writing $G(\xi) = \begin{pmatrix} G_{11}(\xi) & G_{12}(\xi) \\ G_{21}(\xi) & G_{22}(\xi) \end{pmatrix}$ with blocks of the same size as those of A^-, we find $A_{11}G_{12}(\xi) = G_{12}(\xi^p)A_{22}$ from the $(1,2)$-blocks in (56). Expanding $G_{12}(\xi)$ in powers of ξ and using $G_{12}(0) = 0$, we find that $G_{12}(\xi)$ vanishes. Similar, the diagonal blocks in (56) now show that $G_{11}(\xi) = I_m$, $G_{22}(\xi) = I_{n-m}$. Finally, its $(2,1)$-blocks show that $A_{21}^-(\xi^{-r}) + A_{22}G_{21}(\xi) = G_{21}(\xi^p)A_{11}$. Then the transformation $z = G(x^{-1/r})w$ changes $z(x^p) = \tilde{A}(x)z(x)$ into $w(x^p) = \breve{A}(x)w(x)$, where $G(x^{-p/r})\breve{A}(x) = \tilde{A}(x)G(x^{-1/r})$. Using the block structure of $\tilde{A}(x)$ and $G(\xi)$, this implies that $\breve{A}(x) = \begin{pmatrix} A_{11} & 0 \\ \breve{A}_{21}(x) & A_{22} \end{pmatrix}$ where

$$\breve{A}_{21}(x) = A_{21}(x) + A_{22}G_{21}(x^{-1/r}) - G_{21}(x^{-p/r})A_{11} = A_{21}(x) - A_{21}^-(x) = A_{21}^+(x).$$

Thus we have shown that $y(x^p) = A(x)y(x)$ is equivalent over $\hat{K}$ to $w(x^p) = \breve{A}(x)w(x)$ where all entries of the coefficient matrix $\breve{A}(x)$ have positive valuation. As shown in the beginning of the proof of Corollary 25 and stated in Proposition 34 of [39], this implies that $x = 0$ is a regular singular point of the latter system and hence also of the given system $y(x^p) = A(x)y(x)$. This completes the proof of Proposition 26. ∎

Proof of Lemma 27. By Theorem 24 of [39], there exists a gauge transformation $y = T(x)z, T(x) \in \mathrm{GL}_n(\hat{K})$ that changes $y(x^p) = A(x)y(x)$ into $z(x^p) = A^{(1)}(x)z(x)$ where $A^{(1)}(x) \in \mathrm{GL}_n(\hat{K})$ is lower triangular and has constant diagonal entries. Performing the same gauge transformation on the second equation $y(x^q) = B(x)y(x)$, we can assume without loss of generality that the given matrix $A(x)$ additionally has this property. In the sequel, we denote the entries of $A(x)$ by $a_{jk}(x)$, those of $B(x)$ by $b_{jk}(x)$, $j, k = 1, ..., n$, and, without mentioning it again, do the same for other matrices. We omit the argument "(x)" if an entry is constant.

If the diagonal entries a_{jj}, $j = 1, ..., n$, were distinct, it could be shown from the consistency condition that $B(x)$ is also lower triangular thus proving the Lemma – in fact, this would first be done for $b_{1n}(x)$ as the consistency condition implies that $a_{11}b_{1n}(x) = b_{1n}(x^p)a_{nn}$ and thus $b_{1n}(x) = 0$ if $a_{11} \neq a_{nn}$; then the same follows successively for the other entries of $B(x)$ above the diagonal in a similar way. Unfortunately, we have no information about these diagonal entry.

Dividing $A(x)$ by a_{nn}, we can assume that $a_{nn} = 1$. Let $m \geq 1$ be the maximal length of a block I_m in the lower right corner of $A(x)$, *i.e.* we start with

$$A(x) = \begin{pmatrix} A_{11}(x) & 0 \\ A_{21}(x) & I_m \end{pmatrix}, \ A_{11}(x) \text{ lower triangular with constant diagonal.}$$

If all entries $b_{jk}(x)$ vanish for $j = 1, ...n - m, \ k = n - m + 1, ..., n$, then $B(x)$ is lower block triangular with blocks of the same size as $A(x)$ and the lemma is proved. Otherwise there is an entry $b_{r\ell}(x) \neq 0$ with $1 \leq r \leq n - m$, $n - m < \ell \leq n$. By going over to the uppermost nonzero entry in the ℓ-th column of $B(x)$, we can assume that $b_{j\ell}(x) = 0$ for $j = 1, ..., r - 1$ if $r > 1$.

We express the element $p_{r\ell}(x)$ of $P(x) = A(x^q)B(x) = B(x^p)A(x)$ in two ways and find $p_{r\ell}(x) = a_{rr}b_{r\ell}(x) = b_{r\ell}(x^p)a_{\ell\ell}$. This yields that $b_{r\ell}(x)$ is constant and $a_{rr} = a_{\ell\ell} = 1$. We denote $d = b_{r\ell}(x)$.

Consider now the following matrix $S(x)$: Its r-th column is $\frac{1}{d}B_\ell(x^{1/q})$, where $B_\ell(x)$ denotes the ℓ-th column of $B(x)$; the other columns of $S(x)$ are the unit vectors $e_1, ..., e_{r-1}$ and $e_{r+1}, ..., e_n$. Observe that $S(x)$ is lower triangular. We now perform the gauge transformation $y = S(x)z$ on our system and obtain $A^{(2)}(x) = S(x^p)^{-1}A(x)S(x)$ and $B^{(2)}(x) = S(x^q)^{-1}B(x)S(x)$ which still satisfy the consistency condition. $A^{(2)}(x)$ is still lower triangular and has unchanged diagonal entries and lower right block. The right multiplication

$B(x)S(x)$ adds multiples of columns $r+1,...,n$ to the r-th column of $B(x)$ – this is not really interesting. The left multiplication by $S(x^q)^{-1}$ substracts $\frac{1}{d}b_{j\ell}(x)$ times row r from row j of the resulting matrix for $j = r+1,...,n$. This means in particular that the ℓ-th column of $B^{(2)}(x)$ is a multiple of the r-th unit vector.

We consider now the entries $p^{(2)}_{j\ell}(x)$, $j = r+1,...,n$, of the product $P^{(2)}(x) = A^{(2)}(x^q)B^{(2)}(x) = B^{(2)}(x^p)A^{(2)}(x)$. As the ℓ-th columns of $B^{(2)}(x)$ and $A^{(2)}(x)$ are multiples of unit vectors, we find that

$$p^{(2)}_{j\ell}(x) = a^{(2)}_{jr}(x^q)d = b^{(2)}_{j\ell}(x^p)a_{\ell\ell} = 0$$

and hence $a^{(2)}_{jr}(x) = 0$ for $j = r+1,...,n$. Thus the r-th column of $A^{(2)}(x)$ equals the r-th unit vector.

If $r < n-m$ then we can exchange the r-th and $(r+1)$-th rows and columns of $A^{(2)}(x)$ and $B^{(2)}(x)$ and obtain $A^{(3)}(x)$, $B^{(3)}(x)$ which still satisfy the consistency condition, $A^{(3)}(x)$ is lower triangular with constant diagonal and lower right block I_m, but now the additional column that is a unit vector is the $(r+1)$-th column. If $r+1 < n-m$ then we repeat the modification. In this way, we obtain a system $v(x^p) = A^{(4)}(x)v(x)$, $v(x^q) = B^{(4)}(x)v(x)$ equivalent to (37) over $\hat{K}$, where $A^{(4)}(x)$ is still lower triangular with constant diagonal but now has a lower right block I_{m+1}, *i.e.* its size has increased. Thus we can start all over and, after a finite number of steps, we either reach a situation (51), (52) proving the lemma or we stop with $\tilde{A}(x) = I_n$. This proves the Lemma and completes the proof of Theorem 13 in case 2M. ∎

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