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A Class of Vectorfields on S^2 That Are Topologically Equivalent to Polynomial Vectorfields*

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Let X be a C^1 vectorfield on $S^2 = \{(x, y, z) \in \mathbb{R}^3 : x^2 + y^2 + z^2 = 1\}$ such that no open subset of S^2 is the union of closed orbits of X. If X has only a finite number of singular orbits and satisfies one additional condition, then it is shown that X is topologically equivalent to a polynomial vectorfield. A corollary is that a foliation $\mathscr{F}$ of the plane is topologically equivalent to a foliation by orbits of a polynomial vectorfield if and only if $\mathscr{F}$ has only a finite number of inseparable leaves.

1. INTRODUCTION

Two vectorfields X and Y defined on oriented manifolds M and N, respectively, are called *topologically equivalent* if there is an orientation-preserving homeomorphism $h: M \to N$ that sends orbits of X to orbits of Y, preserving the direction of the orbits. In this paper, we investigate the question, which vectorfields on S^2 are topologically equivalent to polynomial vectorfields? Here $S^2 = \{(x, y, z) \in \mathbb{R}^3 : x^2 + y^2 + z^2 = 1\}$. "Vectorfield on S^2" always means a tangent vectorfield to S^2; a polynomial vectorfield on S^2 is, in addition, one each of whose coordinates is a polynomial in x, y, z.

To state our result, we recall from [A] that an orbit γ of a vectorfield X is called *positively* (*resp. negatively*) *stable* if nearby orbits stay near γ in positive (resp. negative) time. More precisely, γ is positively (resp. negatively) stable if for any point p on γ and any $\varepsilon > 0$ there exists $\delta(p, \varepsilon) > 0$ such that if $\| q - p \| < \delta$, then the positive (resp. negative) semiorbit of X through q lies within ε of the positive (resp. negative) semiorbit of X through p. An orbit is called *positively* (resp. *negatively*) *unstable* if it is not

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positively (resp. negatively) stable. An orbit is called *singular* if it is positively or negatively unstable, or if it is an equilibrium. If a vectorfield on S^2 has only a finite number of singular orbits, then these are just equilibria, boundaries of hyperbolic sectors at the equilibria (*separatrices*), and isolated closed orbits [A, p. 258].

Our main result is

THEOREM 1.1. *Let X be a C^1 vectorfield on S^2 such that*

(H1) *no open subset of S^2 is the union of closed orbits*;

(H2) *X has only a finite number of singular orbits*;

(H3) *X satisfies the separatrix cycle condition described in Section* 5.

Then X is topologically equivalent to a polynomial vectorfield.

The hypotheses of Theorem 1.1 allow any finite number of closed orbits; they allow any finite number of equilibria, each with any finite number of elliptic, hyperbolic, and parabolic sectors; and they allow certain patterns of separatrix connections among the equilibria. (Hypothesis (H3) is a restriction on the latter.) However, the hypotheses of Theorem 1.1, while sufficient for the existence of a polynomial model, are certainly not necessary. There exist polynomial (in fact linear) vectorfields that violate (H1), and there exist those with curves of equilibria that trivially violate (H2). Now that Dulac's proof that a polynomial vectorfield on $\mathbb{R}^2$ or S^2 can have only a finite number of isolated closed orbits has been questioned (see [C-S]), the necessity of (H2) is in doubt even for polynomial vectorfields with a finite number of equilibria. More relevant to our result is the existence of polynomial vectorfields that satisfy (H1) and (H2) but violate (H3), and whose existence could not be proved by our approach (Example 2, Sect. 5). On the other hand, our approach, with some additional complication, can be used to show the existence of polynomial models for certain vectorfields that satisfy (H1) and (H2) but violate (H3) (Example 3, Sect. 5).

We remark that the proof of Theorem 1.1 is nonconstructive. Moreover, there is no estimate of the degree of the polynomial model. For explicit construction of polynomial models for structurally stable vectorfields on $\mathbb{R}^2$, see [Sv].

We shall refer to vectorfields on S^2 that satisfy (H1) and (H2) as *vectorfields of finite type*.

The paper is organized as follows. In Section 2, we review the notion of the *scheme* of a vectorfield of finite type on S^2. According to Andronov *et al.* two vectorfields with the same scheme are topologically equivalent. We shall use this fact to show that various vectorfields we construct are topologically equivalent to our original vectorfield X. In Section 3, we discuss a class of easily understood equilibria of vectorfields on $\mathbb{R}^2$ or S^2. This class includes a model for each topological equivalence class of equilibria that we shall encounter. In Section 4, we review the work of Reyn [Re1, Re2] on separatrix cycles joining first-order saddle points, and we extend this work to certain separatrix cycles that pass through saddle-nodes. In Section 5, we construct a C^∞ model Y for X whose closed orbits are of simplest possible type and whose equilibria are of the class described in Section 3. This construction can be carried out provided X is of finite type and satisfies the separatrix cycle condition described in Section 5. After these preliminaries, the proof of Theorem 1.1 is given in Section 6, with proofs of two lemmas postponed until Sections 7 and 8.

The proof of Theorem 1.1 goes as follows. We imbed Y in a q-parameter family of vectorfields $Y(\lambda_1,..., \lambda_q, x)$, in which the topological structure of the equilibria is preserved, but nonequilibrium orbits that are both positively and negatively unstable are broken. Here q is the number of non-equilibrium orbits of Y that are both positively and negatively unstable. This class includes orbits that are both α- and ω-separatrices (see Sect. 2) and closed orbits that are attracting on one side and repelling on the other. We approximate the family $Y(\lambda, x)$ by a polynomial family $\hat{Y}(\lambda, x)$, again preserving the topological structure of the equilibria. For some $\hat{\lambda}$, the orbits of Y that are both positively and negatively unstable all remain unbroken in $\hat{Y}(\hat{\lambda}, \cdot)$. Thus, corresponding to each limit set of Y (these are equilibria, closed orbits, or separatrix cycles for vectorfields of finite type) there is a corresponding limit set of $\hat{Y}(\hat{\lambda}, \cdot)$. It is easy to ensure that each closed orbit of $\hat{Y}(\hat{\lambda}, \cdot)$ has the same attracting or repelling behavior on each side as the corresponding closed orbit of Y. Moreover, because of the separatrix cycle condition, we can ensure that each separatrix cycle of $\hat{Y}(\hat{\lambda}, \cdot)$ has the same attracting or repelling behavior as the corresponding separatrix cycle of Y, and that no new closed orbits are created nearby. It follows that $\hat{Y}(\hat{\lambda}, \cdot)$ has the same scheme as Y and hence is topologically equivalent to Y.

In Section 9, we use Theorem 1.1 to draw conclusions about polynomial models for vectorfields on $\mathbb{R}^2$. In particular we show that a foliation of $\mathbb{R}^2$ with only a finite number of inseparable leaves is topologically equivalent to a foliation by orbits of a polynomial vector field.

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2. Scheme of a Vectorfield of Finite Type

Let p be an isolated equilibrium of a C^1 vectorfield of finite type X on S^2. Then p has arbitrarily small closed *canonical neighborhoods* whose boundaries are composed of curves transverse to X and parts of orbits of X [A, pp. 313–314]; see Fig. 1. The restrictions of X to any two canonical

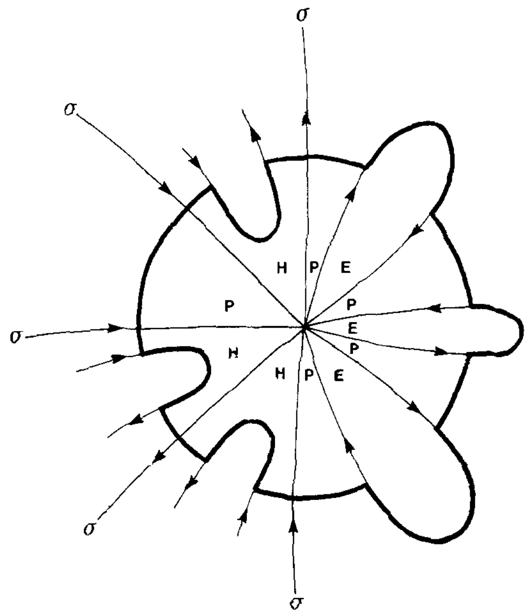

FIG. 1. *E*, elliptic sector; *H*, hyperbolic sector; *P*, parabolic sector; σ, separatrix.

neighborhoods of p are topologically equivalent. There is a familiar division of any canonical neighborhood of p into a finite number of elliptic, hyperbolic, and parabolic sectors [A, Chap. 8]; see Fig. 1. An α- (resp. ω-) *separatrix at p* is a semiorbit of X that approaches p as $t \to -\infty$ (resp. as $t \to \infty$) and that bounds a hyperbolic sector at p. We shall use the shorter expression *separatrix* to refer to an orbit of X that includes an α- or ω-separatrix at any equilibrium.

If F_X is the flow of X, so that $F_X(p, t)$ is the orbit of X that passes through p at $t = 0$, then q belongs to the α- *limit set* (resp. ω- *limit set*) of p if and only if there is a sequence $t_n \to -\infty$ (resp. $t_n \to \infty$) such that $\| F_X(p, t_n) - q \| \to 0$. A *limit set K* is the α- or ω-limit set of some point. It is *nontrivial* if it is not an equilibrium. A limit set is always a compact connected union of orbits. A nontrivial limit set K is a limit set *from the right* (resp. *from the left*) if an arbitrarily small transversal to a nonequilibrium orbit in K contains points on the right (resp. on the left) of K whose α- or ω-limit set is K. Here right and left are determined by the time orientation of the orbit in K. If X has only a finite number of singular orbits, then each nontrivial limit set of X is either a single closed orbit or a compact connected union of equilibria and orbits that are α-separatrices at one end and ω-separatrices at the other. A limit set of the latter type is called a *separatrix cycle*.

Andronov *et al.* define the *scheme* of a vectorfield of finite type to be the following information:

(S1) A list of all equilibria $p_1, \ldots, p_m$ and nonequilibrium singular orbits $\gamma_1, \ldots, \gamma_n$.

(S2) For each equilibrium p_i, a list, in counterclockwise cyclic order, of singular orbits whose α- or ω-limit set is p_i and elliptic sectors at p_i. When a singular orbit is listed we state whether p_i is its α- or ω-limit set; if both, it is listed twice.

(S3) A list of all nontrivial limit sets $K_1, \ldots, K_p$. If a set is a limit set from both right and left, it is listed twice. Thus each K_k is regarded as a limit set from the right or left only.

(S4) For each K_k, the following information:

(a) A list, as follows, of the nonequilibrium singular orbits and equilibria that comprise K_k:

$$p_{i_1}, \gamma_{j_1}, p_{i_2}, \gamma_{j_2}, \ldots, p_{i_c}, \gamma_{j_c},$$

where p_{i_l} is the α- (resp. ω-) limit set of γ_{j_l} (resp. $\gamma_{j_{l-1}}$), $l = 1, \ldots, c$. (Lower subscripts are read mod c. If K_k is a closed orbit, then of course K_k is just some γ_j.)

(b) Whether the counterclockwise traversal of each simple closed curve in K_k agrees with the traversal as t increases or decreases. ("Counterclockwise" is defined by identifying $S^2 \setminus \{p_0\}$ with $\mathbb{R}^2$, where p_0 is a point of S^2 that lies on no singular orbit.)

(c) Whether K_k is a limit set from the right or from the left, and whether K_k is an α- or ω-limit set from that side.

(d) A list, in out-to-in cyclic order, of the singular orbits that limit on K_k from the side in question.

(S5) A list of all *ordered pairs of conjugate free limit sets.* (A, B) is an ordered pair of conjugate free limit sets provided (a) A (resp. B) is either a p_i or a K_k; (b) A (resp. B) is not the α- (resp. ω-) limit set of any γ_j (if A or B is a K_k, this means any γ_j on the correct side); (c) there is an orbit γ such that A (resp. B) is the α- (resp. ω-) limit set of γ (from the correct side if A or B is a K_k).

Two schemes are *equivalent* if there is a bijection between their sets of equilibria and nonequilibrium singular orbits that preserves all the above information.

THEOREM 2.1 [A, p. 442]. *If two C^1 vector fields of finite type on S^2 have equivalent schemes, then they are topologically equivalent.*

We remark that because of the arbitrary choice made in (S4)(b), the scheme of a vectorfield is not unique up to equivalence.

3. MODEL EQUILIBRIA

An earlier, and not quite correct, version of the material in this section appeared in [S-S2].

We call an equilibrium p of a C^∞ vectorfield Y on $\mathbb{R}^2$ or S^2 a *model equilibrium* if either

(E1) p is a first-order node or focus (i.e., both eigenvalues of $DY(p)$ lie on the same side of the imaginary axis); or

(E2) in some local coordinates (x, y) such that p corresponds to $(0, 0)$, we have

$$\dot{x} = X_d(x, y) + X_{d+1}(x, y) + o(|x|^{d+1} + |y|^{d+1}),$$
$$\dot{y} = Y_d(x, y) + Y_{d+1}(x, y) + o(|x|^{d+1} + |y|^{d+1}),$$

with X_i, Y_i homogeneous polynomials of degree i, $d \geqslant 2$, and

(a) $xY_d(x, y) - yX_d(x, y) = (-1)^{d-1}(x^2+y^2)\prod_{j=1}^{d-1}(y - jx)$;
(b) $xY_{d+1}(x, jx) - jxX_{d+1}(x, jx) = 0$, $j = 1, \ldots, d-1$;
(c) if $xX_d(x, jx) + jxY_d(x, jx) = 0$, then $xX_{d+1}(x, jx) + jxY_{d+1}(x, jx) \neq 0$, $j = 1, \ldots, d-1$.

The significance of (E2) is seen by blowing up the equilibrium. Let Φ: $\mathbb{R} \times S^1 \to \mathbb{R}^2$ be the polar coordinate map $\Phi(r, \theta) = (r\cos\theta, r\sin\theta)$. Let $\bar{Y} = r^{1-d}\Phi^* Y$, where $Y = Y(x, y)$ is written in the coordinates of (E2). $\bar{Y}$ is given by

$$\dot{r} = \sum_{i=0}^{1} r^{i+1}[\cos\theta X_{d+i}(\cos\theta, \sin\theta) + \sin\theta Y_{d+i}(\cos\theta, \sin\theta)] + o(r^2),$$

$$\dot{\theta} = \sum_{i=0}^{1} r^{i}[\cos\theta Y_{d+i}(\cos\theta, \sin\theta) - \sin\theta X_{d+i}(\cos\theta, \sin\theta)] + o(r).$$

If $r = 0$, then $\dot{r} = 0$, and, by (E2)(a), $\dot{\theta} = 0$ if and only if $\tan\theta = j$, $j = 1, \ldots, d-1$. If $\tan\theta_0 = j$ for some j between 1 and $d-1$, then

$$\begin{bmatrix} \dfrac{\partial \dot r}{\partial r}(0,\theta_0) & \dfrac{\partial \dot r}{\partial \theta}(0,\theta_0) \\ \dfrac{\partial \dot\theta}{\partial r}(0,\theta_0) & \dfrac{\partial \dot\theta}{\partial \theta}(0,\theta_0) \end{bmatrix}$$

$$= \begin{bmatrix} \begin{array}{l}\cos\theta_0 X_d(\cos\theta_0, \sin\theta_0) \\ \quad +\sin\theta_0 Y_d(\cos\theta_0,\sin\theta_0)\end{array} & 0 \\ \begin{array}{l}\cos\theta_0 Y_{d+1}(\cos\theta_0,\sin\theta_0) \\ \quad -\sin\theta_0 X_{d+1}(\cos\theta_0,\sin\theta_0)\end{array} & \begin{array}{l}\left.\dfrac{d}{d\theta}\right|_{\theta=\theta_0}[\cos\theta Y_d(\cos\theta,\sin\theta) \\ \quad -\sin\theta X_d(\cos\theta,\sin\theta)]\end{array} \end{bmatrix}.$$

By (E2)(a), $\partial\dot\theta/\partial\theta\ (0,\theta_0)\neq 0$; by (E2)(b), $\partial\dot\theta/\partial r(0,\theta_0)=0$. Thus $D\bar Y(0,\theta_0)$ has eigenvectors $\partial/\partial r$ and $\partial/\partial\theta$. The eigenvalue for $\partial/\partial\theta$ is always nonzero. Thus $\dot\theta(0,\theta)$ changes sign at each equilibrium of $\bar Y$ on $r=0$. If $\partial\dot r/\partial r(0,\theta_0)$, the eigenvalue for $\partial/\partial r$, is nonzero, then $(0,\theta_0)$ is a first-order node or saddle. If $\partial\dot r/\partial r(0,\theta_0)=0$, it follows from (E2)(c), the formula for $\dot r$, and [A, p. 340], that $\bar Y$ has a saddle-node at $(0,\theta_0)$; moreover, one can choose local coordinates $(\bar r, \bar\theta)$ near $(0,\theta_0)$ so that $\bar r=0$, $\bar\theta=0$ corresponds to $(0,\theta_0)$ and

$$\bar Y(\bar r,\bar\theta)=(\sigma\bar r^2+\tau\bar r\bar\theta+\eta\bar\theta^2+\psi(\bar r,\bar\theta))\frac{\partial}{\partial\bar r}+(\xi\bar\theta+\phi(\bar r,\bar\theta))\frac{\partial}{\partial\bar\theta} \tag{3.1}$$

with $\xi\neq 0$, $\sigma\neq 0$, $\phi=o(|\bar r|+|\bar\theta|)$, $\psi=o(|\bar r|^2+|\bar\theta|^2)$. We shall refer to a saddle-node for which there exists such a choice of coordinates as a *second-order saddle-node.*

Thus $\bar Y$ has only nodes, saddles, and saddle-nodes of simplest type on the circle $r=0$. If $\bar Y$ has a node (resp. saddle, saddle-node) at $(0, \tan^{-1} j)$, then it has a node (resp. saddle, saddle-node) at $(0, \pi+\tan^{-1} j)$. If $\bar Y$ has a saddle-node at $(0, \tan^{-1} j)$ with two hyperbolic sectors in $r\geqslant 0$, then at $(0, \pi+\tan^{-1} j)$ there is a parabolic sector in $r\geqslant 0$; and vice versa. Note that $\dot\theta$ is positive for $r=0$, $0\leqslant\theta<\tan^{-1} 1$, so that if $(0,\tan^{-1} 1)$ has two hyperbolic sector in $r\geqslant 0$, then there is an α-separatrix at $(0,\tan^{-1} 1)$ in $r>0$, and if $(0,\tan^{-1} 1)$ is a node or has a parabolic sector in $r\geqslant 0$, then all nearby orbits in $r\geqslant 0$ approach $(0,\tan^{-1} 1)$ as $t\to\infty$.

We define the *saddle–node sequence* of a model equilibrium of degree d, $d\geqslant 2$, to be a certain sequence of $2d-2$ symbols from the set $\{S_\alpha, S_\omega, N_\alpha, N_\omega\}$. The jth symbol is determined by the behavior of $\bar Y$ in $r\geqslant 0$ near the jth equilibrium of $\bar Y$ on $r=0$, counting in the counterclockwise direction from $\theta=0$. Let $(0,\theta_0)$ be this jth equilibrium. The jth symbol of the saddle–node sequence is

S_α (resp. S_ω) if there are two hyperbolic sectors of $\bar{Y}$ at $(0, \theta_0)$ in $r \geqslant 0$, bounded by $r=0$ and an α- (resp. ω-) separatrix at $(0, \theta_0)$;

N_α (resp. N_ω) if a neighborhood of $(0, \theta_0)$ in $r \geqslant 0$ is the union of negative (resp. positive) semiorbits of $\bar{Y}$ that converge to $(0, \theta_0)$.

The *saddle–node cycle* of a model equilibrium is just the saddle–node sequence thought of as a cycle: the first term in the sequence follows the last. The following lemma is immediate.

LEMMA 3.1. (1) *The first symbol in the saddle–node sequence of a model equilibrium is S_α or N_ω.* (2) *In the saddle–node cycle of a model equilibrium, S_α is always followed by S_ω or N_α; S_ω by S_α or N_ω; N_α by S_α or N_ω; N_ω by S_ω or N_α.*

Equilibria p and q of vectorfields X and Y on oriented manifolds M and N are called *topologically equivalent* if there are neighborhoods U and V of p and q such that $X | U$ is topologically equivalent to $Y | V$ via a homeomorphism that takes p to q.

THEOREM 3.2. *Every equilibrium of a C^1 vectorfield of finite type is topologically equivalent to a model equilibrium.*

In Section 5, we shall replace our C^1 vectorfield X by a C^∞ vectorfield Y that is topologically equivalent to X and has only model equilibria. The reader will note that this may involve replacing equilibria of X by more degenerate equilibria; for example, we shall replace first-order saddles by degree 3 model equilibria. In this instance, at least, the replacement is done solely to achieve uniformity of exposition, and could easily be avoided.

The remainder of this section is devoted to the proof of Theorem 3.2. Let X be a C^1 vectorfield of finite type having an equilibrium at p. If X has no hyperbolic or elliptic sectors at p, then p is topologically equivalent to a first-order node (or focus). Otherwise, the topological equivalence class of p is determined by the arrangement, in counterclockwise cyclic order, of elliptic sectors at p and α- and ω-separatrices at p [A, p. 315]. Thus we shall represent the topological equivalence class of p by a cycle of symbols from the set $\{E, S_\alpha, S_\omega\}$. This cycle is called the *local scheme* of the equilibrium. From it we construct a saddle–node cycle as follows:

(SN1) If the local scheme includes no S_α or S_ω, then it consists of an even number of E's, say $2k$ E's. The saddle–node cycle includes k N_α's and k N_ω's, with N_α's and N_ω's alternating.

(SN2) Otherwise:

(a) If two S_α's (resp. S_ω's) are adjacent in the local scheme, add on N_α (resp. N_ω) between them.

(b) If a string of E's of length k occurs in the local scheme, replace it by $k+1$ symbols, each an N_α or N_ω, with the N_α's and N_ω's alternating. The new string starts with N_α (resp. N_ω) if the string of E's is preceded by S_α (resp. S_ω).

Thus the equilibrium of Fig. 1 has local scheme

$$EES_\alpha S_\omega S_\omega S_\alpha S_\omega E, \tag{3.2}$$

so its saddle–node cycle is

$$N_\omega N_\alpha S_\alpha S_\omega N_\omega S_\omega S_\alpha S_\omega N_\omega N_\alpha. \tag{3.3}$$

LEMMA 3.3. *The saddle–node cycle constructed from the local scheme of an equilibrium satisfies Lemma* 3.1(2).

Proof. Use the fact that in a local scheme that is not all E's, a string of an even number of E's is surrounded by two S_α's or two S_ω's; a string of an odd number of E's is surrounded by one S_α and one S_ω. ∎

LEMMA 3.4. *The saddle–node cycle constructed from the local scheme of an equilibrium has even length.*

Proof. Let $e=$ number of elliptic sectors at the equilibrium, $h=$ number of hyperbolic sectors at the equilibrium. By a formula of Bendixson, the index i of an equilibrium is given by $i=\frac{1}{2}(e-h+2)$. Since i is an integer, $e \equiv h \bmod 2$. We divide the proof into three cases.

Case 1. If the scheme contains no S_α's or S_ω's, the lemma follows immediately from (SN1). (Note that e is even because $h=0$.)

Case 2. If the scheme contains no E's, let $l=$ length of scheme; $l_\alpha=$ number of S_α's that follow an S_α in the scheme; $l_\omega=$ number of S_ω's that follow an S_ω in the scheme; $m=$ length of saddle–node cycle. Then $h=l-(l_\alpha+l_\omega)\equiv 0 \bmod 2$ (since $e=0$). Therefore $m=l+(l_\alpha+l_\omega)\equiv 0 \bmod 2$.

Case 3. If neither case 1 nor 2 holds, then the scheme divides into $s \geqslant 1$ strings of E's and s strings of S_α's and S_ω's. Let $e_j=$ length of jth string of E's; $l_j=$ length of jth string of S_α's and S_ω's; $l_{\alpha,j}=$ number of S_α's in jth string of S_α's and S_ω's that follow an S_α; $l_{\omega,j}=$ number of S_ω's in jth string of S_α's and S_ω's that follow an S_ω; $h_j=$ number of hyperbolic sectors represented by the jth string of S_α's and S_ω's. Then $h_j = l_j - (l_{\alpha,j} + l_{\omega,j}) - 1$. Therefore

$$\begin{aligned} m &= \sum_{j=1}^{s} (e_j + 1) + \sum_{j=1}^{s} (l_j + l_{\alpha,j} + l_{\omega,j}) \\ &= \sum_{j=1}^{s} (e_j + 1) + \sum_{j=1}^{s} [h_j + 2(l_{\alpha,j} + l_{\omega,j}) + 1] \\ &\equiv e + h \bmod 2 \equiv 0 \bmod 2. \quad \blacksquare \end{aligned}$$

Given the saddle–node cycle constructed from the local scheme of an equilibrium, we choose a saddle–node sequence $\sigma_1 \cdots \sigma_m$ by letting σ_1 be an S_α or N_ω in the saddle–node cycle, and continuing from there.

LEMMA 3.5. *There exists a model equilibrium having this saddle–node sequence.*

Proof. Let $m = 2(d-1)$. By the *abbreviated saddle–node sequence* $\tilde{\sigma}_1 \cdots \tilde{\sigma}_{2(d-1)}$, we shall mean the saddle-node sequence $\sigma_1 \cdots \sigma_{2(d-1)}$ with the subscripts α and ω removed. Thus $\tilde{\sigma}_1 \cdots \tilde{\sigma}_{2(d-1)}$ is a sequence of S's and N's. In order that a model equilibrium have the desired abbreviated saddle–node sequence, it is necessary that X_d and Y_d satisfy

(L1) $xY_d - yX_d = (-1)^{d-1}(x^2+y^2)\prod_{j=1}^{d-1}(y - jx)$,

and for each $j = 1, \ldots, d-1$,

($\mathrm{L2}_j$) if $\theta = \tan^{-1} j$, then $\cos\theta\, X_d(\cos\theta, \sin\theta) + \sin\theta\, Y_d(\cos\theta, \sin\theta)$ has sign

$$\begin{array}{ll} (-1)^j & \text{if } \tilde{\sigma}_j = N \text{ and } \tilde{\sigma}_{d-1+j} = N, \\ (-1)^{j+1} & \text{if } \tilde{\sigma}_j = S \text{ and } \tilde{\sigma}_{d-1+j} = S, \\ 0 & \text{otherwise.} \end{array}$$

Replace ($\mathrm{L2}_j$), $j = 1, \ldots, d-1$, by the requirement that $\cos\theta\, X_d(\cos\theta, \sin\theta) + \sin\theta\, Y_d(\cos\theta, \sin\theta)$ equal a specified value. Let $X_d(x, y) = \sum_{j=0}^{d} a_j x^j y^{d-j}$, $Y_d(x, y) = \sum_{j=0}^{d} b_j x^j y^{d-j}$. Then (L1) and (L2) become a system of $2d+1$ linear equations in the $2d+2$ unknowns $a_0, \ldots, a_d, b_0, \ldots, b_d$. Divide equations ($\mathrm{L2}_j$) by $\cos^{d+1}\theta$, $j = 1, \ldots, d-1$. Then the matrix of the system becomes

$$\begin{array}{cc}
 & \begin{array}{cc} \qquad\qquad d+1 \qquad\qquad\qquad & \qquad\qquad d+1 \qquad\qquad \end{array} \\
\begin{array}{c} \\ \\ d+2 \\ \\ \\ \\ \\ d-1 \\ \\ \end{array} &
\left[\begin{array}{ccccc|cccccc}
-1 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 & \cdots & 0 & 0 \\
0 & -1 & 0 & & 0 & 1 & 0 & 0 & & 0 & 0 \\
\vdots & & & & & \vdots & & & & & \\
0 & 0 & 0 & & -1 & 0 & 0 & 0 & & 1 & 0 \\
0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 & \cdots & 0 & 1 \\
\hline
1 & 1 & 1 & \cdots & 1 & 1 & 1 & 1 & \cdots & 1 & 1 \\
2^d & 2^{d-1} & 2^{d-2} & & 1 & 2^{d+1} & 2^d & 2^{d-1} & & 2^2 & 2 \\
\vdots & & & & & \vdots & & & & & \\
(d-1)^d & (d-1)^{d-1} & (d-1)^{d-2} & \cdots & 1 & (d-1)^{d+1} & (d-1)^d & (d-1)^{d-1} & \cdots & (d-1)^2 & (d-1)
\end{array}\right].
\end{array}$$

To show that this system can be solved, we add the row

$$[d^d \; d^{d-1} \; d^{d-2} \cdots 1 \; d^{d+1} \; d^d \; d^{d-1} \cdots d^2 \; d]$$

to the bottom of the matrix and show that the resulting square matrix has nonzero determinant. To evaluate the determinant, first add column j to column $d+j$, $j=2,\ldots, d+1$. Then expand in turn by rows $1,\ldots, d+2$. The result is $-|A|$, where A is a certain $d \times d$ matrix. If we factor out $2=1^2+1$ from the first row of A, 2^2+1 from the second row,...., d^2+1 from the dth row, we find that $|A| = |B| \prod_{j=1}^{d} (j^2+1)$, where $|B|$ is clearly nonzero: up to permutation of its columns, B is the transpose of a Vandermonde matrix.

In order that a model equilibrium have abbreviated saddle–node sequence $\tilde{\sigma}_1 \cdots \tilde{\sigma}_{2(d-1)}$, it is further necessary that for each $j=1,\ldots, d-1$, X_{d+1} and Y_{d+1} satisfy.

(L3$_j$) if $\theta = \tan^{-1} j$, then $\cos\theta Y_{d+1}(\cos\theta, \sin\theta) - \sin\theta X_{d+1}(\cos\theta, \sin\theta) = 0$;

(L4$_j$) if $\theta = \tan^{-1} j$ and $\cos\theta X_d(\cos\theta, \sin\theta) + \sin\theta Y_d(\cos\theta, \sin\theta) = 0$, then $\cos\theta X_{d+1}(\cos\theta, \sin\theta) + \sin\theta Y_{d+1}(\cos\theta, \sin\theta)$ has sign

$$\begin{array}{ll} (-1)^j & \text{if } \tilde{\sigma}_j = N, \\ (-1)^{j+1} & \text{if } \tilde{\sigma}_j = S. \end{array}$$

To see that this system can be solved, we replace (L4$_j$) by the requirement that $\cos\theta X_{d+1}(\cos\theta, \sin\theta) + \sin\theta Y_{d+1}(\cos\theta, \sin\theta)$ take on the value $\varepsilon_j = \pm 1$ at $\theta = \tan^{-1} j$ for all $j = 1,\ldots, d-1$. Then an equivalent system is

(L5$_j$) $X_{d+1}(\cos\theta, \sin\theta) = \varepsilon_j \cos\theta$ ($\theta = \tan^{-1} j$; $j=1,\ldots, d-1$);

(L6$_j$) $Y_{d+1}(\cos\theta, \sin\theta) = \varepsilon_j \sin\theta$ ($\theta = \tan^{-1} j$; $j=1,\ldots, d-1$),

which can be solved for X_{d+1}, Y_{d+1}.

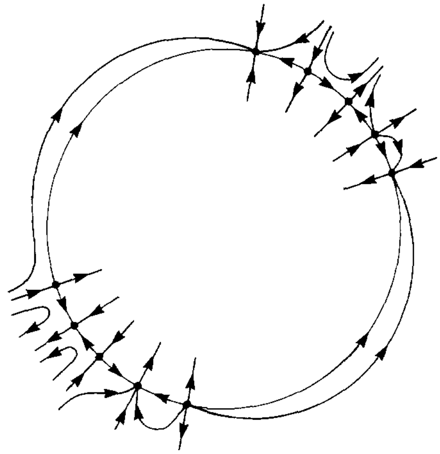

FIGURE 2

Thus we can find a model equilibrium having abbreviated saddle–node sequence $\tilde{\sigma}_1 \cdots \tilde{\sigma}_{2(d-1)}$. That it has saddle–node sequence $\sigma_1 \cdots \sigma_{2(d-1)}$ follows from Lemmas 3.1 and 3.3, and the requirement that σ_1 be S_α or N_ω. ∎

Theorem 3.2 follows from a demonstration that the model equilibrium of Lemma 3.5 has the same local scheme as the original equilibrium. This is left to the reader. In Fig. 2, we have sketched the blow-up of a model equilibrium having saddle–node sequence (3.3) (cf. Fig. 1).

4. SEPARATRIX CYCLES

For any subset A of $\mathbb{R}^2$ or S^2, let $N_\varepsilon(A)$ denote the set of points in $\mathbb{R}^2$ or S^2 whose distance from A is $\leqslant \varepsilon$.

Let Z_0 be a C^s vector field, $2 \leqslant s < \infty$, on $\mathbb{R}^2$ or S^2 with a first-order saddle at p. We may assume local coordinates (x, y) have been chosen so that $p = (0, 0)$ and

$$Z_0(x, y) = (\mu x + \phi(x, y)) \frac{\partial}{\partial x} + (\lambda y + \psi(x, y)) \frac{\partial}{\partial y},$$

$$\mu < 0, \lambda > 0, \phi = o(|x| + |y|), \psi = o(|x| + |y|).$$

If A is a compact subset of $\mathbb{R}^2$ or S^2, we denote by $\mathscr{X}^s(A)$ the Banach space of C^s vectorfields on $\mathbb{R}^2$ or S^2 restricted to A, with the C^s topology. For a fixed $\varepsilon > 0$, let $\mathscr{X}$ be the affine subspace of $\mathscr{X}^s(N_\varepsilon(0, 0))$ consisting of vectorfields that agree to first order with Z_0 at $(0, 0)$. We denote by $C^s[N_\varepsilon(0, 0), \mathbb{R}^2, (0, 0)]$ the space of C^s maps from $N_\varepsilon(0, 0)$ to $\mathbb{R}^2$ that fix the origin, with the C^s topology. By the stable-unstable manifold theorem [C-H, Sect. 9.2] there is a continuous map Ψ from a neighborhood $\mathscr{U}$ of Z_0 in $\mathscr{X}$ to $C^s[N_\varepsilon(0, 0), \mathbb{R}^2, (0, 0)]$, and a neighborhood U of $(0, 0)$ in $\mathbb{R}^2$, such that for all $Z \in \mathscr{U}$, $\Psi(Z) = \Psi_Z$ is a diffeomorphism, and for all $(x, y) \in U$,

$$\Psi_{Z*}Z(x, y) = \mu x(1 + \tilde{\phi}_Z(x, y))\frac{\partial}{\partial x} + \lambda y(1 + \tilde{\psi}_Z(x, y))\frac{\partial}{\partial y},$$

$$\tilde{\phi}_Z = o(|x| + |y|), \tilde{\psi}_Z = o(|x| + |y|).$$

In other words, for each Z near Z_0 there is a local diffeomorphism of $\mathbb{R}^2$ that takes the local stable manifold of Z to the x axis and the local unstable manifold of Z to the y axis; and this diffeomorphism depends smoothly on Z.

For a, b, δ positive and small there is, for each $Z \in \mathscr{U}$, a mapping from $\{a\} \times (0, \delta]$ to $(0, 1) \times \{b\}$ given by following the flow of $\Psi_{Z*}Z$. We write $\beta = \beta_Z(\alpha)$ if (a, α) goes to (β, b) under this mapping. Reyn [Re1] has shown that

$$\beta_Z(\alpha) = A_Z \alpha^{-\mu/\lambda}(1 + f_Z(\alpha)),$$

where $A_Z > 0$ depends continuously on Z, $f_Z(\alpha)$ is continuous for each Z, and $f_Z(\alpha) \to 0$ as $\alpha \to 0$. Thus we can extend f_Z to a continuous mapping from $[0, \delta]$ to $[0, 1)$, still denoted f_Z, that takes 0 to 0. Reyn's argument also shows that the mapping $\mathscr{U} \to C^0([0, \delta], \mathbb{R})$, $Z \to f_Z$, is continuous. (This fact is used in [Re2].)

Now suppose Z_0 has first-order saddles $p_1, ..., p_c$ (not necessarily distinct) and separatrices $\gamma_1, ..., \gamma_c$ such that $K = p_1\gamma_1 p_2\gamma_2 \cdots p_c\gamma_c$ is a separatrix cycle. Let μ_i be the negative eigenvalue and λ_i the positive eigenvalue at p_i. Reyn shows that if $\prod_{i=1}^c |\mu_i/\lambda_i| < 1$, then K is repelling (i.e., an α-limit set); if $\prod_{i=1}^c |\mu_i/\lambda_i| > 1$, then K is attracting. (A separatrix cycle through saddles is a limit set from one side only.) Now let $\mathscr{X}$ be the affine subspace of $\mathscr{X}^s(N_\varepsilon(K))$ consisting of vectorfields that agree to first order with Z_0 at $p_1, ..., p_c$. Assume $\prod_{i=1}^c |\mu_i/\lambda_i| \neq 1$. Then according to Reyn [Re2], for ε small enough there is a neighborhood $\mathscr{U}$ of Z_0 in $\mathscr{X}$ such that if $Z \in \mathscr{U}$ also has a separatrix cycle $K' = p_1\gamma_1' p_2\gamma_2' \cdots p_c\gamma_c'$ in $N_\varepsilon(K)$, then Z has no closed orbits in $N_\varepsilon(K)$. Of course K' is attracting (resp. repelling) if K is.

Reyn's results can easily be extended to certain separatrix cycles passing through saddle-nodes. For our purposes the following extension is sufficient.

Let Z_0 be a C^s vectorfield on $\mathbb{R}^2$ or S^2, $2 \leqslant s < \infty$, with a second-order saddle-node at p. This means (cf. (3.1)) that we can choose local coordinates (x, y) so that $p=(0,0)$ and

$$Z_0(x, y) = (\xi x + \phi(x, y))\frac{\partial}{\partial x} + (\sigma x^2 + \tau xy + \eta y^2 + \psi(x, y))\frac{\partial}{\partial y},$$

$$\xi \neq 0, \eta \neq 0, \phi = o(|x| + |y|), \psi = o(|x|^2 + |y|^2).$$

Let $\mathscr{X}$ be the affine subspace of $\mathscr{X}^s(N_\varepsilon(0,0))$ consisting of vectorfields that agree to second order with Z_0 at $(0,0)$. By the stable–unstable and center manifold theorems [C-H, Sect. 9.2], there is a continuous mapping Ψ from a neighborhood $\mathscr{U}$ of Z_0 in $\mathscr{X}$ to $C^s[N_\varepsilon(0,0), \mathbb{R}^2, (0,0)]$, and a neighborhood U of $(0,0)$ in $\mathbb{R}^2$, such that for all $Z \in \mathscr{U}$, $\Psi(Z) = \Psi_Z$ is a diffeomorphism, and for all $(x, y) \in U$,

$$\Psi_{Z*}Z(x, y) = \xi x(1 + \hat{\phi}_Z(x, y))\frac{\partial}{\partial x} + y(\hat{\tau}x + \hat{\eta}y + \hat{\psi}_Z(x, y))\frac{\partial}{\partial y}$$

with $\eta\hat{\eta} > 0$, $\hat{\phi}_Z = o(1)$, $\hat{\psi}_Z = o(|x| + |y|)$. In other words, for each Z near Z_0 there is a local diffeomorphism of $\mathbb{R}^2$ that takes the local stable or unstable manifold of Z to the x-axis and a local center manifold of Z to the y-axis; and this local diffeomorphism depends smoothly on Z. Divide $\Psi_{Z*}Z$ by $1 + \hat{\phi}_Z(x, y)$ to obtain $\tilde{Z} = \xi x\, \partial/\partial x + y(\tilde{\tau}x + \tilde{\eta}y + \tilde{\psi}_Z(x, y))\, \partial/\partial y$. $\tilde{Z}$ and $\Psi_{Z*}Z$ have the same orbits on U for U sufficiently small. We wish to study the flow of $\tilde{Z}$ in a hyperbolic sector at $(0,0)$. We assume for convenience that $\xi < 0$ and $\tilde{\eta} > 0$, so that the first quadrant is a hyperbolic sector.

Choose $\rho > 0$. If U is small enough, then for all $(x, y) \in U$ with $y > 0$ and for all $Z \in \mathscr{U}$,

$$y(\tilde{\tau}x + \tilde{\eta}y + \tilde{\psi}_Z(x, y)) < \rho y. \tag{4.1}$$

For a, b, δ positive and small there is, for each $Z \in \mathscr{U}$, a mapping from $\{a\} \times (0, \delta]$ to $(0,1) \times \{b\}$ given by following the orbits of $\tilde{Z}$. We write $\beta = \beta_Z(\alpha)$ if (a, α) goes to (β, b) under this mapping.

Let $W = \xi x(\partial/\partial x) + \rho y(\partial/\partial y)$. By (4.1), for all $Z \in \mathscr{U}$,

$$\beta_Z(\alpha) < \beta_W(\alpha) = Ae^{-\xi/\rho} \qquad (A > 0).$$

Intuitively, since ρ may be chosen as small as we like, this estimate shows that if a separatrix cycle K contains a saddle-node p_i, the "contribution" of p_i toward K's being attracting or repelling dominates the contributions of any first-order saddles in K. One can use Reyn's method of proof in [Re2] to formalize this idea and show

THEOREM 4.1. *Let Z_0 be a C^s vectorfield, $2 \leqslant s < \infty$, on $\mathbb{R}^2$ or S^2 having the separatrix cycle $K = p_1\gamma_1 p_2\gamma_2 \cdots p_c\gamma_c$, where each p_i is either a first-order saddle or a second-order saddle-node, and at least one p_i is a saddle-node. Let $\mathscr{X}$ be the affine subspace of $\mathscr{X}^s(N_\varepsilon(K))$ consisting of vectorfields that agree with Z_0 to first order at each saddle and to second order at each saddle-node. Assume that at every saddle-node p_i the nonzero eigenvalue of Z_0 is negative (resp. positive). Then K is attracting (resp. repelling). Moreover, for ε small enough, there is a neighborhood $\mathscr{U}$ of Z_0 in $\mathscr{X}$ such that if $Z \in \mathscr{U}$ also has a separatrix cycle $K' = p_1\gamma_1' p_2\gamma_2' \cdots p_c\gamma_c'$ in $N_\varepsilon(K)$, then K' is attracting (resp. repelling) and Z has no closed orbits in $N_\varepsilon(K)$.*

5. C^∞ MODEL VECTORFIELDS

Let Y be a C^∞ vectorfield of finite type on S^2 each of whose equilibria is a model equilibrium. Choose $p_0 \in S^2$ such that p_0 lies on no singular orbit of Y. We shall identity $Y | S^2 \backslash \{p_0\}$ with a C^∞ vector field Y_0 on $\mathbb{R}^2$ having only model equilibria. Suppose the equilibria of Y_0 of degree $\geqslant 2$ (those that are not nodes or foci) are at $p_1, ..., p_a$ and have degrees $d_1, ..., d_a$. Using the blowing-up construction [D1, D2]) we can find a C^∞ vectorfield $\tilde{Y}$ with the following properties:

(B1) $\tilde{Y}$ is defined on a space that is C^∞ diffeomorphic to $\mathbb{R}^2 \backslash \{p_1, ..., p_a\}$, which we shall identify with the latter.

(B2) There exist C^∞ simple closed curves $\Gamma_1, ..., \Gamma_a$ surrounding $p_1, ..., p_a$ respectively, with disjoint interiors, such that $\tilde{Y} | \mathbb{R}^2 \backslash \bigcup_{i=1}^a \overline{\operatorname{Int} \Gamma_i}$ is C^∞ diffeomorphic to $Y_0 | \mathbb{R}^2 \backslash \{p_1, ..., p_a\}$.

(B3) For each $i = 1, ..., a$, define vectorfields Y_i by $Y_i(x) = Y_0(x - p_i)$. There is an $\varepsilon > 0$ and neighborhoods U_i of Γ_i such that $\tilde{Y} | U_i$ is C^∞ diffeomorphic to $r^{1-d_i}\Phi^* Y_i | (-\varepsilon, \varepsilon) \times S^1$. ($\Phi$ is the polar coordinate map of Section 3.)

Thus all critical points of $\tilde{Y} | \mathbb{R}^2 \backslash \bigcup_{i=1}^a \operatorname{Int} \Gamma_i$ that are not nodes or foci lie on the Γ_i and are first-order saddles or second-order saddle-nodes. Each separatrix cycle K of Y corresponds to a separatrix cycle $\tilde{K}$ of $\tilde{Y} | \mathbb{R}^2 \backslash \bigcup_{i=1}^a \operatorname{Int} \Gamma_i$. $\tilde{K}$ includes arcs of one or more Γ_i that are separatrices of $\tilde{Y}$.

We shall refer to any such $\tilde{Y}$ as a *blow-up* of Y.

Recall that if γ is a closed orbit of Y and $p \in \gamma$, a Poincaré map Π at p is defined by taking a transversal T to γ at p and defining $\Pi: T \to T$ by $\Pi(x) =$ first return of orbit through x to T.

A vectorfield Y on S^2 is called a C^∞ *model vectorfield* provided:

(V1) Y is C^∞ and of finite type.

(V2) All equilibria of Y are model equilibria.

(V3) If γ is a closed orbit of Y, $p \in \gamma$, Π is a Poincaré map at p, and $D\Pi(p) = 1$, then $D^2\Pi(p) \neq 0$.

(V4) If $\tilde{K} = q_1\gamma_1 q_2\gamma_2 \cdots q_c\gamma_c$ is a separatrix cycle of $\tilde{Y} \mid \mathbb{R}^2 \backslash \bigcup_{i=1}^{a}$ Int Γ_i, where $\tilde{Y}$ is a blow-up of Y, then either (1) all q_i in $\tilde{K}$ are saddles and $\prod_{i=1}^{c} \mu_i/\lambda_i \neq 1$ (μ_i = negative eigenvalue of $D\tilde{Y}(q_i)$, λ_i = positive eigenvalue of $D\tilde{Y}(q_i)$); or (2) some q_i in $\tilde{K}$ are saddle-nodes, and at all q_i in $\tilde{K}$ that are saddle-nodes, the nonzero eigenvalues of $\tilde{Y}$ have the same sign.

Condition (V3) is independent of the choices of p and Π; (V4) is independent of the choice of blow-up $\tilde{Y}$.

Let X be a C^1 vectorfield of finite type on S^2. For each equilibrium p_i of X that is not topologically equivalent to a node, we construct a corresponding saddle–node sequence $\Sigma_i = \sigma_{i1}\sigma_{i2} \cdots \sigma_{im_i}$ as in Section 3. Set $m_i = 2(d_i - 1)$. Each separatrix cycle K of X corresponds to a cycle C_K of some of the σ_{ij}. Any σ_{ij} in such a cycle is an S_α or S_ω. Let $\mathscr{S}$ denote the set of all σ_{ij} such that $\sigma_{ij} \in \{S_\alpha, S_\omega\}$ and $\sigma_{i,j+d_i-1} \in \{S_\alpha, S_\omega\}$. Here the second subscript is mod $2(d_i - 1)$. We say X satisfies the *separatrix cycle condition* provided there is a function $f(\sigma_{ij})$ from $\mathscr{S}$ to the positive reals such that

(F1) $f(\sigma_{ij}) = f(\sigma_{i,j+d_i-1})$ if $d_i - 1$ is even; $f(\sigma_{ij}) = [f(\sigma_{i,j+d_i-1})]^{-1}$ if $d_i - 1$ is odd.

(F2) For every one-sided limit set K of X that is a separatrix cycle, either

(1) all σ_{ij} in C_K are in $\mathscr{S}$ and $\prod_{\sigma_{ij} \in C_K} f(\sigma_{ij}) > 1$ (resp. < 1) if K is attracting (resp. repelling); or

(2) some σ_{ij} in C_K are not in $\mathscr{S}$; if K is attracting (resp. repelling), all such σ_{ij} are S_α's (resp. S_ω's).

EXAMPLE 1. Let X be a C^1 vectorfield whose phase portrait near p is given by Fig. 3. There are three separatrix cycles in Fig. 3, each a limit set from one side only: $p\gamma_1$ is attracting, $p\gamma_2$ is repelling, and $p\gamma_1\, p\gamma_2$ is attracting. This vectorfield does not satisfy (F2)(1) of the separatrix cycle condition. The saddle–node sequence of p is

$$\sigma_1\sigma_2\sigma_3\sigma_4 = S_\alpha S_\omega S_\alpha S_\omega, \tag{5.1}$$

with the correspondence between separatrix cycles of X and cycles of the σ_j being

$$p\gamma_1 \to \sigma_1\sigma_4,$$

$$p\gamma_2 \to \sigma_3\sigma_2,$$

$$p\gamma_1\, p\gamma_2 \to \sigma_1\sigma_4\sigma_3\sigma_2.$$

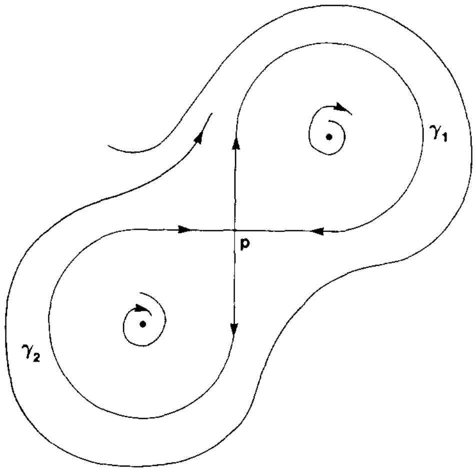

FIGURE 3

Thus we must have $f(\sigma_1)f(\sigma_4) > 1$ and $f(\sigma_3)f(\sigma_2) < 1$. Since $f(\sigma_1) = f(\sigma_3)$ and $f(\sigma_4) = f(\sigma_2)$, this is impossible.

Suppose a C^∞ vectorfield Y has a model equilibrium at $(0, 0)$ that is topologically equivalent to the equilibrium of X at p. Then it must have saddle–node sequence (5.1). Denote the corresponding equilibria of $\bar{Y}$ (notation of Sect. 3) on the circle $r = 0$ by q_1, q_2, q_3, q_4. Let $\mu_j =$ negative eigenvalue of $D\bar{Y}(q_j)$, $\lambda_j =$ positive eigenvalue of $D\bar{Y}(q_j)$, $f(q_j) = |\mu_j/j_j|$. Because of (V4)(1), the nonexistence of f satisfying (F1) and (F2) implies that there is no C^∞ model vectorfield topologically equivalent to X. The reader should also remark that if the equilibrium of X at p is a first-order saddle with eigenvalues $\mu < 0$ and $\lambda > 0$, it must be that $|\mu/\lambda| = 1$. We do not know if there is a polynomial vectorfield part of whose phase portrait is Fig. 3.

EXAMPLE 2. The phase portrait of Fig. 4 also violates (F2)(1) of the separatrix cycle condition. Nevertheless the polynomial vectorfield on $\mathbb{R}^2$,

$$\begin{bmatrix} \dot{x} \\ \dot{y} \end{bmatrix} = ((x-1)^2 + y^2) \begin{bmatrix} (x^2+y^2-1)^2 & -1 \\ 1 & (x^2+y^2-1)^2 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

has this phase portrait. There is no model vectorfield on S^2 part of whose phase portrait is Fig. 4, so the existence of a polynomial vectorfield on $\mathbb{R}^2$ having this phase portrait could not be proved by our method. ((For the relationship between polynomial vectorfields on $\mathbb{R}^2$ and S^2 see Sect. 9.)

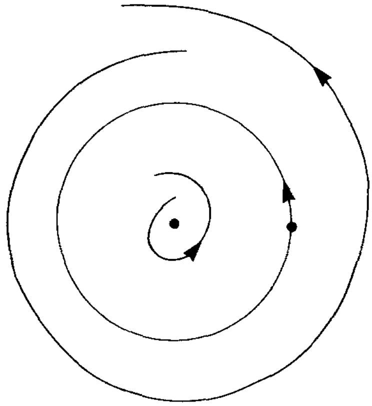

FIGURE 4

EXAMPLE 3. Consider the phase portrait of Fig. 5. Equilibrium p_1 has saddle–node sequence

$$\sigma_{11}\sigma_{12}\sigma_{13}\sigma_{14}=S_\alpha S_\omega N_\omega S_\omega,$$

and p_2 has saddle–node sequence

$$\sigma_{21}\sigma_{22}\sigma_{23}\sigma_{24}\sigma_{25}\sigma_{26}=S_\alpha S_\omega S_\alpha S_\omega N_\omega N_\alpha.$$

The separatrix cycle of Fig. 5 corresponds to the cycle $\sigma_{11}\sigma_{22}\sigma_{23}\sigma_{14}$, which violates (F2)(2) in the separatrix cycle condition. However, p_1 is topologically equivalent to a model equilibrium with saddle–node sequence

$$\sigma'_{11}\sigma'_{12}\sigma'_{13}\sigma'_{14}\sigma'_{15}\sigma'_{16}=S_\alpha S_\omega N_\omega S_\omega N_\omega S_\omega,$$

with the separatrix cycle corresponding to $\sigma'_{11}\sigma_{22}\sigma_{23}\sigma'_{16}$. This cycle satisfies (F2)(2). Thus the proof of Theorem 5.1 will show that there is a model vectorfield part of whose phase portrait is Fig. 5. By means of this trick, our arguments can be used to show the existence of polynomial vectorfields with certain phase portraits that violate the separatrix cycle condition.

THEOREM 5.1. *If X is a C^1 vectorfield of finite type on S^2 that satisfies the separatrix cycle condition, then there is a C^∞ model vectorfield Y on S^2 that is topologically equivalent to X.*

Proof. We shall give the proof under the assumption that X is itself C^∞. The general case requires some additional approximation.

Let $p_1,..., p_m$ be the equilibria of X. Let $\gamma_1,..., \gamma_n$ be the nonequilibrium singular orbits of X with $\gamma_1,..., \gamma_l$ closed and $\gamma_{l+1},..., \gamma_n$ not closed. The

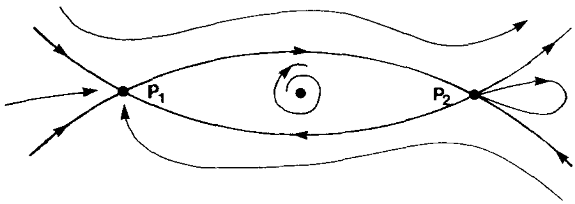

FIGURE 5

notion of a canonical neighborhood of an equilibrium was mentioned in Section 2. A *canonical neighborhood of a closed orbit* γ is a closed annular neighborhood $N(\gamma)$ such that X is transverse to its boundary, and every point of $N(\gamma)$ belongs to a semiorbit that limits on γ without leaving $N(\gamma)$. Canonical neighborhoods of closed orbits having the same attracting or repelling behavior on respective sides are topologically equivalent [A, p. 382]. Choose disjoint canonical neighborhoods $N(p_i)$, $i=1,\dots,m$, $N(\gamma_j)$, $j=1,\dots,l$, so small that:

(P1) Each γ_j, $j=l+1,\dots,n$, that has p_i as its α- or ω-limit set intersects $N(p_i)$ in one or two semiorbits that limit on p_i. These semiorbits meet $\partial N(p_i)$ transversally.

(P2) Let $\{s_{ik}\}$ denote the set of semiorbits in $N(p_i)$ that are α- or ω-separatrices at p_i. Let $s_{ik} \cap \partial N(p_i) = \{a_{ik}\}$. There is a disjoint set of connected subarcs I_{ik} of $\partial N(p_i)$ such that $a_{ik} \in I_{ik}$, X is transverse to $\bigcup I_{ik}$, each γ_j other than those of (P1) meets $\partial N(p_i)$ only in $\bigcup I_{ik}$, and if such a γ_j meets I_{ik}, then $s_{ik} \subset \alpha(\gamma_j) \cup \omega(\gamma_j)$.

(P3) On each separatrix cycle K that is a limit set on the right (resp. left) there is a point p on $K \backslash \bigcup N(p_i)$; a right (resp. left) transversal T to K at p, $T \subset S^2 \backslash [\bigcup_{i=1}^m N(p_i) \cup \bigcup_{i=1}^l N(\gamma_j)]$, parameterized by ϕ: $[0, a] \to S^2$ with $\phi(0)=p$; and a Poincaré map Π for K, $\Pi: \phi((0, a_1]) \to \phi(0, a_2]$, $\Pi(\phi(a_1)) = \phi(a_2)$, such that if K is attracting (resp. repelling) on the side in question, then $(\phi^{-1} \circ \Pi \circ \phi)(u) - u < 0$ (resp. >0) for all $u \in (0, a_1]$. Moreover, the orbit from $\phi(a_1)$ to $\phi(a_2)$ does not meet any $N(p_i)$.

Let $\tilde{N}(p_i)$ be a canonical neighborhood of p_i contained in Int $N(p_i)$, and let $A(p_i) = \text{Int}\, N(p_i) \backslash \tilde{N}(p_i)$. Let Z be a C^∞ vectorfield on $S^2 \backslash \bigcup A(p_i)$ such that:

(Z1) $Z \mid \tilde{N}(p_i)$ is a canonical neighborhood of a model equilibrium at p_i topologically equivalent to the equilibrium of X at p_i. Thus if the equilibrium of X at p_i is not topologically equivalent to a node, then $Z \mid \tilde{N}(p_i)$ is C^∞ equivalent to a C^∞ vectorfield Z_i, defined on a closed neighborhood of (0, 0) in $\mathbb{R}^2$, that satisfies (E2)(a), (b), (c) of Section 3. Let $\sum_i = \sigma_{i1} \cdots \sigma_{im_i}$ be the saddle–node sequence constructed from the local scheme of X at p_i. Denote the corresponding equilibria of $\bar{Z}_i$ on $r=0$ by $q_{i1},\dots, q_{im_i}$. At each q_{ij} that is a saddle, let μ_{ij} (resp. λ_{ij}) denote the negative (resp. positive) eigenvalue. We require that $|\mu_{ij}/\lambda_{ij}| = f(\sigma_{ij})$. (The proof of Theorem 3.2 shows that there exist model equilibria with the desired saddle–node sequence and the desired λ_{ij}, μ_{ij}. The fact that a canonical neighborhood of such a model equilibrium can be mapped onto $\tilde{N}(p_i)$ by a C^∞ diffeomorphism is geometrically obvious.)

(Z2) $Z|N(\gamma_j)$ is topologically equivalent of $X|N(\gamma_j)$; moreover, $Z|N(\gamma_j)$ equals $X|N(\gamma_j)$ except in a small neighborhood of some point of γ_j, and Z satisfies (V3) on γ_j. (This is easily arranged by altering X on a small flow box around a point $p\in\gamma_j$. If γ_j is attracting or repelling from both sides, we can arrange that $D\Pi(p)\neq 1$; if γ_j is attracting from one side and repelling from the other, we can arrange that $D\Pi(p)=1$ and $D^2\Pi(p)\neq 0$.)

(Z3) Outside $\bigcup_{i=1}^m \operatorname{Int} N(p_i)\cup\bigcup_{j=1}^l \operatorname{Int} N(\gamma_j)$, $Z=X$.

We now extend Z to $\bigcup A(p_i)$.

1. *Elliptic Sectors.* Each elliptic sector $\tilde{E}$ of Z in $\tilde{N}(p_i)$ corresponds to an elliptic sector E of X in $N(p_i)$. Let b_1, b_2 denote the points of $\partial N(p_i)$ that are common to E and the adjacent parabolic sectors, and let $\tilde{b}_1$ and $\tilde{b}_2$ be points of $\partial\tilde{N}(p_i)$ belonging to the corresponding parabolic sectors (see Fig. 6).

Join $\tilde{b}_1$ to b_1 and b_2 to $\tilde{b}_2$ by curves across $A(p_i)$ that fit together with the integral curves of Z through $\tilde{b}_1$, b_1, b_2, $\tilde{b}_2$ to form a C^∞ curve, and extend Z along these curves in a nonzero C^∞ manner. Extend Z in a C^∞ manner across the open region R_E of Fig. 6 so that Z is tangent to the constructed curves and Z is nonzero in R_E. (This can be done in a C^0 manner since $Z|\partial R_E$ has index 0, then C^∞ approximated; see [H] on C^∞

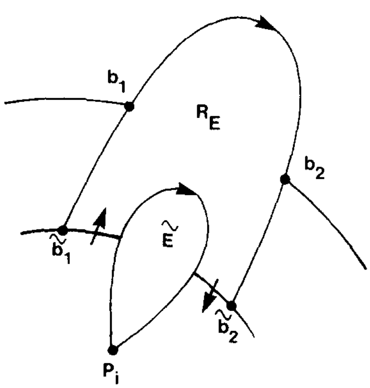

FIGURE 6

approximation.) Then every orbit of Z that enters R_E on one side of $\tilde{E}$ leaves on the other (or there would be an equilibrium of Z in R_E). This construction is to be done so that the regions R_E are disjoint.

2. *Separatrices.* Let $\{\tilde{s}_{ik}\}$ be the set of semiorbits of Z in $\tilde{N}(p_i)$ that are α- or ω-separatrices of Z at p_i. Here $\tilde{s}_{ik} \sim s_{ik}$ under the equivalence between $Z|\tilde{N}(p_i)$ and $X|N(p_i)$. Let $\tilde{s}_{ik}$ meet $\partial\tilde{N}(p_i)$ in $\tilde{a}_{ik}$. Join each a_{ik} to $\tilde{a}_{ik}$ by a curve across $A(p_i)$ that fits together with $\tilde{s}_{ik}$ and the integral curve of Z through a_{ik} to form a C^∞ curve. Extend Z along these constructed curves in a nonzero C^∞ manner.

3. *Hyperbolic Sectors.* Corresponding to each hyperbolic sector $\tilde{H}$ of Z in $\tilde{N}(p_i)$ there is now an open region R_H of $A(p_i)$ as shown in Fig. 7. Extend Z over R_H in a nonzero C^∞ manner.

4. *Parabolic Sectors.* Corresponding to each parabolic sector of Z at p_i there is now an open region R_P of $A(p_i)$ as shown in Fig. 8. Extend Z over R_P in a nonzero C^∞ manner.

5. If Z has a node or focus at p_i, then Z is transverse to $\partial A(p_i)$. Extend Z over $A(p_i)$ in a nonzero C^∞ manner so that each orbit of Z that meets $A(p_i)$ enters through one boundary curve and leaves through the other. This can be done, e.g., by first transferring the radial flow on an annulus to $A(p_i)$ by a diffeomorphism and then smoothing.

Each separatrix cycle K of X corresponds to a separatrix cycle $\tilde{K}$ of Z. The separatrix cycle condition and (Z1) guarantee that K and $\tilde{K}$ have the

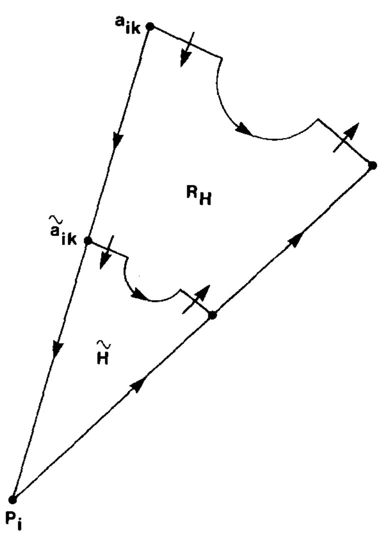

FIGURE 7

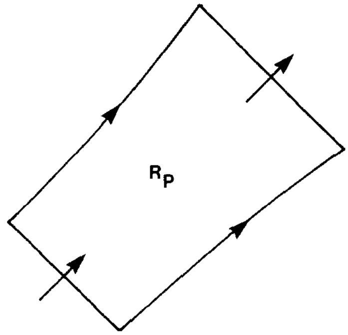

FIGURE 8

same stability. A Poincaré map $\tilde{\Pi}$ for $\tilde{K}$ is defined on the transversal T of (P3); $\tilde{\Pi}$ takes $\phi((0, a_1])$ to $\phi((0, a_2])$. (To see that $\tilde{\Pi}$ is defined on $\phi((0, a_1])$, assume for definiteness that $a_2 > a_1$ and consider the open "annulus" R_K bounded by $\tilde{K}$, the orbit of Z(or X) from $\phi(a_1)$ to $\phi(a_2)$, and $\phi([a_1, a_2])$. Since $\tilde{K}$ is repelling by (P3) and there are no equilibria of Z in R_K, the ω-limit set of a point in $\phi((0, a_1])$ must be a closed orbit in R_K or must lie at least partly outside R_K [A, p. 92]. But any closed orbit in R_K must cross $\phi((0, a_1])$ or there would be an equilibrium of Z in R_K. It follows that the positive semiorbit through any point in $\phi((0, a_1])$ either returns to $\phi((0, a_1])$ or leaves R_K through $\phi(a_1, a_2])$.

If there are closed orbits of Z in R_K, we now alter Z near T in order to break them. Assume for definiteness that K (hence $\tilde{K}$) is repelling. Then for some $\rho > 0$, $(\phi^{-1} \circ \tilde{\Pi} \circ \phi)(u) > u$ for $u \in (0, \rho] \cup [a_1 - \rho, a_1]$. Let $G(u, t) = F_Z(\phi(u), t)$, where F_Z is the flow of Z (Sect. 2). By altering Z on $G([\rho, a_1 - \rho] \times [0, \delta])$ we can ensure that $(\phi^{-1} \circ \tilde{\Pi} \circ \phi)(u) > u$ for all $u \in (0, a_1]$. Thus the new vectorfield has no closed orbits in R_K. Let Y be the vectorfield that results from carrying out this construction for each separatrix cycle.

We claim that Y has no nonequilibrium singular orbits other than those corresponding to $\gamma_1,..., \gamma_n$. Any nonequilibrium singular orbit of Y other than those corresponding to $\gamma_1,..., \gamma_n$ must have nonempty intersection with $\bigcup A(p_i)$, and in fact must contain a semiorbit that meets $\bigcup A(p_i) \cup \bigcup N(\gamma_j)$ only in regions R_H that are not included in regions R_K. Let us consider for definiteness a positively unstable semiorbit γ of Y that meets the interior of such a region R_H. By construction no positively unstable semiorbit of X meets R_H, so all orbits of X that meet R_H are positively asymptotic to the same equilibrium or closed orbit [A, p. 274]. In fact, all must ultimately enter $A(p_i)$, where p_i is a node or focus, or all must ultimately enter a region R_P, or all must ultimately enter a region $N(\gamma_j)$. But by our constructions the same is true of orbits of Y that meet R_H, a contradiction.

Thus the only singular orbits of Y are the equilibria $p_1,\dots, p_m$, their separatrices, and the closed orbits $\gamma_1,\dots,\gamma_l$. The proof of Theorem 5.1 is completed by showing that Y has the same scheme as X (Sect. 2) and hence is topologically equivalent to X. ▮

6. Proof of Theorem 1.1

Suppose the model vectorfield Y of Theorem 5.1 has equilibria $p_1,\dots, p_m$ of degrees $d_1,\dots, d_m$. If $d_i \geqslant 2$, the topological equivalence class of p_i is determined by its jet of order d_i+1. Choose $s \geqslant \max(d_i+2)$. Let $\mathscr{X}$ be the set of C^s vectorfields Z on S^2 such that for $i=1,\dots, m$, $Z-Y$ vanishes to order d_i+1 at p_i if $d_i \geqslant 2$, and to 1st order at p_i if $d_i=1$. $\mathscr{X}$ is an affine subspace of $\mathscr{X}^s(S^2)$.

Let ψ, χ: $\mathbb{R}\to\mathbb{R}$ be two C^∞ functions such that $0\leqslant\psi$, $\chi\leqslant 1$, and moreover

(1) $\psi(t)>0$ if and only if $0<t<1$, and $\int_0^1 \psi(t)\,dt=1$;

(2) $\chi(u)=1$ if $|u|\leqslant\frac{1}{2}$ and $\chi(u)=0$ if $|u|\geqslant 1$.

We renumber the nonequilibrium singular orbits of Y so that $\gamma_1,\dots,\gamma_q$ denote the orbits that are both positively and negatively unstable. The numbering is chosen so that $\gamma_1,\dots,\gamma_M$ are closed and $\gamma_{M+1},\dots,\gamma_q$ are not.

For $j=1,\dots, q$ let B_j be a flow box around a point of γ_j, with the B_j disjoint. The B_j are chosen so small that no singular orbit meets B_j except those that limit on a limit set containing γ_j. On each B_j we choose coordinates (t_j, u_j) such that $Y|B_j=\partial/\partial t_j$. By multiplying Y by a small positive constant if necessary, we can ensure that B_j contains $\{(t_j, u_j): 0\leqslant t_j\leqslant 1, -1\leqslant u_j\leqslant 1\}$, with $u_j=0$ corresponding to $\gamma_j\cap B_j$.

Define a q-parameter family $\mathscr{Y}(\lambda)=\mathscr{Y}(\lambda_1,\dots,\lambda_q)$ of vector fields in $\mathscr{X}$ as follows:

$$\text{If } x\notin\bigcup_{j=1}^{q} B_j, \text{ then } \mathscr{Y}(\lambda)(x)=Y(x).$$

$$\text{If } x\in B_j, \text{ then } \mathscr{Y}(\lambda)(x)=\mathscr{Y}(\lambda)(t_j, u_j)=\frac{\partial}{\partial t_j}+\lambda_j\psi(t_j)\,\chi(u_j)\,\frac{\partial}{\partial u_j}.$$

For $j=1,\dots, M$ we define $\mathscr{F}_j$: a neighborhood of Y in $\mathscr{X}\to\mathbb{R}$ as follows. In B_j, consider the line segment T_j defined by $t_j=0$. Let $T_j^\delta=\{(0,u_j): |u_j|<\delta\}$. For all Z near Y in $\mathscr{X}$, Z is transverse to T_j, and, if δ is small enough, a first return map Π_Z: $T_j^\delta\to T_j$ is defined. Use u_j as the coordinate on T_j and define $\mathscr{G}_j(Z,u_j)=\Pi_Z(u_j)-u_j$. Define $u_j(Z)$ implicitly by $\partial\mathscr{G}_j/\partial u_j(Z, u_j(Z))=0$. This can be done since $\partial\mathscr{G}_j/\partial u_j(Y,0)=0$ because γ_j is both positively and negatively unstable, and, by (V3) in the definition of model vectorfield (Sect. 5), $\partial^2 \mathscr{G}_j/\partial u_j^2(Y, 0) \neq 0$. Finally, define $\mathscr{F}_j(Z) = \mathscr{G}_j(Z, u_j(Z))$.

LEMMA 6.1. *There is a neighborhood $\mathscr{U}$ of Y in $\mathscr{X}$ and canonical neighborhoods $N(\gamma_j)$, $j = 1,..., M$, such that all $\mathscr{F}_j$, $j = 1,..., M$, are defined on $\mathscr{U}$, and for each j,*

(1) *if $Z \in \mathscr{U}$, then $Z | N(\gamma_j) \neq 0$ and Z is transverse to $\partial N(\gamma_j)$;*

(2) *$\mathscr{F}_j | \mathscr{U}$ is C^{s-1};*

(3) *if $Z \in \mathscr{U}$ and $\mathscr{F}_j(Z) = 0$, then $Z | N(\gamma_j)$ is topologically equivalent to $Y | N(\gamma_j)$.*

Moreover, $\partial/\partial \lambda_k(\mathscr{F}_j \circ \mathscr{Y})(0) = \pm \delta_{kj}$ (Kronecker delta; $k = 1,..., q$; $j = 1,..., M$).

Proof. [So, pp. 9–12]. ∎

For each $j = M + 1,..., q$, γ_j contains an α-separatrix at some $p_{i_\alpha(j)}$ and an ω-separatrix at $p_{i_\omega(j)}$. Let $T_j \subset B_j$ be defined as above, with coordinate u_j. For each $Z \in \mathscr{X}$ there is a unique orbit $\gamma_{j,\alpha}(Z)$ that contains an α-separatrix at $p_{i_\alpha(j)}$ having the same tangent at $p_{i_\alpha(j)}$ as an α-separatrix contained in γ_j. Let $u_{j,\alpha}(Z) =$ first intersection of $\gamma_{j,\alpha}(Z)$ with T_j. Similarly, there is a unique orbit $\gamma_{j,\omega}(Z)$ that contains an ω-separatrix at $p_{i_\omega(j)}$ having the same tangent at $p_{i_\omega(j)}$ as an ω-separatrix contained in γ_j. Let $u_{j,\omega}(Z) =$ last intersection of $\gamma_{j,\omega}(Z)$ with T_j.

LEMMA 6.2. *If $\mathscr{U}$ is a small enough neighborhood of Y in $\mathscr{X}$, then for each $j = M + 1,..., q$, the map $Z \to u_{j,\alpha}(Z)$ (resp. $u_{j,\omega}(Z)$) is defined for all $Z \in \mathscr{U}$ and is $C^{s-d_{i_\alpha(j)}}$* (resp. $C^{s-d_{i_\omega(j)}}$).

Lemma 6.2 is proved by applying the stable-unstable-center manifold theorem to blow-ups of the Z in $\mathscr{U}$. Details are in Section 7.

Let $\mathscr{F}_j(Z) = u_{j,\alpha}(Z) - u_{j,\omega}(Z)$, $j = M + 1,..., q$. By Lemma 6.2, $\mathscr{F}_j$ is at least C^2.

LEMMA 6.3. $\partial/\partial \lambda_k(\mathscr{F}_j \circ \mathscr{Y})(0) = \delta_{kj}$ *($k = 1,..., q$; $j = M + 1,..., q$).*

Let $\mathscr{F} = (\mathscr{F}_1,..., \mathscr{F}_q)$.

PROPOSITION 6.4. *$\mathscr{F} \circ \mathscr{Y}$ is C^2, $(\mathscr{F} \circ \mathscr{Y})(0) = 0$, and $D(\mathscr{F} \circ \mathscr{Y})(0)$ is nonsingular.*

Proposition 6.4 follows from the definitions of $\mathscr{F}$ and $\mathscr{Y}$ and the foregoing lemmas.

It follows that $\mathscr{F} \circ \mathscr{Y}$ maps a neighborhood V of 0 in $\mathbb{R}^q$, which we take to be compact, diffeomorphically to a neighborhood of 0 in $\mathbb{R}^q$. Let $Y(\lambda, x) = \mathscr{Y}(\lambda)(x)$.

LEMMA 6.5. *$Y(\lambda, x) | VxS^2$ can be approximated arbitrarily closely in the C^s topology by a polynomial family $\hat{Y}(\lambda, x)$, where $\hat{Y}(\lambda, \cdot) \in \mathscr{X}$ for each λ in V.*

The proof of Lemma 6.5 is in Section 8.

Set $\hat{\mathscr{Y}}(\lambda) = \hat{Y}(\lambda, \cdot)$. By Lemma 6.5 $\mathscr{F} \circ \hat{\mathscr{Y}}: V \to \mathbb{R}^q$ can be made arbitrarily close to $\mathscr{F} \circ \mathscr{Y} | V$ in the C^2 topology. Hence for $\hat{\mathscr{Y}}$ sufficiently close to $\mathscr{Y}$, there exists $\hat{\lambda} \in V$ such that $(\mathscr{F} \circ \hat{\mathscr{Y}})(\hat{\lambda}) = 0$. Thus if $\hat{\mathscr{Y}}$ is sufficiently close to $\mathscr{Y}$, $\hat{\mathscr{Y}}(\hat{\lambda})$ has closed orbits and separatrix connections corresponding to those of Y. For $\hat{\mathscr{Y}}$ sufficiently close to $\mathscr{Y}$, the closed orbits of $\hat{\mathscr{Y}}(\hat{\lambda})$ near those of Y have the same stability. It follows from the definition of model vectorfield, the definition of $\mathscr{X}$, and the results of Section 4 that separatrix cycles of $\hat{\mathscr{Y}}(\hat{\lambda})$ have the same stability as the corresponding separatrix cycles of Y. Moreover, every $Z \in \mathscr{X}$ has equilibria at $p_1,..., p_m$ topolgically equivalent to those of Y at $p_1,..., p_m$, and every $Z \in \mathscr{X}$ that is sufficiently close to Y has no other equilibria. We may assume this is true of $\hat{\mathscr{Y}}(\hat{\lambda})$. By using Lemma 6.2 to see that separatrices of $\hat{\mathscr{Y}}(\hat{\lambda})$ are close to those of Y, we can see that if $\hat{\mathscr{Y}}(\hat{\lambda})$ is close enough to Y, then the schemes of $\hat{\mathscr{Y}}(\hat{\lambda})$ and Y are isomorphic provided $\hat{\mathscr{Y}}(\hat{\lambda})$ has no singular orbits other than those already mentioned. Since $\hat{\mathscr{Y}}(\hat{\lambda})$ has only the equilibria $p_1,..., p_m$ with known separatrices, according to [A, p. 258], it is enough to check that $\hat{\mathscr{Y}}(\hat{\lambda})$ has no closed orbits other than those already mentioned.

Let $\hat{\gamma}_1,..., \hat{\gamma}_n$ be the nonequilibrium singular orbits of $\hat{\mathscr{Y}}(\hat{\lambda})$ that are close to $\gamma_1,..., \gamma_n$. There is a homeomorphism $h: S^2 \to S^2$ close to the identity that fixes $p_1,..., p_m$ and takes γ_j to $\hat{\gamma}_j$, $j = 1,..., n$. The complement in S^2 of the set of singular orbits of Y is a finite union of connected open subsets of S^2, called the *cells* of Y, each of which is simply or doubly connected [A, p. 276]. If C is a simply connected cell, then $h(C)$ contains no closed orbit of $\hat{\mathscr{Y}}(\hat{\lambda})$ (otherwise there would be an equilibrium of $\hat{\mathscr{Y}}(\hat{\lambda})$ in $h(C)$). If C is doubly connected then each component of ∂C is a node, focus, closed orbit, or separatrix cycle; moreover, one component of ∂C is the α-limit set of every point in C, and the other is the ω-limit set of every point in C[A, p. 279].

Let C be a doubly connected cell of Y, and let A_1 and A_2 be the two components of ∂C. Choose closed neighborhoods N_i of A_i as follows:

(1) If A_i is a node, focus, or attracting or repelling closed orbit, then N_i is a small canonical neighborhood of A_i. Therefore [P-P, pp. 153–155], if Z is sufficiently close to Y, then Z has a node, focus, or attracting or

repelling closed orbit respectively in N_i, and N_i is a canonical neighborhood of it.

(2) If A_i is a closed orbit that is both positively and negatively unstable, then N_i is a canonical neighborhood of A_i that satisfies (1), (2), and (3) of Lemma 6.1.

(3) If A_i is a separatrix cycle, then N_i is a neighborhood of A_i contained in the neighborhood $N_\varepsilon(A_i)$ of Theorem 4.1, such that Y is transverse to $\partial N_i \cap C$. It follows that if Z is sufficiently close to Y and Z has a separatrix cycle in N_i, then every point of $\partial N_i \cap C$ belongs to a semiorbit of Z that limits on A_i without leaving N_i.

Let D denote the closure of $C \backslash (N_1 \cup N_2)$, a compact set. If $p \in D$, then there exist finite numbers $t_\pm(p)$ such that $F_Y(p, t) \in D$ iff $t_-(p) \leqslant t \leqslant t_+(p)$, and the sets $\{t_\pm(p): p \in D\}$ are bounded. Therefore if $\hat{\mathscr{Y}}$ is sufficiently close to $\mathscr{Y}$ then $D \subset h(C)$ and each orbit of $\hat{\mathscr{Y}}(\hat{\lambda})$ through a point of D leaves D in forward and backward time. It follows from the choice of N_1, N_2 that $\hat{\mathscr{Y}}(\hat{\lambda})$ has no closed orbit in $h(C)$. ∎

7. Proof of Lemma 6.2.

For any vectorfield W on $\mathbb{R}^2$, let $j^k W(q)$ denote the k-jet of W at q. Parameterize a neighborhood of p_i in S^2 by $\Psi: N_\delta(0, 0) \to S^2$, with Ψ chosen so that $j^{d_i+1}\Psi^* Z(0, 0)$ satisfies (E2)(a)–(c) (see Sect. 3) for all $Z \in \mathscr{X}$. ($\mathscr{X}$ is defined in Sect. 6.) Let $P_{d_i+1}(x, y)$ be the polynomial vectorfield of degree $d_i + 1$ having this $(d_i + 1)$-jet.

Let $\mathscr{X}_\delta$ denote the affine subspace of $\mathscr{X}^s(N_\delta(0, 0))$ consisting of vectorfields W with $j^{d_i+1} W(0, 0) = P_{d_i+1}$. The map $\mathscr{X} \to \mathscr{X}_\delta$ given by $Z \to \Psi^* Z$ is bounded linear.

Recall that $\Phi: \mathbb{R} \times S^1 \to \mathbb{R}^2$ is the polar coordinate map $\Phi(r, \theta) = (r \cos\theta, r \sin\theta)$. Since any $W \in \mathscr{X}_\delta$ vanishes to order $d_i - 1$ at $(0, 0)$, we can blow up W to $r^{1-d_i}\Phi^* W$, a C^{s-d_i} vectorfield on $[-\delta, \delta] \times S^1$. If we put $W' = r^{1-d_i}\Phi^* W$, then

$$j^1 W' \mid \{0\} \times S^1 = j^1 r^{1-d_i}\Phi^* P_{d_i+1} \tag{7.1}$$

and

$$\dot{r} \text{ component of } j^2 W' \mid \{0\} \times S^1 = \dot{r} \text{ component of } j^2 r^{1-d_i}\Phi^* P_{d_i+1}. \tag{7.2}$$

Let $\mathscr{X}'_\delta$ denote the affine subspace of $C^{s-d_i}([-\delta, \delta] \times S^1)$ consisting of vectorfields W' on $[-\delta, \delta] \times S^1$ that satisfy (7.1) and (7.2). The map $\mathscr{X}_\delta \to \mathscr{X}'_\delta$ given by $W \to r^{1-d_i}\Phi^* W$ is bounded linear. To see this, note first that the map $\mathscr{X}_\delta \to \mathscr{X}^{s-1}([-\delta, \delta] \times S^1)$ that takes W to $\Phi^* W$ is bounded linear

[D2, p. 55]), and $\Phi^* W$ vanishes to order $d_i - 1$ along $\{0\} \times S^1$. Then repeatedly use

LEMMA 7.1. *Let C_l^k denote the space of C^k functions on $[-\delta, \delta] \times S^1$ that vanish to order l on $\{0\} \times S^1$, with the C^k topology. Then the map $C_l^k \to C_{l-1}^{k-1}$ given by $f(r, \theta) \to r^{-1} f(r, \theta)$ is bounded linear.*

Proof. Use $f(r, \theta) = r \int_0^1 \partial f/\partial r(sr, \theta)\, ds$, so $r^{-1} f(r, \theta) = \int_0^1 \partial f/\partial r(sr, \theta)\, ds$. ∎

If γ is an α- or ω-separatrix of Y at p_i and the tangent direction of $\Psi^{-1}\gamma$ at $(0, 0)$ is θ_0, then $r^{1-d_i}\Phi^* P_{d_i+1}(0, \theta_0) = 0$. Therefore for each $Z \in \mathscr{X}$, $r^{1-d_i}\Phi^*\Psi^* Z(0, \theta_0) = 0$, and moreover $r^{1-d_i}\Phi^*\Psi^* Z(0, \theta_0)$ has $\partial/\partial\theta$ as an eigenvector with eigenvalue $\sigma_1 \neq 0$, σ_1 independent of Z. We assume without loss of generality that $\sigma_1 < 0$. Then the other eigenvalue is $\sigma_2 \geqslant 0$, σ_2 independent of Z. The corresponding eigenvector is $\partial/\partial r$.

If $\sigma_2 = 0$, then by the center manifold theorem [C-H, Sect. 9.2], there is a C^{s-d_i} functional $\theta(Z, r)$, defined on a neighborhood of $(Y, 0)$ in $\mathscr{X} \times \mathbb{R}$, such that $\theta(Z, 0) = \theta_0$ for all Z, and $\{(r, \theta(Z, r))\}$ is a center manifold for $r^{1-d_i}\Phi^*\Psi^* Z$ at $(0, \theta_0)$. For each Z near Y, this local center manifold includes the α- or ω-separatrix of $\Phi^*\Psi^* Z$ at $(0, \theta_0)$ in $r > 0$. This fact implies the lemma. If $\sigma_2 > 0$, the same result follows from the center-unstable manifold theorem [C-H, Sect. 9.2]. ∎

8. PROOF OF LEMMA 6.5

LEMMA 8.1. *Let $x_1, ..., x_m$ be points in a closed ball B in $\mathbb{R}^p$; $k_1, ..., k_m$ nonnegative integers; $s > \max k_i$ an integer; and $g\colon [-1, 1]^q \times B \to \mathbb{R}$ a C^∞ function such that for each $i = 1, ..., m$, the k_i-jet of g at (λ, x_i) is independent of λ. Given $\varepsilon > 0$, there exists a polynomial $P\colon [-1, 1]^q \times B \to \mathbb{R}$ such that $j^{k_i} P(\lambda, x_i) = j^{k_i} g(\lambda, x_i)$ (all $\lambda \in [-1, 1]^q$; $i = 1, ..., m$) and $\| P - g \|_s < \varepsilon$, where $\| \ \|_s$ is the C^s norm for functions on $[-1, 1]^q \times B$.*

Proof. We use induction on m. For $m = 0$ just approximate g in the C^s topology by a polynomial. Proceeding inductively, we assume without loss of generality that $x_m = (0, ..., 0)$. Write $g(\lambda, x) = P_m(x) + \sum_{|I| = k_m + 1} x^I \tilde{g}_I(\lambda, x)$, where $I = (i_1, ..., i_p)$, $|I| = i_1 + \cdots + i_p$, $x^I = x_1^{i_1} \cdots x_p^{i_p}$, $P_m(x) = j^{k_m} g(\lambda, 0)$ (any $\lambda \in [-1, 1]^q$), and the $\tilde{g}_I$ are C^∞ functions. Approximate each $\tilde{g}_I$ by a polynomial P_I so that P_I and g_I have the same k_i-jet on $[-1, 1]^q \times \{x_i\}$, $i = 1, ..., m-1$. Let $P = P_m + \sum_{|I| = k_m + 1} x^I P_I$. If the P_I are sufficiently close to $\tilde{g}_I$, then P will satisfy the conclusion of the lemma. ∎

Lemma 6.5 follows from

PROPOSITION 8.2. *Let Y: $[-1, 1]^q \times S^2 \to \mathbb{R}^3$ be a C^∞ mapping such that for each λ, $Y(\lambda, \cdot)$ is a tangent vector field to S^2. Let $x_1,..., x_m$ be a finite collection of points on S^2; $k_1,..., k_m$ nonnegative integers; $s > \max k_i$ an integer. Assume that for each $i = 1,..., m$, the k_i-jet of Y at (λ, x_i) is independent of λ. Then given $\varepsilon > 0$ there is a polynomial $\hat{Y}$: $[-1, 1]^q \times S^2 \to \mathbb{R}^3$ such that*:

(1) $\| Y - \hat{Y} \|_s < \varepsilon$.

(2) *For each λ, $\hat{Y}(\lambda, \cdot)$ is a tangent vectorfield to S^2.*

(3) $j^{k_i}\hat{Y}(\lambda, x_i) = j^{k_i}Y(\lambda, x_i)$ *(all $\lambda \in [-1, 1]^q$; $i = 1,..., m$).*

Proof. Without loss of generality we may assume that for each $i = 1,..., m$, if $x_i = (x_{i1}, x_{i2}, x_{i3})$, then $x_{i1} \neq 0$, $x_{i2} \neq 0$, $x_{i3} \neq 0$. Let B be a closed ball in $\mathbb{R}^3$ that contains S^2. We extend Y to a C^∞ map Y $[-1, 1]^q \times B \to \mathbb{R}^3$, $Y(\lambda, x) = (g_1(\lambda, x), g_2(\lambda, x), g_3(\lambda, x))$, such that the k_i-jet of Y at (λ, x_i) is independent of λ and $x_1 g_1 + x_2 g_2 + x_3 g_3 = 0$ identically. When $x_3 = 0$, we have

$$x_1 g_1(\lambda, x_1, x_2, 0) + x_2 g_2(\lambda, x_1, x_2, 0) = 0, \tag{8.1}$$

so we can write

$$g_1 = x_2 g_{12}(\lambda, x_1, x_2) + x_3 g_{13}(\lambda, x_1, x_2, x_3),$$
$$g_2 = x_1 g_{21}(\lambda, x_1, x_2) + x_3 g_{23}(\lambda, x_1, x_2, x_3),$$

where g_{12}, g_{13}, g_{21}, g_{23} are C^∞ functions whose k_i-jets at (λ, x_i) $(i = 1,..., m)$ are independent of λ. We shall think of g_{12}, g_{21} as functions on $[-1, 1]^q \times B$. By Lemma 8.1 we can approximate g_{12}, g_{13}, g_{23} by polynomials $\hat{g}_{12}$, $\hat{g}_{13}$, $\hat{g}_{23}$ having the same k_i-jet at (λ, x_i) (all $\lambda \in [-1, 1]^q$; $i = 1,..., m$) as the functions they approximate. Let $\hat{g}_{21} = -\hat{g}_{12}$. By (8.1), $\hat{g}_{21}$ approximates g_{21} and has the same k_i-jet at (λ, x_i) (all $\lambda \in [-1, 1]^q$; $i = 1,..., m$). Let $\hat{g}_1 = x_2 \hat{g}_{12} + x_3 \hat{g}_{13}$; $\hat{g}_2 = x_1 \hat{g}_{21} + x_3 \hat{g}_{23}$. Then

$$x_1 \hat{g}_1 + x_2 \hat{g}_2 = x_3(x_1 \hat{g}_{13} + x_2 \hat{g}_{23}).$$

Let $\hat{g}_3 = -(x_1 \hat{g}_{13} + x_2 \hat{g}_{23})$. Then $\hat{Y} = (\hat{g}_1, \hat{g}_2, \hat{g}_3)$ satisfies the conclusion of the proposition. ∎

9. Planar Vectorfields

Let $Z(z_1, z_2)$ be a C^1 vectorfield on $\mathbb{R}^2$. Using sterographic projection

$$\begin{aligned} x_1 &= 2z_1(1+z_1^2+z_2^2)^{-1}, \\ x_2 &= 2z_2(1+z_1^2+z_2^2)^{-1}, \\ x_3 &= (z_1^2+z_2^2-1)(1+z_1^2+z_2^2)^{-1}; \\ z_1 &= x_1(1-x_3)^{-1}, z_2 = x_2(1-x_3)^{-1}, \end{aligned} \tag{9.1}$$

we can transform Z to a C^1 vectorfield on $S^2 \backslash \{(0, 0, 1)\}$. Multiplying this vectorfield by a suitable C^1 function that is zero only at (0, 0, 1), we obtain a vectorfield on $S^2 \backslash \{(0, 0, 1)\}$ that is the restriction of a C^1 vectorfield X on S^2 having an equilibrium at (0, 0, 1). Whether or not X satisfies hypotheses (H1)–(H3) of Theorem 1.1 depends only on Z. If X does satisfy these hypotheses, we can find a topologically equivalent polynomial vectorfield $\hat{Y}(x_1, x_2, x_3) = (\hat{f}_1, \hat{f}_2, \hat{f}_3)$. Using (9.1) we transform $\hat{Y}$ to a vectorfield on $\mathbb{R}^2$, which becomes a polynomial vectorfield $\hat{Z}$ when multiplied by a sufficiently high power of $1+z_1^2+z_2^2$. $\hat{Z}$ is topologically equivalent to Z.

As a special case we consider C^1 vectorfields Z on $\mathbb{R}^2$ that vanish nowhere. The orbits of Z give a foliation of $\mathbb{R}^2$. Recall that a leaf L_1 of a foliation is *inseparable* if there exists another leaf $L_2 \neq L_1$ such that for any neighborhoods O_1 and O_2 of L_1 and L_2 there is a leaf L that intersects both O_1 and O_2.

Corollary 9.1. *Let $\mathscr{F}$ be a foliation of the plane given by the orbits of a C^1 vectorfield Z. $\mathscr{F}$ has a finite number of inseparable leaves if and only if $\mathscr{F}$ is topologically equivalent to a foliation of the plane by orbits of a nowhere zero polynomial vectorfield.*

Proof. Let X be a C^1 vectorfield on S^2 constructed from Z as described above. The only equilibrium of X is at (0, 0, 1), and every orbit of X approaches (0, 0, 1) as $t \to \pm\infty$. It is easy to show that if a leaf L of $\mathscr{F}$ is not inseparable, then the corresponding orbit of X is both positively and negatively stable. Therefore X has only a finite number of singular orbits. X has no closed orbits or separatrix cycles (otherwise Z would have equilibria). By Theorem 1.1 and the contructions of this section, Z is topologically equivalent to a polynomial vectorfield $\hat{Z}$.

The converse is proved in [M] or [S-S1]. ∎

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