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ELEMENTARY SOLUTIONS OF DIFFERENTIAL EQUATIONS

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In this paper we deal with the problem: when does a differential equation have an elementary solution, that is a solution which can be expressed in terms of algebraic operations, logarithms and exponentials? As an application of our theorem, we give necessary and sufficient conditions for a certain class of first order differential equations to have elementary solutions.

For the simplest differential equation $y' = \alpha$, where α is an algebraic function, Liouville showed that if such an equation has an elementary solution, then this solution is an algebraic function plus a sum of constant multiples of logarithms of algebraic functions. In his paper, "Liouville's Theorem on Functions with Elementary Integrals", Pacific J. of Math., 24 No. 1, Rosenlicht showed how this theorem can be handled algebraically and generalized. We will use Rosenlicht's methods to show that if an arbitrary algebraic differential equation has an elementary solution, this solution must be of a special form.

An (ordinary) differential field is a field K and a map $': K \to K$ called a derivation, which satisfies $(a + b)' = a' + b'$ and $(ab)' = a'b + ab'$ for all a, b in K. For example, a field of functions, meromorphic in some region of the plane, with the usual differentiation, is such a field. A differential subfield k of K is a subfield which is closed under the derivation. If c is in K and $c' = 0$ then c is called a constant of K. The set of constants can be seen to form a subfield of K. In this paper all fields will be of characteristic 0. By a differential equation of order n over k, we mean an expression of the form $f(y, y', \cdots, y^{(n)}) = 0$ where f is a polynomial, with coefficients in k, in the variables $y, y', \cdots, y^{(n)}$ with $y^{(n)}$ actually appearing. An element u of K is said to satisfy such an equation if $f(u, u', \cdots, u^{(n)}) = 0$ where $u^{(i)}$ is the ith derivative of u. We note that if u satisfies a differential equation of order n, then the field $k(u, u', \cdots, u^{(n)})$ is a differential field of transcendence degree at most n. To see that it is closed under the derivation, note that by differentiating the equation $f(u, u', \cdots, u^{(n)}) = 0$ we can solve for $u^{(n+1)}$ in terms of lower derivatives of u.

If $k \subset K$ are differential fields an element $u (u \neq 0)$ in K is called an elementary integral (exponential of an elementary integral) with respect to k if there exist elements $v_0, v_1, \cdots, v_n$ in k and $c_1, c_2, \cdots, c_n$

constants of k such that

$$u' = v_0' + \sum_{i=1}^{n} c_i \frac{v_i'}{v_i}$$

$$\left(\frac{u'}{u} = v_0' + \sum_{i=1}^{n} c_i \frac{v_i'}{v_i}\right).$$

For elements u, v in K with $u \neq 0$, we say u is an exponential of v or equivalently v is a logarithm of u if $v' = u'/u$. Note that if u is in k (v is in k) then v is an elementary integral (u is an exponential of an elementary integral) with respect to k. K is called a generalized elementary extension of k if

(1) K and k have the same field of constants, and

(2) there exists a tower of differential fields $k = K_0 \subset K_1 \subset \cdots \subset K_n = K$ where $K_i = K_{i-1}(u_i)$ and u_i is either algebraic over K_{i-1} or an elementary integral with respect to K_{i-1} or an exponential of an elementary integral with respect to K_{i-1}.

If K is a generalized elementary extension of k and satisfies the additional property that each of the above u_i is either algebraic over K_{i-1} or a logarithm or exponential of an element in K_{i-1}, we say K is an elementary extension of k. It was shown in [8] that every generalized elementary extension of k lies in an elementary extension of k. Intuitively, one just has to add enough logarithms into a generalized elementary extension K, making sure not to extend the constants, to get an elementary extension containing K. We say an element w of some differential extension of k is elementary with respect to k if it lies in some elementary extension of k.

Generalized elementary extensions were introduced to deal with the following phenomenon: Liouville's theorem [5], says that if u is elementary with respect to k and u' is in k, then

$$u' = v_0' + \sum_{i=1}^{n} c_i \frac{v_i'}{v_i}$$

for some elements v_i in k and constants c_i. Another way of saying this is: if an integral u of an element of k is elementary, then u is an elementary integral, (this, of course, was why elementary integrals were so named). In general we have no bound on the transcendence degree of the smallest elementary extension of k containing u, because we don't know how many logarithms (all possibly algebraically independent) we need to adjoint to k to insure that we get u. Yet, we know that u lies in a generalized elementary extension of transcendence degree at most one over k. The main theorem of this paper says that this is true in general; if a differential equation over k of order n has a solution in a generalized elementary extension of k, this solution lies in a generalized elementary extension of transcendence degree less than or equal to n over k.

Before I proceed, I need a technical fact about generalized elementary extensions. We say an element u in K is a good elementary integral with respect to $k \subset K$ if

$$u' = v_0' + \sum_{i=1}^{n} c_i \frac{v_i'}{v_i}$$

with v_i in k and $\{c_1, c_2, \cdots, c_n\}$ a Q-linearly independent set of constants. An element $u \neq 0$ in K is said to be a good exponential of an elementary integral with respect to $k \subset K$ if

$$\frac{u'}{u} = v_0' + \sum_{i=1}^{n} c_i \frac{v_i'}{v_i}$$

with the v_i in k and $\{1, c_1, c_2, \cdots, c_n\}$ a Q-linearly independent set of constants. A generalized elementary extension K of k is said to be good if the elementary integrals and exponentials of elementary integrals used in building up the tower to K are good. I claim that for any generalized elementary extension K of k there is a tower of fields $k = K_0^* \subset K_1^* \subset \cdots \subset K_n^* = K$ which turns K into a good elementary extension of k. It is enough to show that for $E = F(\theta)$, where θ is an elementary integral or exponential of an elementary integral, we can make E into a good elementary extension of F. Let

$$\theta' = v_0' + \sum_{i=1}^{k} c_i \frac{v_i'}{v_i}$$

and assume that c_i's are linearly dependent over Q. We can assume that c_k depends linearly on $c_1, \cdots, c_{k-1}$ and let

$$c_k = \frac{(n_1 c_1 + \cdots + n_{k-1} c_{k-1})}{n}$$

with $n, n_1, \cdots, n_{k-1}$ integers, then

$$\theta' = v_0' + \sum_{i=1}^{k-1} \frac{c_i}{n} \frac{(v_i^n v_k^{n_i})'}{v_i^n v_k^{n_i}} .$$

Continuing in this way we can eventually arrive at an expression

$$\theta' = v_0' + \sum_{i=1}^{j} d_i \frac{u_i'}{u_i}$$

where $\{d_1, \cdots, d_j\}$ is a Q-linearly independent set of constants and $u_0, u_1, \cdots, u_j$ are elements of k. So using this expression we see that $F(\theta)$ is a good elementary extension of F. Now assume that θ satisfies

$$\frac{\theta'}{\theta} = v_0' + \sum_{i=1}^{k} c_i \frac{v_i'}{v_i}$$

and suppose $\{1, c_1, \cdots, c_k\}$ are Q-linearly dependent. As before assume that c_k depends linearly on $1, c_1, \cdots, c_{k-1}$, and let, $c_k = (n_0 + n_1 c_1 + \cdots + n_{k-1} c_{k-1})/n$ where $n, n_0, n_1, \cdots, n_{k-1}$ are integers, We can then write

$$\frac{(\theta^n v_k^{-n_0})'}{\theta^n v_k^{-n_0}} = n v_0' + \sum_{i=1}^{k-1} c_i \frac{(v_i^n v_k^{n_i})}{v_i^n v_k^{n_i}} .$$

Letting $\theta^* = \theta^n v_k^{-n_0}$, $u_i = v_i^n v_k^{n_i}$, $u_0 = n v_0$, we get

$$\frac{\theta^{*\prime}}{\theta^*} = u_0' + \sum_{i=1}^{k-1} c_i \frac{u_i'}{u_i} .$$

Note that θ is algebraic over $F(\theta^*)$. Continuing in this way we eventually get to the stage where $\{1, c_1, \cdots, c_j\}$ are linearly independent over Q, and θ algebraic over $F(\theta^*)$ so $F \subset F(\theta^*) \subset F(\theta)$ is a good generalized elementary extension of F. In conclusion, we have shown that if θ is an elementary integral with respect to F, then it is a good elementary integral with respect to F, and that if θ is an exponential of an elementary integral with respect to F then some $a\theta^n (n \in Z, n \neq 0, a \in F^*)$ is a good exponential of an elementary integral with respect to F. This allows us to exhibit any elementary extension K of k as a good elementary extension of k.

The proof of the theorem relies on a fact about algebraic dependence of certain elements in generalized elementary extension (the lemma below). This in turn relies on the following proposition, whose proof can be found in [6].

PROPOSITION. *Let L be a differential field and K a differential extension field with the same constants as L, which is furthermore algebraic over $k(\zeta)$ for some ζ in K. Suppose that $c_1, \cdots, c_n$ are constants which are linearly independent over Q and $u_1, \cdots, u_n, v$ in K ($u_i \neq 0$* for all *i). Suppose further that*

$$v' - \sum_{i=1}^{n} c_i \frac{u_i'}{u_i}$$

is in L. If ζ' is in L, then $u_1, \cdots, u_n$ are in $\bar{L}$ (the algebraic closure of L in K) *and $v = c\zeta + d$, where c is a constant and d is in $\bar{L}$.*

If ζ'/ζ is in L, then we have v in $\bar{L}$ and there are integers $\nu_0, \nu_1, \cdots, \nu_n$ with $\nu_0 \neq 0$ such that $u_i^{\nu_0}\zeta^{\nu_i}$ is in $\bar{L}$ for $1 \leqq i \leqq n$.

LEMMA. *Let $k \subset L \subset K$ be differential fields with the same constants. Let ζ, η be elements of K, algebraically dependent over L such that ζ is an elementary integral or an exponential of an elementary integral with respect to $\bar{k}$, the algebraic closure of k in K and η is a good elementary integral or a good exponential of an elementary integral with respect to $\overline{k(\zeta)}$. We can then find a ξ in K such that*

(1) *ξ is an elementary integral or an exponential of an elementary integral with respect to $\bar{k}$.*

(2) *ζ and η are algebraically dependent over $k(\xi)$.*

(3) *ξ is algebraic over L.*

Furthermore, if ζ, η are both elementary integrals, so is ξ and if ζ, η are both exponentials of elementary integrals then so is ξ.

Proof. First note that we can assume that ζ is not algebraic over L, otherwise we could take ζ for our ξ. From this we can conclude that $\overline{k(\zeta)} \cap \bar{L} = \bar{k}$. The proof now proceeds by considering the following four cases:

Case 1. Assume

$$\zeta' = \Sigma a_i \frac{s_i'}{s_i} + t' \quad \text{and} \quad \eta' = \Sigma b_i \frac{u_i'}{u_i} + v'$$

where s_i, t are in $\bar{k}$ and u_i, v are in $\overline{k(\zeta)}$ and $\{b_i\}$ is a Q-linearly independent set of constants. We can apply the proposition to the expression

$$\Sigma b_i \frac{u_i'}{u_i} + (v - \eta)' = 0$$

with respect to the fields $L \subset L(\zeta) \subset K$ and conclude that $u_1, \cdots, u_n$ are in $\bar{L}$ and $v - \eta = c\zeta + d$ where c is a constant and d is in $\bar{L}$. Let $\xi = d = v - \eta - c\zeta$. Conditions (2) and (3) are then clearly satisfied. Note that since $u_1, \cdots, u_n$ are in $\overline{k(\zeta)} \cap \bar{L}$, they are in $\bar{k}$. We have

$$\xi' = v' - \eta' - c\zeta' = -\Sigma b_i \frac{u_i'}{u_i} - c\Sigma a_i \frac{s_i'}{s_i} - ct' .$$

Since u_i, s_i, t are in $\bar{k}$, ξ is an elementary integral with respect to $\bar{k}$, so we have (1).

Case 2. Assume that

$$\zeta' = \Sigma a_i \frac{s_i'}{s_i} + t' \quad \text{and} \quad \frac{\eta'}{\eta} = \Sigma b_i \frac{u_i'}{u_i} + v'$$

where s_i, t, u_i, v are as before and $\{1, b_i\}$ is a Q-linearly independent set of constants. We can apply the proposition to the expression

$$-\frac{\eta'}{\eta} + \Sigma b_i \frac{u_i'}{u_i} + v' = 0$$

and conclude that the u_i and η are in $\bar{L}$ and $v = c\zeta + d$ with d in $\bar{L}$ and c a constant. Let $\xi = \eta$. Conditions (2) and (3) are then satisfied. Since the s_i and $d = V - c\zeta$ are in $\bar{L} \cap \overline{k(\zeta)} = \bar{k}$ and

$$\begin{aligned}\frac{\xi'}{\xi} = \frac{\eta'}{\eta} &= \Sigma b_i \frac{u_i'}{u_i} + v' = \Sigma b_i \frac{u_i'}{u_i} + c\zeta' + d' \\ &= \Sigma b_i \frac{u_i'}{u_i} + c\Big(\Sigma a_i \frac{s_i'}{s_i} + t'\Big) + d'\end{aligned}$$

we have that ξ is an exponential of an elementary integral over $\bar{k}$.

Case 3. Assume

$$\frac{\zeta'}{\zeta} = \Sigma a_i \frac{s_i'}{s_i} + t' \quad \text{and} \quad \eta' = \Sigma b_i \frac{u_i'}{u_i} + v'$$

with u_i, v, s_i, t as before and $\{b_i\}$ a Q-linearly independent set of constants. We can apply the proposition to the expression

$$\Sigma b_i \frac{u_i'}{u_i} + (v - \eta)' = 0$$

and conclude that $v - \eta$ is in $\bar{L}$ and that there are integers $\nu_0, \cdots, \nu_n$ with $\nu_0 \neq 0$ such that $u_i^{\nu_0}\zeta^{\nu_i}$ is in $\bar{L}$. Let $\xi = \nu_0(\eta - v)$. ξ then satisfies (2) and (3). Each $u_i^{\nu_0}\zeta^{\nu_i}$ is in $\overline{k(\zeta)} \cap \bar{L} = \bar{k}$ and

$$\xi' = \nu_0(\eta' - v') = \Sigma b_i \frac{(u_i^{\nu_0}\zeta^{\nu_i})'}{u_i^{\nu_0}\zeta^{\nu_i}} - (\Sigma \nu_i b_i)\Big(\Sigma a_i \frac{s_i'}{s_i} + t'\Big)$$

so ξ is an elementary integral with respect to $\bar{k}$.

Case 4. Assume

$$\frac{\zeta'}{\zeta} = \Sigma a_i \frac{s_i'}{s_i} + t' \quad \text{and} \quad \frac{\eta'}{\eta} = \Sigma b_i \frac{u_i'}{u_i} + v'$$

with u_i, v, s_i, t as before and $\{1, b_i\}$ a Q-linearly independent set of constants. We can apply the proposition to the expression

$$-\frac{\eta'}{\eta} + \Sigma b_i \frac{u_i'}{u_i} + v' = 0$$

and conclude that v is in $\bar{L}$ and that there are integers $\nu_0, \nu_1, \cdots, \nu_{n+1}$ with $\nu_0 \neq 0$ such that the $u_i^{\nu_0}\zeta^{\nu_i}$ and $\eta^{\nu_0}\zeta^{\nu_{n+1}}$ are in $\bar{L}$. Let $\xi = \eta^{\nu_0}\zeta^{\nu_{n+1}}$. (2) and (3) are then satisfied

$$\frac{\xi'}{\xi} = \frac{(\eta^{\nu_0}\zeta^{\nu_{n+1}})'}{\eta^{\nu_0}\zeta^{\nu_{n+1}}} = \Sigma b_i \frac{(u_i^{\nu_0}\zeta^{\nu_i})'}{u_i^{\nu_0}\zeta^{\nu_i}} + \nu_0 v' + (\nu_{n+1} - \Sigma b_i \nu_i)\left(\Sigma a_i \frac{s_i'}{s_i} + t'\right)$$

so ξ is an exponential of an elementary integral.

THEOREM. *Let E be a differential field of transcendence degree n over a differential field F. If E lies in a generalized elementary extension K of F, then E lies in a generalized elementary extension of F of transcendence degree n over F. Furthermore, if $K = F(\theta_1, \cdots, \theta_N)$ where each θ_i is algebraic or an elementary integral with respect to $F(\theta_1, \cdots, \theta_{i-1})$, then E lies in a generalized elementary extension of F of transcendence degree n which is likewise generated only by elements which are algebraic or elementary integrals. A similar statement holds if we restrict each θ_i to be algebraic or an exponential of an elementary integral with respect to $F(\theta_1, \cdots, \theta_{i-1})$.*

Proof. If $n = 0$, then the theorem is a triviality. Therefore, we can assume that the transcendence degree of E over F is $\geqq 1$. If F consisted only of constants, then any generalized elementary extension would coincide with F. So we can assume F contains a nonconstant. Furthermore, by the primitive element theorem for differential fields [7], we can conclude that $E = F(y, y', \cdots, y^{(n)})$ for some element y of E. Note that $y, y', \cdots, y^{(n-1)}$ forms a transcendence base for E over F. Now the proof proceeds by induction on n. Although the proof could be written to suppress the $n = 1$ step, we include it here in the hope that it will aid in understanding the induction step.

$n = 1$. Let $E = F(y, y')$ with y' algebraic over $F(y)$. Let K be a generalized elementary extension of F, containing E, whose transcendence degree over F is minimal with respect to all such extensions. Using the facts about good elementary extension developed in the paragraphs preceding the Proposition, we can pick a transcendence basis $\theta_1, \cdots, \theta_m$ such that each θ_i is a good elementary integral or a good exponential of an elementary integral with respect to some algebraic extension of $F(\theta_1, \cdots, \theta_{i-1})$. Now assume $m > 1$

and we will work towards a contradiction. We will apply the lemma, so let k be the algebraic closure of $F(\theta_1, \cdots, \theta_{m-2})$ in K, $\zeta = \theta_{m-1}$, $\eta = \theta_m$ and $L = k(y, y')$. Since m was picked as small as possible, y is not algebraic over k. Therefore, ζ and η are algebraic ally dependent over L. We can conclude that there exists a ξ satisfying the conclusions of the lemma. While ξ is algebraic over $k(y, y')$, it is not algebraic over k, for otherwise $\zeta = \theta_{m-1}$ and $\eta = \theta_m$ would be algebraically dependent over $F(\theta_1, \cdots, \theta_{m-2})$, contradicting the way they were chosen. Therefore, y is algebraic over $F(\theta_1, \cdots, \theta_{m-2}, \xi)$ and so $F(y, y')$ would lie in a generalized elementary extension of F of transcendence degree $\leqq m - 1$.

Induction Step. Assume that the theorem is true for differential fields E^*, F^* such that the transcendence degree of E^* over F^* is less than n. Again let K be a generalized elementary extension of F, containing E, whose transcendence degree m over F is minimal with respect to all such extensions. Assume $m > n$ and choose a transcendence basis $\theta_1, \cdots, \theta_m$ of K over F such that each θ_i is a good elementary integral or a good exponential of an elementary integral with respect to an algebraic extension of $F(\theta_1, \cdots, \theta_{i-1})$.

I will first show that for each j, with $0 \leqq j \leqq n - 1$, θ_{m-j} is algebraic over $F(\theta_1, \cdots, \theta_{m-j-i}, y, y', \cdots, y^{(j)})$ and that this last field has transcendence degree m over F. This is a standard replacement argument with the above induction hypothesis in a supporting role. For $j = 0$, we know y is not algebraic over $F(\theta_1, \cdots, \theta_{m-1})$, for otherwise $F(y, y', \cdots, y^{(n)})$ would lie in a generalized elementary extension of F of transcendence degree $m - 1$. Since y is algebraic cover $F(\theta_1, \cdots, \theta_m)$, we get θ_m algebraic over $F(\theta_1, \cdots, \theta_{m-1}, y)$ which then must have transcendence degree m over F. Now assume θ_{m-k} is algebraic over $F(\theta_1, \cdots, \theta_{m-k-1}, y, y', \cdots, y^{(k)})$ for $k < j$ and that this latter field has transcendence degree m over F. $F(\theta_1, \cdots, \theta_m)$ is therefore algebraic over $F(\theta_1, \cdots, \theta_{m-j}, y, y', \cdots, y^{(j-1)})$ and therefore $y^{(j)}$ is algebraic over this latter field. If $y^{(j)}$ were algebraic over $F(\theta_1, \cdots, \theta_{m-j-1}, y, y', \cdots, y^{(j-1)})$, then letting F^* be the algebraic closure of $F(\theta_1, \cdots, \theta_{m-j-1})$ in K and $E^* = F^*(y, y', \cdots, y^{(j)})$ we would have a field E^* of transcendence degree $j < n$ over F^* which lies in K, a generalized elementary extension of F^*, so $F^*(y, y', \cdots, y^{(n)})$ would lie in a generalized elementary extension K^* of F^* of transcendence degree j over F^*. K^* would then be a generalized elementary extension of F of transcendence degree $m - 1$ over F which contains $F(y, y', \cdots, y^{(n)})$, a contradiction. So $y^{(j)}$ must not be algebraic over $F(\theta_1, \cdots, \theta_{m-j-1}, y, y', \cdots, y^{(j-1)})$ and therefore θ_{m-j} is algebraic over $F(\theta_1, \cdots, \theta_{m-j-1}, y, \cdots, y^{(j)})$ and this latter field still has transcendence degree m over F.

In particular, we can conclude that θ_{m-n+1} is algebraic over $F(\theta_1, \cdots, \theta_{m-n}, y, y', \cdots, y^{(n-1)})$ and so θ_m and θ_{m-n+1} are algebraically dependent over $F(\theta_1, \cdots, \theta_{m-n-1}, y, y', \cdots, y^{(n-1)})$. We will now apply the lemma. Let k be the algebraic closure of $F(\theta_1, \cdots, \theta_{m-n-1})$ in K, L be the algebraic closure of $k(y, \cdots, y^{(n-1)})$ in $K, \zeta = \theta_{m-n}$, and $\eta = \theta_{m-n+1}$. By the lemma, there is a ξ in K such that ξ is an elementary integral or an exponential of an elementary integral with respect to $\bar{k}$ and $\theta_{m-n+1}, \theta_{m-n}$ are algebraically dependent over $k(\xi)$. ξ is algebraic over $F(\theta_1, \cdots, \theta_{m-n-1}, y, \cdots, y^{(n-1)})$ but it is not algebraic over $F(\theta_1, \cdots, \theta_{m-n-1}, y, \cdots, y^{(n-2)})$. If it were, then $F(\theta_1, \cdots, \theta_{m-n-1}, \xi, \theta_{m-n}, \theta_{m-n+1}, y, \cdots, y^{(n-2)})$ would have transcendence degree $m-1$ over F, contradicting the fact proven in the previous paragraph. Thus $y^{(n-1)}$ is algebraic over $F(\theta_1, \cdots, \theta_{m-n-1}, \xi, y, \cdots, y^{(n-2)})$. If we now let E^* be the algebraic closure of this latter field in K and F^* be the algebraic closure of $F(\theta_1, \cdots, \theta_{m-n-1}, \xi)$ in K, we see E^* is a differential field of transcendence degree $n-1$ over F^* which lies in a generalized elementary extension of F^*. Therefore E^* lies in a generalized elementary extension of F^* of transcendence degree $n-1$ over F^*. We can conclude that $E = F(y, \cdots, y^{(n)})$ lies in a generalized elementary extension of F of transcendence degree $m-n+n-1 = m-1$, contradicting our choice of m.

The proofs of the final two assertions of this theorem are the same as the one above, keeping in mind the final sentence of the lemma. These last two assertions were first noticed by Koenigsberger [1], who outlined an analytic proof for the cases $n = 1$ and 2.

In the next two corollaries, $C(x)$ will be the field of rational functions over the complex numbers whose derivation is given by $x' = 1$ and $c' = 0$ for all c in C.

COROLLARY 1. *If a first order differential equation over $C(x)$ has a solution which is elementary but not algebraic over $C(x)$, then the equation has either a one parameter family of solutions of the type*

$$y = G(x, \varphi_0(x) + a_1 \log \varphi_1(x) + \cdots + a_r \log \varphi_r(x) + c)$$

with c an arbitrary constant, the a_i's constants and G and the φ_i's algebraic functions or the equation has a one parameter family of solutions

$$y = G(x, \exp(\varphi_0(x) + a_1 \log \varphi_1(x) + \cdots + a_r \log \varphi_r(x) + c))$$

of similar descriptions

Proof. This theorem was first proven by Mordukhai-Boltovski,

[2] and [4, p. 86], using analytic techniques. Let y be an elementary but nonalgebraic solution of the equation $F(x, y, y') = 0$. By the theorem, we know that y lies in a generalized elementary extension of $C(x)$, of transcendence degree one over $C(x)$. This means that y can be considered as an algebraic function $G(x, \theta)$ where θ is an elementary integral or exponential of an elementary integral with respect to some algebraic extension K of $C(x)$. If θ is an elementary integral with respect to K and c is any element of C, I claim that $G(x, \theta + c)$ is a solution of the same differential equation as y. This can be stated algebraically. The map which takes θ to $\theta + c$ induces differential automorphism (i.e. a field theoretic isomorphism which preserves the differential structure) of $K(\theta)$. This map can be extended to a (field theoretic) isomorphism is $K(\theta, y)$ into $\overline{K(\theta)}$, the algebraic closure of $K(\theta)$. It is known [5], that the differential structure of an algebraic extension of a differential field of characteristic zero, is uniquely determined, so the isomorphism of $K(\theta, y)$ into $\overline{K(\theta)}$ is a differential isomorphism. Therefore the image of y satisfies the same differential relationships over K as y does. The image of $y = G(x, \theta)$ is just $G(x, \theta + c)$ so this proves the claim. If θ is an exponential of an elementary integral, then for any nonzero d in C the map which takes θ to $d\theta$ induces a differential automorphism of $K(\theta)$. The same kind of reasoning tells us that $G(x, d\theta)$ satisfies the same equation as $G(x, \theta)$. We therefore have the conclusion of the corollary.

COROLLARY 2. *Let $f(y)$ be a rational function in the indeterminant y with coefficients in the complex numbers. If $y' = f(y)$ has a nonconstant solution, elementary over $C(x)$, then $1/f(y)$ is either of the form*

$$\frac{d(v(y))}{dy} \quad or \quad c\frac{d(u(y))dy}{u(y)}$$

where $v(y)$, $u(y)$ are rational functions of y with coefficients in the complexes and c is a nonzero complex number. Conversely, if $f(y) \neq 0$ and $1/f(y)$ is of one of the two above forms, then $y' = f(y)$ has a nonconstant solution, elementary over $C(x)$.

Proof. We prove the converse first. If $1/f(y)$ is of the first form, we let y be defined by the equation $v(y) = x$. If y is of the second form, we let y be defined by the equation $u(y) = \exp(x/c)$. In both cases, y will be a nonconstant solution elementary of $C(x)$.

Now let y be a nonconstant element of some elementary extension of $C(x)$ such that $y' = f(y)$. We then can conclude that the simple transcendental extension $C(y)$, with the induced derivation $'$, is a differential field. Expanding $1/f(y)$ in partial factions with respect to y we get

$$1 = \frac{y'}{f(y)} = \frac{d(v(y))}{dy}y' + \Sigma c_i \frac{d(u_i(y))/dy}{u_i(y)}y'$$
$$= v(y)' + \Sigma c_i \frac{(u_i(y))'}{u_i(y)}$$

where $u_i(y)$, $v(y)$ are in $C(y)$ and the c_i's are in C. As before, we can assume that the c_i's are linearly independent over Q. Since y lies in an elementary extension of $C(x)$ we are reduced to one of the following three cases:

(1) y is algebraic over $C(x)$. In this case we have a relation of the form $v' + \Sigma c_i(u_i'/u_i) = 1$ in an algebraic extension of $C(x)$. Applying the proposition, with $L = C$ and $\zeta = x$, we see that the u_i are in C, so $u_i' = 0$. Therefore

$$\frac{y'}{f(y)} = v' = \frac{d(v(y))}{dy}y' \text{ or } \frac{1}{f(y)} = \frac{d(v(y))}{dy}\,.$$

(2) y is algebraic over $L(\theta)$ where L is an algebraic extension of $C(x)$ and θ is an elementary integral with respect to L. Since $\theta' \in L$, we can apply the proposition to $\zeta = \theta$ and get that each u_i is algebraic over $C(x)$. If $d(u_i(y))/dy \neq 0$ for some i, then y would be algebraic over $C(x)$ and we would be reduced to the provious case. So we can assume $d(u_i(y))/dy = 0$ for all i and therefore $(u_i(y))' = 0$. As before we can then conclude that $1/f(y) = d(v(y))/dy$.

(3) y is algebraic over $L(\theta)$ where L is an algebraic extension of $C(x)$ and θ is a good exponential of an elementary integral with respect to L. First notice that if we write

$$\frac{\theta'}{\theta} = \Sigma d_i \frac{s_i'}{s_i} + t'$$

with s_i and t in L and d_i in C, then since θ and x are algebraically dependent over $C(y)$ and $x' = 1 \in C(y)$, we can apply the proposition and get that each s_i must be algebraic over $C(y)$. Thus each s_i is algebraic over both $C(x)$ and $C(y)$ and so must be in C if we are not to be reduced to the case where y is algebraic over $C(x)$. So $s_i' = 0$ for each s_i, and $\theta'/\theta = t'$. Our next step is to notice that we have a relation of the form

$$\Sigma c_i \frac{u_i'}{u_i} + v' = 1$$

in an algebraic extension of $L(\theta)$. Since θ'/θ is in L, we can apply the proposition and conclude that there are integers $\nu_0 \neq 0, \nu_1, \cdots, \nu_k$ such that each $u_i^{\nu_0}\theta^{\nu_i}$ and v is algebraic over $C(x)$. Again if $d(v(y))/dy \neq 0$ then y would be algebraic over $C(x)$ and we would be reduced to the first case. So we can assume $v' = (d(v(y))/dy)y' = 0$. If we let $u_i^{\nu_0}\theta^{\nu_i} = A_i$ then

$$\frac{u_i'}{u_i} = \frac{1}{\nu_0}\left(\frac{A_i'}{A_i} - \nu_i\frac{\theta'}{\theta}\right) = \frac{1}{\nu_0}\left(\frac{A_i'}{A_i} - \nu_i t'\right)$$

so

$$1 = \Sigma c_i\frac{u_i'}{u_i} + v' = \frac{1}{\nu_0}\Sigma c_i\frac{A_i'}{A_i} - \frac{1}{\nu_0}(\Sigma c_i\nu_i)t' .$$

Each A_i and t is algebraic over $C(x)$ and $x' \in C$ so applying the lemma for the final time we can conclude that each A_i is in C. Thus, for each u_i,

$$\frac{u_i'}{u_i} = -\frac{\nu_i}{\nu_0}\frac{\theta'}{\theta} .$$

If all the ν_i where 0, then each $u_i' = 0$, so we would have

$$1 = \Sigma c_i\frac{u_i'}{u_i} + v' = 0$$

a contradiction. We can therefore assume $\nu_1 \neq 0$ and then $u_i'/u_i = (\nu_i/\nu_1)(u_1'/u_1)$ so

$$\Sigma c_i\frac{u_i'}{u_i} = \left(\Sigma c_i\frac{\nu_i}{\nu_1}\right)\frac{u_1'}{u_1} .$$

Finally we get

$$\frac{y'}{f(y)} = \Sigma c_i\frac{u_i'}{u_i} + v' = c\frac{u_1'}{u_1} \quad \text{where} \quad c = \Sigma c_i\frac{\nu_i}{\nu_1} .$$

Thus $1/f(y) = c(d(u_1(y)/dy)/u_1(y)$.

We can use Corollary 2 to show certain differential equations have no elementary solutions. First notice that if we can write $1/f(y) = d(v(y))/dy$ for some $v(y)$ in $C(y)$, then by expanding $v(y)$ in powers of y and differentiating term by term we see that $1/f(y)$ could not have any term of the form $1/y$, when we expand $1/f(y)$ in powers of y. Similarly if $1/f(y) = (d(u(y))/dy)/u(y)$, when we expand $1/f(y)$ in powers of y, no terms of the form y^i for $i < -1$ can appear. In particular $y' = y^2/(y+1)$ has no nonconstant elementary solutions since $(y+1)/y^2 = (1/y^2) + (1/y)$. In general if $f(y) \in Q(y)$, we can decide if $1/f(y)$ is of one of the two forms described in the corollary as was shown by Risch [3].

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