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Formal Solutions of Differential Equations

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We give a survey of some methods for finding formal solutions of differential equations. These include methods for finding power series solutions, elementary and liouvillian solutions, first integrals, Lie theoretic methods, transform methods, asymptotic methods. A brief discussion of difference equations is also included.

In this paper, I shall discuss the problem of finding formal expressions that represent solutions of differential equations. By using the term "formal", I wish to emphasize the fact that most of the time I will not be concerned with questions of where power series converge or in what domains the expressions represent functions. I shall talk about power series solutions, solutions that can be expressed in terms of special functions such as exponentials, logarithms, or error functions, solutions given implicitly in terms of elementary first integrals and Lie theoretic techniques. I shall briefly mention transform methods, asymptotic expansions and devote a final section to a short discussion of formal solutions of difference equations.

There are many open problems in these areas and I have included my favorite ones. I hope they will stimulate further work.

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I. Power Series Solutions of Differential Equations

My aim here is to contrast what is known about linear differential equations with what is known about non-linear differential equations. Good general references for information about linear differential equations are Poole (1960) and Schlesinger (1895). Consider the linear differential equation

$$L(y) = y^{(n)} + a_{n-1}(x)y^{(n-1)} + \ldots + a_0(x)y = 0$$

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where the $a_i(x) \in \mathbb{C}((x))$, the field of formal Laurent series with finite principal parts. The point $x = 0$ is called an ordinary point of $L(y)$ if 0 is not a pole of any of the $a_i(x)$. When this is the case $L(y) = 0$ will have n linearly independent solutions $y_i = \Sigma a_{ij}x^j$, $0 \le i \le n-1$, in $\mathbb{C}[[x]]$, the ring of formal power series (furthermore, each of these solutions will converge in some neighborhood of 0, if the $a_i(x)$ converge in this neighborhood). Such a fundamental set of solutions can be found by setting $a_{ij} = 0$ for $0 \le i,\ j \le n-1,\ j \ne i$, and $a_{ii} = 1$, and using the differential equation to find $a_{ij},\ j \ge n$, by recursion. If some $a_i(x)$ has a pole at 0, we say 0 is a singular point of $L(y)$. We say the 0 is a regular singular point if in any open angular sector Ω at 0 all solutions y of $L(y) = 0$, analytic in Ω, satisfy $\lim\limits_{z \to 0} z^N y = 0$ for some $N \ge 0$. Fuchs showed that this is equivalent to saying that the order of the pole of each $a_i(x)$ at 0 is $\le n - i$. If we let $\delta = x\frac{d}{dx}$, we may write

$$x^n L(y) = \delta^n y + b_{n-1}(x)\delta^{n-1}y + \ldots + b_0(x)y$$

for some $b_i(x) \in \mathbb{C}((x))$. In these terms, 0 is a regular point if and only if 0 is not a pole of any of the $b_i(x)$. Let $b_i(x) = \Sigma b_{ij}x^j$ and

$$P(\lambda) = \lambda^n + b_{n-1,0}\lambda^{n-1} + \ldots + b_{0,0}.$$

$P(\lambda)$ is called the indicial polynomial of $L(y)$ at 0. If 0 is a regular singular point of $L(y)$, then there exist n linearly independent solutions of $L(y) = 0$ of the form

$$y_i = x^{\lambda_i}(\varphi_{i0} + \varphi_{i1}\log x + \ldots + \varphi_{is_i}(\log x)^{s_i})$$

with $\varphi_{ij} \in \mathbb{C}((x))$ and λ_i a root of $P(\lambda) = 0$ of multiplicity s_i (Coddington & Levinson (1955), Ch. 4). Once the λ_i are determined, the φ_{ij} can be found using a recursive procedure, due to Frobenius, similar to the ordinary point case. This method has been implemented by several people (e.g. Lafferty (1977), Davenport (1988), Della Dora (1981a), (1981b), Watanabe (1970), Tournier (1987)). When 0 is not a regular singular point, we say that 0 is an irregular singular point. In this case, there exists n linearly independent solutions of the form

$$y_i = e^{Q_i(x)}x^{\gamma_i}(\varphi_{i0} + \varphi_{i1}\log x + \ldots + \varphi_{is_i}(\log x)^{s_i}) \tag{1}$$

where $Q_i(x)$ is a polynomial in x^{-1/q_i}, q_i a positive integer, $\gamma_i \in \mathbb{C}$, s_i a positive integer, and $\varphi_{ij} \in \mathbb{C}[[x^{1/q_i}]]$. Schlesinger (1987) (Vol. I, Sec. 110) describes a method for finding q_i and Q_i and a more modern algorithm based on Newton polygon calculations is given in Della Dora (1981c) (see also Levelt (1975)). Once q_i and Q_i are found, one makes a change of variable $y = e^{Q_i(x)}z$ and proceeds as in the regular singular point case. An implementation, in the DESIR system, is described in Della Dora (1981c) and Tournier (1987). One can make similar definitions with respect to any point $x = \alpha$ or even the point of infinity (this latter case reduces to the point $t = 0$ after we make a change of coordinates $t = \frac{1}{x}$ and

$\frac{d}{dt} = -t^2 \frac{d}{dt}$. All the above algorithms force one to work with algebraic numbers, even if the original equation has coefficients that are polynomials with rational coefficients. A method that minimizes the amount of factorizations needed to do these calculations is presented in (Della Dora *et al.*, (1985).

We now consider a system of linear differential equations

$$Y' = \frac{A(x)}{x^q} Y \tag{2}$$

where $A(x) \in M_n(\mathbb{C}[[x]])$, the ring of $n \times n$ matrices with entries in $\mathbb{C}[[x]]$, and q is a non-negative integer. If $q = 0$, we say that 0 is an ordinary point and if $q \geq 1$, we say that 0 is a singular point. The definitions (in terms of the growth of solutions near 0) of regular and irregular singular point carry over to this case, but there is no analogue of the criteria of Fuchs to distinguish between these two. To do so, we can proceed in several ways. One way (the cyclic vector method) is to convert the system (2) to a single nth order equation and then use Fuchs' criteria (Adjamagbo (1988), Bertrand (1985), Katz (1986), Malgrange (1974), (1981), Ramis (1978), (1984)). A second method is due to Moser (1960). This method considers transforms $Y \to BY$ with $B \in M_n(\mathbb{C}(x))$ and their effect on (2). One gets an equation of a similar form with a possibly different value of q. One tries to find a matrix B so that the resulting q is minimal. When this happens, it is known that $q = 1$ if and only if 0 is a regular singular point. Both methods are discussed, with implementations in mind, in Hillali (1982), (1983), (1986), (1987a), (1987b), (1987c), Hillali & Wazner (1983), (1986a), (1986b). Other criteria and methods for determining if a singular point is regular are discussed in Gerard and Levelt (1973) (c.f. in particular Theorem 4.5). These papers also discuss how one can use either method to calculate other invariants of (2) (e.g. Malgrange index, Katz invariants).

The formal solutions given by (1) do not necessarily involve convergent series. It is known that if the $a_i(x)$ are analytic in a neighborhood of the origin, then in any sufficiently small sector at the origin, there are analytic solutions having (1) as asymptotic expansions. Questions regarding calculating these solutions are addressed in Loday-Richaud (1988), Ramis (1978), (1984), (1985a), (1985b), and Ramis & Thomas (1981).

Before leaving linear differential equations, it should be noted that some work has been done to implement methods of expressing solutions of linear differential equations in terms of series involving Chebyshev polynomials Geddes (1977) or other special functions Cabay & Labahn (1989) and Chaffy (1986).

We now turn to nonlinear differential equations. Although some work regarding algorithms for finding series solutions of nonlinear differential equations has been done in the past (e.g. Fitch, Norman & Moore (1981), (1986) and Geddes (1981)) the first general algorithm was presented in Denef & Lipshitz (1984). They show that given a set S of ordinary polynomial differential equations in $y_1, \ldots, y_m$ with coefficients in $\mathbb{Q}[x]$ and initial conditions, one can decide if S has a solution in $K[[x]]$ satisfying these initial conditions, where $K = \mathbb{C}, \mathbb{R}$, or $\mathbb{Q}_p$. Their basic idea in this is to show how one can find an integer N such that the system S is solvable if and only if S has a solution $\bmod x^N$. This latter condition reduces to checking the solvability of a system of linear equations. Although their algorithm is very explicit, it does not seen to be efficient.

Deciding if a system of ordinary polynomial differential equations has a power series solution is a delicate question. In Denef & Lipshitz (1984), it is also shown that there is no algorithm to decide if such a system has a convergent solution or if such a system has a non-zero solution. The situation for partial differential equations is worse. Denef and Lipshitz show that there is no algorithm to decide if a <u>linear</u> partial differential equation with coefficients in $\mathbb{Q}[x_1, \dots, x_9]$ has a solution in $\mathbb{C}[[x_1, \dots, x_9]]$. Furthermore, there are systems of partial differential equations having infinitely many power series solutions, none of which are computable (i.e., the sequence of coefficients cannot be generated by a Turing machine).

In Grigor'ev & Singer (1988), the authors consider a Newton polygon method to find solutions of differential equations of the form $y = \sum_{i=0}^{\infty} \alpha_i x^{\beta_i}$ where the $\alpha_i \in \mathbb{C}$ and the β_i are real with $\beta_0 > \beta_1 > \dots$. They show that if such an expression satisfies a polynomial differential equation $p(x, y, y', \dots) = 0$, then $\lim \beta_i = -\infty$. Furthermore, given any such y and $p(x, y, y', \dots)$, there exists an N such that for any $z = \Sigma\gamma_i x^{\delta_i}$ satisfying $p(x, z, z', \dots) = 0$ with $\alpha_i = \gamma_i$ and $\beta_i = \delta_i$ for all i with $\beta_i > N$, then $\alpha_i = \gamma_i$ and $\beta_i = \delta_i$ for all i (that is, each y is finitely determined). The authors give a method for enumerating solutions of this form of a differential equation and show that it is an undecidable problem to determine if a system of polynomial differential equations has a solution of this form.

We have not yet mentioned power series solutions of algebraic equations. Algorithms for finding the Puiseux expansions (power series in rational powers of x) of algebraic functions are well known Knuth (1981), Ch. 4.7. The fastest to date is due to the Chudnovskys (1985). They have shown how algorithms for finding power series solutions of linear differential equations can be used to find Puiseux expansions of algebraic functions. The key observation is that if y satisfies an irreducible equation $f(x, y) = 0$ of degree n over $\mathbb{C}(x)$, then $[\mathbb{C}(x, y) : \mathbb{C}(x)] = n$ and $y' = -f_x/f_y \in \mathbb{C}(x, y)$, so $\mathbb{C}(x, y)$ is closed under the derivation $'$. This implies that $y, y', \dots, y^{(n)}$ must be linearly dependent over $\mathbb{C}(x)$, so y satisfies nth order linear differential equation over this field. This equation can be calculated from $f(x, y)$ and then using an efficient version of the Frobenius algorithm one can calculate the Puiseux expansion of y. They are able to show that one can compute the first N terms of this expansion in $O(dN)$ operations and $O(dN)$ space, where d is the total degree of $f(x, y)$.

Other papers concerning power series solutions of differential equations are Bogen (1977), Fateman (1977), Lamnabhi-Lagarrigue & Lamnabhi (1982), (1983), Norman (1975), and Stoutemyer (1977).

II. Closed Form Solutions

We are concerned here with expressing the solutions of differential equations in terms of some given class of functions (e.g., exponentials, integrals and algebraic functions). We begin by considering the simplest differential equation

$$y' = \alpha$$

and ask when a solution (i.e., $y = \int \alpha$) can be expressed in terms of elementary functions, that is, in terms of sin, cos, exp, log, arctan, etc.; the functions of elementary calculus. For example, $y' = (2x)\exp(x^2)$ has an elementary solution $y = \exp(x^2)$ but $y' = \exp(x^2)$ does not (although this is not obvious). We wish to give the informal notion of expressible in elementary terms some mathematical rigor. This is done using the notion of a differential field. A field F is said to be a <u>differential field</u> with derivation $'$, if $' : F \to F$ satisfies $(a+b)' = a' + b'$ and $(ab)' = a'b + ab'$ for all $a, b \in F$. The constants of F are $\{c \mid c \in F \text{ and } c' = 0\}$ and are denoted by $C(F)$. For example, $\mathbb{C}(x)$ with the derivation d/dx in a differential field as is the field of meromorphic functions on a connected open set in $\mathbb{C}$ with the usual derivation. To formalize the notion of elementary function, first notice that if one thinks in terms of functions of a complex variable, then sin, cos, tan, arctan, etc. can all be expressed in terms of exp and log. This motivates the following definition. Let $F \subset E$ be differential fields. We say E is an <u>elementary extension</u> of F if there is a tower of fields $F = E_0 \subset \ldots \subset E_n = E$ where $E_i = E_{i-1}(t_i)$ and either (i) t_i is algebraic over E_{i-1}, or (ii) $t_i'/t_i = u_i'$ for some $u_i \in E_{i-1}$ (i.e., $t_i = \exp(u_i)$), or (iii) $t_i' = u_i'/u_i$ for some $u_i \in E_{i-1}$ (i.e., $t_i = \log(u_i)$). We say that y is <u>elementary</u> over F if y belongs to an elementary extension of F. For example, $y = \exp(x\log(x + \sqrt{x}))$ is elementary over $\mathbb{C}(x)$, since y belongs to the last member of the tower $\mathbb{C}(x) \subset \mathbb{C}(x, \sqrt{x}) \subset \mathbb{C}(x, \sqrt{x}, \log(x+\sqrt{x})) \subset \mathbb{C}(x, \sqrt{x}, \log(x+\sqrt{x}), \exp(x\log(x+\sqrt{x})))$.

Our naive question "When can we express a solution of $y' = \alpha$ in terms of elementary functions?" can now be formalized as "Given a differential field F and an element α of F, when does $y' = \alpha$ have a solution in an elementary extension of F?" The answer is given by Liouville's Theorem: Let F be a differential field of characteristic zero and $\alpha \in F$. If $y' = \alpha$ has a solution in an elementary extension K of F, with $C(F) = C(K)$, then

$$\alpha = v' + \sum_{i=1}^{m} c_i \frac{u_i'}{u_i}$$

where v and the u_i are in F and c_i are constants of F. In other words, if α has an elementary antiderivative, then $\int \alpha = v + \Sigma c_i \log(u_i)$, where v and the u_i only involve those functions that already appear in α. The condition on the constants is technical but necessary (if we work over the complex numbers, there is no problem; see Risch (1969) and Davenport, Siret & Tournier (1988) for a further discussion of this issue). Special cases of the above theorem were originally proved by Liouville (1833), (1835). Ostrowski gave a proof of this theorem in the context of differential fields in Ostrowski (1946). The work of Liouville and Ostrowski is discussed in Ritt (1948), along with additional work of Mordukhai-Boltovski and Ritt. A completely algebraic proof was first given by Rosenlicht in Rosenlicht (1968) (see also Rosenlicht (1976)). The best place to read a proof of this theorem is Rosenlicht (1972).

To get a feeling for why Liouville's Theorem is true, one should consider the following pieces of evidence. First, the theorem is true when α is in $\mathbb{C}(x)$. In this case we may expand α in partial fractions $\alpha = p(x) + \sum_i \sum_j a_{ij}(x - b_i)^{-n_{ij}}$. When we integrate α, each term contributes something in $\mathbb{C}(x)$, except if $n_{ij} = 1$, in which case we get $\log(x - b_i)$, which appears linearly. Secondly, we can look at the general case and ask: If we need a new algebraic, log or exponential to integrate an expression, how can this new function appear in the antiderivative. For example, if $\int \alpha$ is an algebraic function of α, then we can sum the conjugates of $\int \alpha$ and divide by their number and get a new antiderivative of α that is a rational function of α. Since the antiderivative is unique up to additive constant, the original algebraic function must be a rational function of α (i.e. no non-rational algebraic functions are needed). Now assume that we needed a new logarithm or an exponential to express our antiderivative. For example, assume that $\int \alpha = (\exp(u))^n + \ldots$. When we differentiate both sides of this equation, we get $\alpha = nu'(\exp(u))^n + \ldots$. Since we are assuming that $\exp(u)$ does not already occur in α, we must have $n = 0$. If $\int \alpha = (\log(u))^n \ldots$, then $\alpha = n(u'/u)(\log(u))^{n-1} + \ldots$. Since we assume that $\log(u)$ does not appear in α, we must have $n = 1$, i.e. the new log appears linearly. This heuristic argument can be formalized and is the basis of the argument in Rosenlicht (1972).

Liouville's Theorem gives a criterion for a function to have an elementary antiderivative and in Rosenlicht (1972) this is used to show that $\int \exp(x^2)$ is not elementary. A general algorithm to decide if a function, elementary over $\mathbb{C}(x)$ has an elementary antiderivative was given by Risch in a series of paper (Risch (1968), (1969), (1970)). The algorithm takes as input an elementary tower $K(x) \subset E_1 = K(x, t_1) \subset \ldots \subset E_m = K(x, t_1, \ldots, t_m)$ (where K is a finitely generated field of characteristic zero) and an element α in E_m and decides if it is of the form prescribed by Liouville's Theorem. If it is, the algorithm produces such an expression. Risch (1969) treated the case of a purely transcendental integrand. Improvements of this algorithm were made by many people (Bronstein (1988), Davenport, Siret & Tournier (1988), Davenport (1983) (this has a large and useful bibliography), Davenport (1986), Epstein (1975), Geddes & Stefanus (1989), Horowitz (1969), (1971), Kaltofen (1984), Norman (1983), Norman & Davenport (1979), Norman & Moore (1977), Rothstein (1976), (1977), Trager (1976), (1984), Yun (1977)). In Risch (1968) and Risch (1970), Risch outlined an algorithm for the mixed case; the case where algebraics are also allowed in the defining tower of α. This algorithm is much more complex than the previous one. When α is algebraic over $\mathbb{C}(x)$, new ideas and improvements were given by Trager and Davenport (Trager (1979), (1984), Davenport (1981)). Bronstein has generalized and applied these ideas to the general case in Bronstein (1990). The Risch algorithm for purely transcendental elementary functions has been implemented in most computer algebra systems. Bronstein's algorithm is being implemented at present in the SCRATCHPAD system.

All algorithms proceed by induction on the length of the defining elementary tower for α (the method of Norman & Moore (1977) does not, but it is known not to be an algorithm, see Norman & Davenport (1977) and Davenport (1986)). A particular α can belong to several different elementary towers. For example $\sqrt{x}\exp(x)$ belongs to both $\mathbb{C}(x, \sqrt{x}, \exp(x))$ and $\mathbb{C}(x, \log(x), \exp(x+(1/2)\log(x))$. The first of these fields is built up using algebraic elements, while the second is purely transcendental. The efficiency of the algorithms depends heavily on the particular choice of defining tower. Some work has been done with regards to selecting a good defining tower (Davenport (1986) and Bronstein (1988)) but much more can be done. This motivates the following problem:

Problem 1. What is the "best" field of definition for an elementary function? Can one decide if a given elementary function belongs to a purely transcendental elementary extension of $\mathbb{C}(x)$?

Several generalizations of the Liouville Theorem have been made. Risch (1976) gives a Liouville type theorem for integration in terms of real elementary functions and Bronstein gives an algorithm in Bronstein (1989). In Singer, Saunders & Caviness (1985), a Liouville type theorem is presented, along with algorithmic considerations, that deals with integration in terms of a class of functions that includes the elementary functions as well as the error function and the logarithmic integral. This work has been generalized by Cherry (Cherry (1985), (1986)) and Knowles (1986). In these papers the structure of the defining tower plays a crucial role in the algorithmic results and these algorithms only treat certain classes of functions (in particular, they do not handle functions that are built up using algebraic functions). Using ideas developed in algebraic K-theory, Baddoura (1989) gives a Liouville type theorem and algorithm for integration in terms of elementary functions and dilogarithms. Baddoura's work also only deals with a restricted class of functions. There are still many open problems concerning generalizations of Liouville's Theorem and the interested reader is referred to the above papers. Some heuristics are also given in Picquette (1989).

So far we have only considered indefinite integrals. Heuristic techniques for evaluating definite integrals are discussed in Geddes & Scott (1989), Kolbig (1985) and Wang (1971). Recently, Almkvist & Zeilberger (1989) have proposed a method for evaluating expressions of the form $f(x) = \int_a^b F(x,y)dy$, for example

$$\int_{-\infty}^{+\infty} \exp\left(-\frac{x^2}{y^2} - y^2\right) dy = \sqrt{\pi}\exp(-2x).$$

They consider functions $F(x,y)$ that satisfy a pair of linear partial differential equations of the form

$$P(x,y,\partial/\partial x)F = p_n(x,y)\frac{\partial^n F}{\partial x^n} + \ldots + p_0(x,y)F = 0, \text{ and}$$
$$Q(x,y,\partial/\partial y)F = q_m(x,y)\frac{\partial^m F}{\partial x^m} + \ldots + q_0(x,y)F = 0,$$

with coefficients that are polynomials in x and y (these functions are called D-finite (Lipshitz (1988)). In this case it is known that $f(x)$ will satisfy an ordinary linear differential equation

$$L(x,d/dx)f = a_N(x)\frac{d^N f}{dx^N} + \ldots + a_0(x)f = 0$$

(see Lipshitz (1988)). L can be found using an elimination algorithm. One then solves $L(x,d/dx)f = 0$ in terms of some class of functions (if this is possible, see below) and compares initial conditions to get a closed form expression for $f(x)$.

We now turn to the problem of solving more complicated differential equations in closed form. We start by considering linear differential equations

$$L(y) = y^{(n)} + a_{n-1}y^{(n-1)} + \ldots + a_0 y = 0.$$

When the a_i are constants, we teach our undergraduates how to express all solutions as sums of products of polynomials and exponentials. An implementation of an algorithm to do this is described in Tournier (1979). When the a_i are rational functions, some heuristics and special cases are discussed in Malm (1982) and Schmidt (1979) and implementations of variation of parameters and the method of undetermined coefficients are discussed in Schmidt (1976) and Rand (1984).

We now turn to some general algorithms. Assume that the $a_i \in k(x)$, where k is some finitely generated extension of $\mathbb{Q}$. The question of when $L(y) = 0$ has only solutions that are algebraic over $k(x)$ was originally treated by F. Klein in 1877 when $n = 2$. Klein showed that if $L(y) = 0$ has only algebraic solutions then there is a change of variables $x = \varphi(t)$ such that the new equation is of a very special form, that is it appears in a list of all linear differential equations with three singular points and only algebraic solutions discovered by H. A. Schwarz around 1870 (see Gray (1986) for a discussion of the work of Klein, Schwarz and their contemporaries). A modern discussion of Schwarz's list and related material appears in Matsuda (1985). Klein's method was made effective by Baldassarri and Dwork in Baldassarri & Dwork (1979) and Baldassarri (1980). For $n \geq 2$, P. Painlevé and his student A. Boulanger gave a decision procedure in 1898 (a similar procedure was rediscovered by the present author in 1979, see Singer (1980)).

The next natural class of functions are the liouvillian functions. These are the functions that can be built up from $k(x)$ using integration, exponentiation, algebraic functions and composition (a formal definition is given below). These functions are named after J. Liouville, who was the first to give necessary and sufficient conditions for a second order homogeneous linear differential equation to have a solution of this form, Liouville (1839), (1841) and Ritt (1948). When $n = 2$, Kovacic (1986) gave an algorithm to decide if all solutions of $L(y) = 0$ can be expressed in terms of liouvillian functions and showed how to exhibit a basis when this is the case. Kovacic's algorithm is very explicit and parts of it have been implemented in MACSYMA (Saunders (1981)) and MAPLE (Char (1986)) (see also Smith (1984)). Improvements to this algorithm have been given in Duval & Loday-Richaud (1989). For $n \geq 2$, an algorithm is presented in Singer (1981) to decide if $L(y) = 0$ has a non-zero liouvillian solution and, if so, shows how to construct a vector space basis for the space of all such solutions (some of the ideas already occur in Marotte (1898), but I was not aware of this at the time Singer (1981) was written). A natural generalization of this is to find an effective procedure to produce for a given linear differential equation $L(y)$, with coefficients in a liouvillian extension of $\mathbb{Q}(x)$, a basis for the liouvillian solutions of $L(y) = 0$. I have recently shown (Singer (1988c)) that one can do this if the linear differential equation has coefficients in a purely transcendental liouvillian extension of $\mathbb{C}(x)$ or in an elementary extension of $\mathbb{C}(x)$. The algorithm presented there is extremely inefficient and can use improvement and generalization to handle the complete liouvillian case.

Problem 2. Find an efficient algorithm to decide if an nth order linear differential equation with rational function (or liouvillian) coefficients has a liouvillian solution.

Some progress has been made on this problem. A problem that comes up in Singer (1981) is the problem of factoring linear differential equations. Schwarz discusses an algorithm (with implementation) for this in Schwarz (1989) and Grigor'ev discusses another algorithm and gives complexity bounds in Grigor'ev (1988). In Singer (1981), group theoretical methods were used to obtain certain bounds (see below) and Ulmer (1989) shows how stronger techniques from group theory yield better bounds.

Other work on deciding if linear differential equations have liouvillian solutions appears in Watanabe (1981), where techniques are developed to transform a given linear equation to a hypergeometric equation and Watanabe (1984), where change of variable techniques are discussed that will take a linear differential equation with coefficients in a liouvillian extension of $\mathbb{C}(x)$ to one with coefficients in $\mathbb{C}(x)$.

I will now give a sketch of some of the ideas involved in Kovacic (1986) and Singer (1981), and start by defining some notions from differential algebra (Kaplansky (1957) and Kolchin (1973) are good references for this). Let F be a differential field of characteristic 0. If $L(y) = 0$ is an nth order linear differential equation with coefficients in F, we can formally adjoin to F a set of n solutions $y_1, \ldots, y_n$ of $L(y) = 0$, linearly independent over $\mathbb{C}$, and their derivatives. When C is algebraically closed, we can choose $y_1, \ldots, y_n$ so that the field $K = F(y_1, \ldots, y_n, y_1', \ldots, y_n', \ldots, y_1^{(n-1)}, \ldots, y_n^{(n-1)})$ contains no new constants (note that this field is closed under $'$ since $y_i^{(m)}$, $m \geq n$, can be expressed in terms of lower order derivatives of y_i using $L(y_i) = 0$). Such a K is unique up to a differential F-isomorphism and is called the Picard-Vessiot extension of F corresponding to $L(y) = 0$. Let $G = \{\sigma \mid \sigma$ is an automorphism of K, $\sigma(u)' = \sigma(u')$ for all $u \in K$ and $\sigma(v) = v$ for all $v \in F\}$. G is called the galois group of the equation $L(y) = 0$ over F (or of the field K over F). If $y \in K$ is any solution of $L(y) = 0$ and $\sigma \in G$, then $\sigma(y)$ is also a solution of $L(y) = 0$. One can show that this implies that $y = \Sigma c_j y_j$ for some $c_j \in C$. Therefore, for each i, $\sigma(y_i) = \Sigma c_{ij} y_j$ for some $c_{ij} \in K$. In this way we may associate a matrix (c_{ij}) with every $\sigma \in G$. (c_{ij}) is invertible, so this gives us an isomorphism of G into $GL(n, C)$, the group of invertible $n \times n$ matrices over C. Identifying G with its image, it can be shown that $G = GL(n, C) \cap V$, where $V \subset C^{n^2}$ is the zero set of some collection of polynomials (such a set is said to be closed in the Zariski topology). There is a galois theory that identifies differential subfields K_1, $F \subset K_1 \subset K$, with Zariski closed subgroups of G (a closed subgroup corresponds to the field of elements left fixed by all its members; in particular F corresponds to G). We can formalize the notion of solvable in terms of liouvillian functions. K is said to be a liouvillian extension of k if there is a tower of fields $k = K_0 \subset \ldots \subset K_n = K$ such that $K_i = K_{i-1}(t_i)$, where either $t_i' \in K_{i-1}$ or $t_i'/t_i \in K_{i-1}$ or t_i is algebraic over K_{i-1} (the first two cases correspond to t_i being an integral or an exponential). A fundamental theorem states that $L(y) = 0$ is solvable in terms of liouvillian functions (i.e. its Picard-Vessiot extension lies in a liouvillian extension of F) if and only if its galois group contains a solvable subgroup of finite index (Kaplansky (1957), Kolchin (1973), Singer (1988b)).

Let us now consider the problem of finding liouvillian solutions of $L(y) = 0$. For simplicity, let us just try to decide if all solutions of $L(y) = 0$ are liouvillian. The galois theory implies that this is the case if and only if the galois group of $L(y)$ has a solvable subgroup of finite index. An effective version of the Lie-Kolchin Theorem asserts that in this case G will have a subgroup H such that the elements of H can simultaneously be put in upper triangular form and such that the index of H is bounded by $I(n)$, a computable function of n. If y is a common eigenvector of H, then $\sigma(y'/y) = cy'/cy = y'/y$ so y'/y is left fixed by H. This implies that y'/y is algebraic over F of degree bounded by $I(n)$. Therefore if $L(y) = 0$ is solvable in terms of liouvillian functions, $L(y) = 0$ will have a solution y such that y'/y is algebraic over F of degree bounded by $I(n)$. We now must decide if $L(y) = 0$ has such a solution. The idea is to look for candidates for the minimal polynomial of $u = y'/y$. If $p(u) = u^N + b_{N-1}(x)u^{N-1} + \ldots + b_0(x)$ $(N \leq I(n))$ is the minimal polynomial of such a u, then one can show that there exist solutions $z_1, \ldots, z_N$ of $L(y) = 0$ such that each b_i will be the ith symmetric function of the z_j'/z_j. By studying the poles of the coefficients of $L(y) = 0$, we can bound the number and order of the poles and zeroes of the b_i. This allows us to bound the degrees of the numerators and denominators of the b_i. Therefore if $L(y) = 0$ has only liouvillian solutions, it will have a solution y such that $u = y'/y$ satisfies a polynomial over $k(x)$ of degree $\leq I(n)$ whose coefficients have numerators and denominators of effectively bounded degrees. Elimination theory then allows us to decide if such a solution exists and produces u. We then use the change of variables $y = ze^{\int u}$ to get a new equation $L^*(z) = 0$ of lower order and proceed via induction. Actually, to make the induction work we prove a stronger result: given a linear differential equation with coefficients in an algebraic extension of $\mathbb{Q}(x)$, one can find in a finite number of steps a basis for the space of liouvillian solutions of $L(y) = 0$. This is done in Singer (1981).

So far, we have only been considering homogeneous linear differential equations, but one can ask the same questions about non-homogeneous linear differential equations $L(y) = b$. Such questions are considered in Davenport (1984), (1985), Davenport & Singer (1985), (1986), where in addition some open problems are mentioned.

We now turn to the general problem of solving a homogeneous linear differential equation $L(y) = 0$ of order n in terms of algebraic combinations and superpositions of solutions of linear differential equations of lower order (not necessarily homogeneous). In this context, asking for liouvillian solutions of a linear differential equation is the same as asking: When can it be solved in terms of first order linear equations (all solutions of first order linear equations are liouvillian and liouvillian functions are built up using algebraic combinations of solutions of $y' = a$ and $y' - ay = 0$)?

One can next ask: When can the solutions of a homogeneous linear differential equation be expressed in terms of solutions of linear differential equations of order at most two. Special cases of this question have been considered by Clausen, Goursat, Bailey, Ramanujan and others (Erdelyi *et al.* (1953)), who tried to understand when the product of two generalized hypergeometric functions is again a generalized hypergeometric function. They discovered beautiful formulas, such as

$$ {}_1F_1(\alpha,\rho;z)\,{}_1F_1(\alpha,\rho;-z) = {}_2F_3(\alpha,\rho-\alpha;\rho,(1/2)\rho,(1/2)(\rho+1);z^2/4) $$

I formalized the notion of solvability in terms of second order linear differential equations in Singer (1985). Briefly, a homogeneous linear differential equation is said to be solvable in terms of second order linear differential equations if the associated Picard-Vessiot extension lies in a tower of fields, each generated over the previous one by either an algebraic element or a solution of second order linear differential equation (we consider first order linear differential equations to be degenerate second order equations and allow them as well). In Singer (1985), I gave a criterion in terms of the galois group, for a homogeneous linear differential equation $L(y) = 0$, with coefficients in an arbitrary differential field k of characteristic 0, to be solvable in terms of second order linear differential equations. For example, if $L(y)$ has order 3, then it is solvable in terms of second order linear differential equations if and only if one of the following holds: (i) $L(y) = L_1(L_2(y))$ or $L(y) = L_2(L_1(y))$, where $L_1(y)$ and $L_2(y)$ are linear homogeneous differential polynomials of orders 1 and 2 respectively, with coefficients algebraic over k, or (ii) $L(y) = 0$ has a basis of its solution space of the form

$$\begin{aligned} y_1 &= b_0u^2 + b_1(u^2)' + b_2(u^2)'' \\ y_2 &= b_0uv + b_1(uv)' + b_2(uv)'' \\ y_3 &= b_0v^2 + b_1(v^2)' + b_2(v^2)'' \end{aligned}$$

where the b_i are algebraic over k and $\{u, v\}$ is a basis of the solution space of a second order homogeneous linear differential equation of order 2 with coefficients in k (for example, the solution space of $y''' - 4xy' - 6y = 0$ is spanned by $(u^2)', (uv)'$, and $(v^2)'$, where $\{u, v\}$ is a basis for the solutions of $y'' - xy = 0$). In Singer (1985), I show how this can be used to give a decision procedure to determine if an arbitrary third order homogeneous linear differential with coefficients in $\mathbb{Q}(x)$ can be solved in terms of second order linear differential equations.

The general problem of solving homogeneous linear differential equations in terms of lower order linear differential equations is considered in Singer (1988a) (see Singer & Tretkoff (1985) for a discussion of a related problem). Again this notion can be formalized in terms of towers of fields. Necessary and sufficient conditions can be given in terms of the Lie algebra of the galois group. One result is that a homogeneous linear differential equation cannot be solved in terms of lower order linear differential equations if and only if the Lie algebra of its galois group is simple and has no non-zero representations of smaller degree. I do not know of any general algorithms, and pose

Problem 3. Give an effective procedure to decide if a homogeneous linear differential equation can be solved in terms of linear differential equations of lower order. (One can show that a solution of Problem 6 below would yield a solution of this.)

When we consider the question of solving a third order homogeneous linear differential equation in terms of second order linear equations, the algorithm given in Singer (1985) does not allow us to restrict in advance the kind of second order equations we can use. This suggests

Problem 4. Give a procedure to decide if a homogeneous linear differential equation can be solved in terms of solutions of a restricted class of linear differential equations (e.g., Bessel functions).

Recall that a power series $F(x, y)$ in two variables is D-finite if f satisfies a system of non-zero differential equations of the form $P(x, y, \partial/\partial x)F = 0$ and

$Q(x, y, \partial/\partial y)F = 0$. For example, algebraic functions of two variables are D-finite. As mentioned above if F is such a function then $f(x) = \int_b^a F(x,y)dy$ will satisfy a linear differential equation over $\mathbb{C}(x)$. The solutions of the hypergeometric equation can be expressed in this form

$$f(a,b,c;x) = \frac{\Gamma(c)}{\Gamma(b)\Gamma(c-b)} \int_0^1 t^{b-1}(1-t)^{c-b-1}(1-xt)^{-a}dt$$

Poole (1960). This leads to the following question.

Problem 5. Find a procedure to decide if a linear differential equation has a nonzero solution of the form $\int_b^a f(x,y)dy$, where f is an algebraic function of two variables, and produce one if it does.

Related to the problem of solving linear differential equations in finite terms is the problem of deciding if two linear differential equations are equivalent under a change of coordinates (Berkovich, Gerdt, Kostova&Nechaevsky (1989), Kamran & Olver (1986), Neuman (1984), (1985)) and finding linear differential operators that commute with a given linear differential operator (which then can be used to find solutions of the original operator, see Gerdt & Kostov (1989)).

In most of the above considerations, the galois group of a homogeneous linear differential equation plays a crucial role. Yet unlike the situation with algebraic equations, there is no known algorithm to calculate the galois group of a homogeneous linear differential equation (i.e. produce a set of polynomials defining this group in $GL(n,C) \subset C^{n^2}$) or even its dimension as an algebraic variety (for $n = 2$ or 3 this can be done as a consequence of the algorithms described above, but for $n > 3$, nothing is known). This suggests

Problem 6. Give an algorithm that will find the galois group of any homogeneous linear differential equation with coefficients in $\mathbb{Q}(x)$, or at least calculate its dimension.

There has been some recent activity concerning calculation of the differential galois groups of certain classes of linear differential equations. In Beukers, Bronawell & Heckman (1988), Beukers & Heckman (1987) and Katz (1987), the authors are able to extract representation theoretic information about the galois groups from information at the singular points of the differential equation and combining this with information about root systems of simple Lie groups, can give usable sufficient conditions for an nth order linear differential equation to have a "large" galois group (i.e. the galois group contains $SL(n,\mathbb{C})$ or $SP(n,\mathbb{C})$). Katz is able to refine these techniques in Katz (1989) to calculate the Lie algebra of the galois groups of many differential equations. In Duval & Mitschi (1988) and Mitschi (1989a), (1989b), the authors use the theory of the "savage π_1" (see below) developed by Ramis (1985a), (1985b), (1988) and Martinet & Ramis (1988) to explicitly calculate the galois groups of generalized confluent hypergeometric equations.

Related to the galois group is the notion of the monodromy group. Given a homogeneous linear differential equation $L(y)$ with coefficients in $\mathbb{C}(x)$, let $\{a_1, \ldots, a_m\}$ be the singular points (possibly including ∞) and let $y_1, \ldots, y_n$ be a fundamental set of solutions at a regular point a_0. Given any path γ in the Riemann sphere $S^2 - \{a_1, \ldots, a_m\}$, we can analytically continue $y_1, \ldots, y_n$ around γ and get a new fundamental set of solutions. This new set is a linear combination of the old set, so we can associate to γ an invertible matrix A_γ. A_γ depends only on the homotopy class of γ and we get a homomorphism from $\pi_1(S^2 - \{a_1, \ldots, a_m\})$ to $GL(n, \mathbb{C})$ called the monodromy representation of the differential equation. The image of this homomorphism is called the monodromy group (see Poole (1960) and Katz (1976)). In general, it is very difficult to compute this group. Problem 6 can therefore be restated for monodromy groups.

When all the singular points of $L(y)$ are regular singular points, we know, (e.g., Tretkoff & Tretkoff (1979)) that the Zariski closure of the monodromy group is the galois group. This is not the case when we have irregular singular points (e.g. the monodromy group of $y' - y = 0$ is trivial but the galois group is $\mathbb{C}^*$). Recall that at a singular point (for simplicity, we assume this to be 0), there are n linearly independent solutions of the form

$$y_i = e^{Q_i(x)} x^{\gamma_i} (\varphi_{i0} + \varphi_{i\ell} \log x + \ldots + \varphi_{is_i} (\log x)^{s_i})$$

where $Q_i(x)$ is a polynomial in x^{-1/q_i}, q_i a positive integer, $\gamma_i \in \mathbb{C}$, s_i a positive integer, and $\varphi_{ij} \in \mathbb{C}[[x^{1/q_i}]]$. Let $\nu = LCM\{q_i\}$ and $t = x^{1/\nu}$. Let $K = \mathbb{C}\{t\}[t^{-1}]$, the ring of meromorphic functions in t and $\hat{K} = \mathbb{C}[[t]][t^{-1}] = \mathbb{C}((t))$. In this situation, Ramis defines a group to replace the classical monodromy group. This group is generated by three subsets: the exponential torus, the formal monodromy and the Stokes matrices. Let E be the Picard-Vessiot extension of $\mathbb{C}\{x\}[x^{-1}]$ generated by the y_i. The exponential torus is defined as follows: $K(e^{Q_1(x)}, \ldots, e^{Q_n(x)})$ is a Picard-Vessiot extension of K whose galois group over K is $(\mathbb{C}^*)^r$ for some r. Ramis calls this group the exponential torus $\mathcal{T}$ and shows that it is a subgroup of the galois group of E over $\mathbb{C}\{x\}[x^{-1}]$. One can also form the extension $F = \hat{K}(\log t, \{t^{\gamma_i}\}, \{e^{Q_i}\})$ of $\hat{K}$. Note that E is a subfield of this extension. The map $t \to t \cdot \exp(2\pi i/v)$ induces an automorphism of F, which in turn induces an automorphism of E. In this way we can consider $\mathbb{Z}/\nu\mathbb{Z}$ a subgroup of the galois group of E over $\mathbb{C}\{x\}[x^{-1}]$; this is called the formal monodromy. Although the φ_{ij} above are formal series, it is known that in sufficiently small angular sectors, they are the asymptotic expansions of analytic functions. Ramis shows that by demanding a special kind of asymptotic expansion (this is the notion of k-summability) then one can canonically select the sectors and canonically select the analytic functions representing these formal solutions (strictly speaking this statement is only true under an additional assumption on the Newton polygon of the linear differential equation dual to the one under consideration. The technically correct statement can be found in the above references, but the above statement gives the flavor of the result). These sectors overlap and on the overlap the respective solutions are related to each other by a matrix change of basis. The matrices gotten in this way are called the Stokes matrices and Ramis shows that they are also in the galois group of E over $\mathbb{C}\{x\}[x^{-1}]$. Ramis is finally able to show that the Zariski closure of the group generated by the exponential torus, the formal monodromy and the Stokes matrices is the local galois group, i.e. the galois group of E over $\mathbb{C}\{x\}[x^{-1}]$. Ramis also shows that one can formally construct a group $\prod$, the "savage π_1" such that any group generated by the exponential torus, formal monodromy and Stokes matrices of a singular point is a representation of $\prod$. This gives a generalization of the classical monodromy representation at a point.

The exponential torus and the formal monodromy can be calculated from the formal expressions (above) for the solutions y_i. When one has integral representations of the solutions y_i (for example as G-functions) then one can also calculate the Stokes matrices. Furthermore, if the differential equation has only two singular points, one regular and one irregular, then the local galois group at the irregular singular point is the same as the global galois group (i.e. the galois group over $\mathbb{C}(x)$). This is the ideal used in Duval (1989), Duval & Mitschi (1988) and Mitschi (1989a), (1989b).

We now turn to non-linear differential equations. A liouville type theorem describing the form of elementary solutions of such equations was given by Mordukhai-Boltovski (see Ritt (1948)) for first order non-linear differential equations with coefficients in $\mathbb{C}(x)$, and generalized to higher order equations in Singer (1975), Risch (1979) and Rosenlicht (1977). Mordukhai-Boltovski's theorem states that if $f(x, y, y') = 0$ is a polynomial first order differential equation with coefficients in $\mathbb{C}$ that has an elementary solution, then the equation has a solution of the form

$$y = G(x, \varphi_0 + a_1 \log \varphi_1 + \ldots + a_r \log \varphi_r)$$

or

$$y = G(x, \exp(\varphi_0 + a_1 \log \varphi_1 + \ldots + a_r \log \varphi_r))$$

where the a_i are in $\mathbb{C}$, G is an algebraic function of two variables and the φ_i are algebraic functions of one variable. Except in special cases, I do not know how to make this result effective.

Problem 7. Give a procedure to decide if a polynomial first order differential equation $f(x, y, y') = 0$ has an elementary solution and to find one if it does.

The final issue I wish to bring up in this section is the general question of deciding if a set of polynomial differential equations $\{p_\alpha = 0\}$ in $y_1, \ldots, y_n$ (say with coefficients in $\mathbb{Q}$) is consistent, that is, if the equations have any solutions at all. Closely related to this problem is the problem of determining if every solution of a set of differential equations $\{p_\alpha = 0\}$ is also a solution of another differential equation $q = 0$. Ritt gave an effective procedure for this (Ritt (1966)) and in the process initiated the study of differential ideals and differential algebra in general.

Note that when we say solution, we mean an analytic solution (Rubel (1983) discusses the failure of differential algebra to deal with non-analytic solutions). This procedure was generalized by Seidenberg (1956) and Grigor'ev (1989). Recently Wu has implemented parts of Ritt's procedure (Wu (1987a), (1987b), (1989)) (see also Wang (1987)). In particular, he can show that Newton's laws can be mechanically derived from Kepler's laws. Besides considering the efficiency of Ritt's algorithms, there are still problems in effective differential ideal theory that are open and deserve more attention. We mention one and refer the reader to Ritt (1966) and Kolchin (1973) for relevant definitions

Problem 8. Give an algorithm that finds the minimal prime components of a radical differential ideal.

There are well known algorithms that find the prime components of a radical ideal of (nondifferential) polynomials, but this problem is open in the differential case.

Related to the ideal theory of differential equations is the question of finding Groebner basis for systems of linear partial differential equations (Galligo (1985) and Chen (1989), Kandri-Rody & Weispfenning (1990)), the general problem of simplifying systems of differential equations (Wolf (1985a), (1985b)), and the problem of generating all integrability conditions for systems of partial differential equations (Schwarz (1984)). In Galligo (1985), the author also mentions other problems concerning D-modules, that is modules over the ring $\mathbb{C}[x_1, \dots, x_n, \partial/\partial x_1, \dots, \partial/\partial x_n$ These modules have been useful in studying properties of solutions of systems of linear differential equations.

III. First Integrals

In elementary courses in differential equations, I discuss the predator-prey equations

$$\begin{aligned}\dot{x} &= ax - bxy\\ \dot{y} &= -cy + dxy\end{aligned}$$

and show that the function

$$F(x, y) = dx + by - c\log x - a\log y$$

is constant on solution curves $(x(t), y(t))$. By studying the critical points of $F(x, y)$ one can then show that all solution curves are closed, that is, all solutions are periodic. A non-constant function that is constant on solution curves is called a first integral. In Singer (1977) and Prelle & Singer (1983), we showed that if differential equations have elementary first integrals, they must be of a very special form. For example, if

$$\begin{aligned}\dot{x} &= P(x, y)\\ \dot{y} &= Q(x, y)\end{aligned} \tag{3}$$

where P and Q are polynomials with complex coefficients, has an elementary first integral, it has one of the form

$$F(x,y) = v_0(x,y) + \Sigma c_i \log(v_i(x,y))$$

where the c_i are constants and the v_i are algebraic functions of two variables. Furthermore, we showed in Prelle & Singer (1983), that if (3) has an elementary integral then there exists an R with $R^n \in \mathbb{C}(x,y)$ for some nonzero integer n, such that $d(RQ\,dx - RP\,dy) = 0$ (i.e. $\frac{\partial(RQ)}{\partial y} + \frac{\partial(RP)}{\partial x} = 0$). Such an R is called an integrating factor and once one is determined, we show in Prelle & Singer (1983) how to determine if (3) has an elementary first integral. Let R be an integrating factor and write $R^n = \prod f_i^{n_i}$ where f_i are irreducible polynomials and n_i are nonzero integers. One can show (Prelle & Singer (1983)) that since R is an integrating factor of (3) we must have $f_i | Df_i$ where $D = P\frac{\partial}{\partial x} + Q\frac{\partial}{\partial y}$. Conversely, Darboux showed that if one could find all irreducible f such that $f \,|\, Df$, then one could decide if there is an integrating factor (see Ince (1944), p. 31). We also know, (Jouanolou (1979), p. 109 and Singer (1988)) that for each system (3) there is an integer N such that if f is irreducible and $f \,|\, Df$, then the degree of f is less than N, but we do not know any effective procedure for determining N. N does not depend only on the degrees of P and Q in (3) but also on the coefficients as the following example shows. Let $P = (n+1)x$ and $Q = ny$, then $D = (n+1)x\frac{\partial}{\partial x} + ny\frac{\partial}{\partial y}$. One checks that $f = x^n - y^{n+1}$ satisfies $f = n(n+1)Df$. The problem of finding integrating factors and elementary first integrals reduces to

Problem 9. Given $D = P\frac{\partial}{\partial x} + Q\frac{\partial}{\partial y}$, with $P, Q \in \mathbb{C}[x,y]$, effectively bound the degrees of all f in $\mathbb{C}[x,y]$ that are irreducible and satisfy $f \,|\, Df$.

Both Poincaré (1934) and Painlevé (1972) worked on this problem and gave partial results. A modern account of related work appears in Jouanolou (1979).

Even without solving Problem 9, one can use the algorithm outlined in Prelle & Singer (1983) by arbitrarily assigning a bound to the degree of the f's such that $f \,|\, Df$. The drawback is that the algorithm will sometimes not find a first integral when one exists. This approach has been implemented in Shtokhamer, Glinos & Caviness (1986) with surprising success.

Prelle & Singer (1983) also contains results that imply that if an nth order differential equation $f(x,y,y',\ldots,y^{(n)}) = 0$ has an elementary first integral, it must be of a very special form. These other results have not been made effective. Risch (1976) contains related results.

Singer (1988) contains the foundations of a theory of liouvillian first integrals, that is liouvillian functions of several variables that are constant on solution curves of differential equations. This paper also contains algorithmic considerations. For example, I show that one can decide if (3) has a liouvillian first integral if one can decide the following question:

Problem 10. Given P, Q in $\mathbb{C}[x, y]$ and a, b in $\mathbb{C}(x, y)$, decide if $DU + aU = b$ has a solution a in $\mathbb{C}(x, y)$, where $D = P(\partial/\partial x) + Q(\partial/\partial y)$, and if so find such a solution.

Except in special cases I am unable to give such a procedure, nor am I able to reduce this question to the previous question.

There are several other approaches to finding first integrals. The approach using Lie methods is described below. Schwarz (1985) and Wolf (1987a), (1987b) describe methods that search for polynomial first integrals of an a priori bounded degree. In Goldman (1987a), (1987b) and Sit (1989), the authors describe a method to find polynomial first integrals (or more generally, first integrals that are sums of monomials with real or complex exponents) with an a priori bounded number of terms.

IV. Lie Methods

Both the problem of finding closed form solutions of differential equations and the problem of finding integrating factors can be attacked using Lie group methods. The basic idea is to find a group of symmetries of the differential equations and then use this group to reduce the order or the number of variables appearing in the equation. I will exhibit this idea by discussing Lie's discovery that the knowledge of a one-parameter group of symmetries of an ordinary differential equation of order n allows us to reduce the problem of solving this equation to that of solving a new differential equation of order $n - 1$ and integrating. In the case of a first order equation, I will also discuss how the knowledge of a one-parameter group of symmetries allows one to construct an integrating factor. I will be closely following the expositions in Markus (1960), pp. 1–80 and Olver (1979), (1986), Ch. 2, although most of the results mentioned here can be found (in one form or another) in Lie's original works (for example, the comments following Example IV.5 appear as Satz 3 of Lie (1922).

There seem to be no totally general methods for finding the symmetry group of a differential equation, but there are methods that do handle large classes of equations. In Schwarz (1988), Schwarz gives an introduction to Lie methods and differential equations with a special emphasis on the use of computer algebra in computing symmetries. Sample programs and many examples, including symmetries of partial differential equations are also given there. Implementations are also discussed in Char (1980) and the works of Steinberg. Olver (1986), Schwarz (1988) and Steinberg (1983), (1985) are a good source of additional references.

I start with several key definitions. A local one-parameter group acting on $\mathbb{R}^2$ is an open set V, $\{0\} \times \mathbb{R}^2 \subset V \subset \mathbb{R} \times \mathbb{R}^2$ and a C^∞ map $\phi : V \to \mathbb{R}^2$ such that (1) $\phi(0,(x,y)) = (x,y)$ for all $(x,y) \in \mathbb{R}^2$, and (2) $\phi(g,\phi(h,(x,y))) = \phi(g+h,(x,y))$ whenever $g,h \in \mathbb{R}$, $(x,y) \in \mathbb{R}^2$ and $(h,(x,y))$, $(g,\phi(h,(x,y)))$ and $(g+h,(x,y))$ are in V (i.e., whenever (2) makes sense). If $V = \mathbb{R} \times \mathbb{R}^2$, we say ϕ is global. We sometimes will write $\phi_t(x,y)$ for $\phi(t,(x,y))$.

<u>EXAMPLE IV.1</u>: (Olver (1979), p. 204) Let $V = \{(t,(x,y))\,|\,ty \neq 1\}$ and let

$$\phi(t,(x,y)) = \left(\frac{x}{1-ty}, \frac{y}{1-ty}\right)$$

Note that this cannot be extended to a global group acting on $\mathbb{R}^2$.

An <u>infinitesimal one-parameter group</u> is a system of differential equations $\frac{dx}{dt} = f(x,y)$, $\frac{dy}{dt} = g(x,y)$ (or, more geometrically, the vector field $f(x,y)\frac{\partial}{\partial x} + g(x,y)\frac{\partial}{\partial y}$). Given a local one parameter group $\phi(t,(x,y)) = (F(t,(x,y)), G(t,(x,y)))$, we can define an infinitesimal one parameter group by $\frac{dx}{dt} = \frac{\partial}{\partial t}(F(t,(x,y))\Big|_{t=0}$, $\frac{\partial y}{\partial t} = \frac{\partial}{\partial t}(G(t,(x,y))\Big|_{t=0}$. Conversely, given an infinitesimal one-parameter group, if $x(t,(x_0,y_0))$, $y(t,(x_0,y_0))$ are the solutions corresponding to $x(0) = x_0$ and $y(0) = y_0$, then $\phi(t,(x_0,y_0)) = (x(t,(x_0,y_0)), y(t,(x_0,y_0)))$ defines a local one-parameter group acting on $\mathbb{R}^2$. This allows us to move back and forth between these two notions.

<u>EXAMPLE IV.2</u>: In Example IV.1, the infinitesimal one-parameter group is

$$xy\frac{\partial}{\partial x} + y^2\frac{\partial}{\partial y}.$$

If $X = f\frac{\partial}{\partial x} + g\frac{\partial}{\partial y}$ is an infinitesimal one-parameter group, we say that (x_0,y_0) is a critical point if $f(x_0,y_0) = g(x_0,y_0) = 0$. If (x_0,y_0) is not a critical point, it is called a regular point and one can show (Markus (1960), p. 14) that there is a change of coordinates $u(x,y), v(x,y)$ near (x_0,y_0) such that in these new coordinates $X = \frac{\partial}{\partial v}$.

Given a local one-parameter group ϕ_t and a differential equation $F(x,y,y') = 0$, we say that <u>ϕ_t is a symmetry group of $F(x,y,y') = 0$</u> if the following holds: if Γ is the graph of a solution $y(x)$ of $F(x,y,y') = 0$ through (x_0,y_0) then, if t is close to 0, there is an open neighborhood U_t of (x_0,y_0) such that $\phi_t(\Gamma \cap U_t)$ is the graph of a solution of $f(x,y,y') = 0$ through $\phi_t(x_0,y_0)$ (i.e. ϕ_t takes solutions to solutions). Luckily, one never needs to verify this condition directly. If $X = f\frac{\partial}{\partial x} + g\frac{\partial}{\partial y}$ is the infinitesimal one-parameter group associated with ϕ_t, one can show that ϕ_t is a symmetry group of $F(x,y,y') = 0$ if and only if $X_1(F(x,y,y')) = 0$ whenever $F(x,y,y') = 0$ where

$$X_1 = f\frac{\partial}{\partial x} + g\frac{\partial}{\partial y} + \left(\frac{\partial g}{\partial x} + \left(\frac{\partial g}{\partial y} - \frac{\partial f}{\partial x}\right)y' - \left(\frac{\partial f}{\partial y}\right)(y')^2\right)\frac{\partial}{\partial y'}$$

Here we are thinking of x, y, and y' as three independent variables. (To understand what is happening geometrically, it is convenient to think in terms of manifolds. A local one parameter group acting on $\mathbb{R}^2$ is a local action of the Lie group $(\mathbb{R}, +)$ on $\mathbb{R}^2$. One can define the local action ϕ_t of $\mathbb{R}$ on any manifold. As with one-parameter groups, such an action corresponds to a vector field X on the manifold. The action ϕ_t induces an action of $\mathbb{R}$ on the 1st jet space of the manifold and X_1 is the corresponding vector field on this jet space. $F(x, y, y') = 0$ defines a submanifold of the jet space and the condition that the action of $\mathbb{R}$ leave this invariant is precisely that $X_1(F(x, y, y')) = 0$. For details and generalizations of this approach, see Olver (1986).)

<u>EXAMPLE IV.3</u>: (Olver (1986), p. 136) Let ϕ_t be the one-parameter group defined by $\phi_t(x, y) = (e^t x, e^t y)$. The associated infinitesimal one-parameter group is $X = x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y}$. Consider a differential equation $y' = F\left(\frac{y}{x}\right) = 0$, that is, a homogeneous equation. One easily checks that $X = X_1$ and $X_1\left(y' - F\left(\frac{y}{x}\right)\right) = 0$. One can also see directly that solutions of a homogeneous equation are mapped to other solutions under the groups of dilations.

We have already mentioned that at a regular point (x_0, y_0), one can choose coordinates $u(x, y), v(x, y)$ so that $X = \frac{\partial}{\partial v}$. In this coordinate system we also have $X_1 = \frac{\partial}{\partial v}$. Assume that $y' = F(x, y)$ is a differential equation such that $X_1(y' - F(x, y)) = 0$ when $y' - F(x, y) = 0$. If we write the differential equation in the new coordinates, say $\frac{dv}{du} = G(u, v)$, then the condition $X_1\left(\frac{dv}{du} - G(u, v)\right) = 0$ when $\frac{dv}{du} = G(u, v)$ implies that $\frac{\partial}{\partial v}G(u, v) = 0$. Therefore, $G(u, v) = H(u)$ is independent of v and $v = \int H\,du + c$. Rewriting this in terms of the original coordinates gives us a solution of the differential equation.

<u>EXAMPLE IV.4</u>: This is a continuation of the previous example. If we let $u = \frac{y}{x}$ and $v = \log x$, then $X = x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y}$ becomes $X = \frac{\partial}{\partial v}$. Assuming that $y = y(x)$ and $v = v(u)$, we have that

$$\frac{dy}{dx} = \frac{1 + u\frac{dv}{du}}{\frac{dv}{du}}$$

so the equation $\frac{dy}{dx} = F\left(\frac{y}{x}\right)$ becomes $\frac{dv}{du} = \frac{1}{F(u) - u}$. This has a solution $v = \int \frac{du}{F(u) - u} + c$.

For example, if

$$\frac{dy}{dx} = \frac{y^2 + 2xy}{x^2} = \left(\frac{y}{x}\right)^2 + 2\frac{y}{x}$$

then $F(u) = u^2 + 2u$. In the coordinates $u = \frac{y}{x}$, $v = \log x$, we have

$$\frac{dv}{du} = \frac{1}{u^2 + u}$$

The solution is $v = -\log\left(1 + \frac{1}{u}\right) + c$, so $y = \frac{x^2}{d - x}$.

This idea can be generalized to higher order equations. Let $F(x, y, \dots, y^{(n)}) = 0$ be an nth order differential equation. The definition of a one-parameter group being a symmetry group of $F(x, y, \dots, y^{(n)}) = 0$ is the same as before. This again can be stated in terms of the associated infinitesimal one-parameter group $f\frac{\partial}{\partial x} + g\frac{\partial}{\partial y}$: ϕ_t is a local one-parameter symmetry group of $F(x, y, \dots, y^{(n)}) = 0$ if and only if

$$X_n F = f\frac{\partial}{\partial x} + \sum_{j=0}^{n} g_j \frac{\partial F}{\partial y^{(j)}} = 0 \text{ when } F(x, y, \dots, y^{(n)}) = 0,$$

$$\text{where } g_0 = g$$

$$\text{and } g_j = \frac{\partial g_{j-1}}{\partial x} + \sum_{k=0}^{j-1} \frac{\partial g_{k-1}}{\partial y^{(y)}} y^{(k+1)} - \left(\frac{\partial f}{\partial x} + \frac{\partial f}{\partial y} y'\right) y^{(j)}$$

When this happens, one chooses local coordinates $u(x, y), v(x, y)$ such that $X = \frac{\partial}{\partial v}$ and writes the equation in the new coordinates as $G(u, v, \dots, v^{(n)}) = 0$ (where $v' = \frac{dv}{du}$). The condition $X_n G = 0$ becomes $\frac{\partial G}{\partial v} = 0$, so the equation really is $G(u, v', \dots, v^{(n)}) = 0$. Letting $w = v'$, we see that finding a solution of $G(u, w, \dots, w^{(n-1)}) = 0$, integrating $w = \int v\, du$ and rewriting in the old coordinate system, solves the original equation. Therefore, the existence of a one-parameter group of symmetries of the equation allows us to reduce the order of the equation.

<u>EXAMPLE IV.5</u>: (Olver (1986), p. 142) Consider the equation $y'' + p(x)y' + q(x)y = 0$. The group $\phi_t(x, y) = (x, e^t y)$ is a one-parameter group of symmetries of this equation. The associated one-parameter infinitesimal group is $X = y\frac{\partial}{\partial y}$. If we let $u = x$ and $v = \log y$, then $X = \frac{\partial}{\partial v}$. Since $y = e^v$, $y' = v'e^v$ and $y'' = (v'' + (v')^2)e^v$, the equation becomes $v'' + (v')^2 + pv' + q = 0$. Letting $w = v'$ we get the usual Riccati equation $w' + w^2 + pw + q = 0$. Solving this and letting $y = e^v = e^{\int w}$ solves the original equation.

I now mention a result related to Section III. Consider a differential equation $y' = \frac{Q(x,y)}{P(x,y)}$ which we write as $Q(x, y)dx - P(x, y)dy = 0$. One can show (Olver (1986), p. 139 or Markus (1960), p. 18) that if this differential equation has a local one parameter symmetry group with associated infinitesimal group $f(x, y)\frac{\partial}{\partial x} + g(x, y)\frac{\partial}{\partial y}$, then

$$R(x, y) = \frac{1}{f(x, y)Q(x, y) - g(x, y)P(x, y)}$$

is an integrating factor, that is $d(RQ\, dx - RP\, dy) = 0$.

<u>EXAMPLE IV.6</u>: Consider again a homogeneous equation $y' = F\left(\frac{y}{x}\right)$ but write this as $F\left(\frac{y}{x}\right) dx - dy = 0$. Since the group associated with $x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y}$ is a symmetry group of this equation, $R = \left(xF\left(\frac{y}{x}\right) - y\right)^{-1}$ is an integrating factor.

This result is the basis of many heuristics (Char (1980)). The main problem with applying the above ideas is that it is difficult, in general, to find an infinitesimal one-parameter group satisfying the appropriate conditions and once such a group is found, finding the change of coordinates to make $X = \frac{\partial}{\partial v}$. This problem is discussed in Olver (1979), (1986) and Char (1980), (1981). One can also find non-trivial applications in these references as well as the works of Miller, Schwarz and Steinberg listed in the references. Other works of interests are Belinfante & Kolman (1979), Beyer (1979), Bluman & Cole (1974), Campbell (1966), Cohen (1911), Fushchich & Kornyala (1989), Kersten (1986), Ovsiannikov (1982), Reiman (1981), Roseman & Schwarzmeier (1979), and Winternitz (1983).

I close this section by mentioning the equivalence problem and the method of Cartan. The equivalence problem is the problem of determining when two systems of ordinary or partial differential equations can be mapped to each other by an appropriate change of coordinates and the method of Cartan is a method to solve this problem. This method was turned into an algorithm by Gardner and applied to a diverse collection of problems (Gardner (1983), (1989), Kamran (1988)). Cartan's equivalence method has been used to determine possible symmetry groups of differential equations in Hsu & Kamran (1988) and Kamran & Olver (1988).

V. Transform Methods

The basic idea behind transform methods is to transform a differential equation into an algebraic equation, solve the algebraic equation and then transform back (occasionally, one just transforms the original equation into a simpler differential equation and then tries to solve the simpler equation). An elementary example is the effect of the Laplace Transform on linear differential equations with constant coefficients. The Laplace Transform of a function f, defined on $[0, \infty)$, is $L(f) = F(z) = \int_0^\infty e^{-zt} f(t)dt$. Using the fact that $L(f^{(n)}) = z^n L(f) - \sum_{k=1}^{n} z^{n-k} f^{(k-1)}(0)$, one can easily transform any system of linear differential equations with constant coefficients into a system of linear (algebraic) equations with polynomial coefficients. One solves this and inverts the transform to get solutions of the original equations. This has been implemented in MACSYMA, see Avgoustis (1977), Clarkson (1989) and Rand (1984). More general transform techniques are discussed in Glinos & Saunders (1984), where implementations of techniques from the operational calculus are discussed.

VI. Asymptotics

A problem here is to find algorithms that will generate formulas such as

$$\int_a^x \frac{dt}{\log t} = \frac{x}{\log x} + \frac{x}{(\log x)^2} + \ldots + (n-1)!\frac{x}{(\log x)^n} + o\left(\frac{x}{(\log x)^n}\right)$$

or other expressions that describe the growth behavior of solutions of linear differential equations. There have been various attempts to give algebraic substance to asymptotic expansions and estimates, that is, make a calculus of asymptotic expressions. Early work includes the considerations of du Bois-Reymond (see the bibliography in Hardy (1910), (1912)). Recently, this area has been given a firm algebraic footing in the works of Boshernitzan, Rosenlicht and van den Dries (see the references). I hope that some of their work can be made effective. Along these lines I propose the following problems

Problem 11. Find an algorithm that solves the following: Given a real elementary function f, find a real elementary function F such that $\int_a^x f \sim F$ (i.e. $\lim\limits_{x \to \infty} \frac{\int_a^x f}{F} = 1$) if such an F exists.

Some work on this problem appears in Bourbaki (1961) and Rosenlicht (1980), and a solution of this would be a first step towards algorithmically generating expressions like (4). For an overview of the many pitfalls associated with attempts to make a calculus of these generalized asymptotic expansions, as well as other useful information on asymptotics, see Olver (1974), especially Ch. 1, Sec. 10 and Olver (1980), especially Sec. 3.

Let P and Q be polynomials in y with coefficients that are real liouvillian functions. All solutions of

$$y' = \frac{P(y)}{Q(y)}$$

that are differentiable in a neighborhood of $+\infty$ are ultimately monotonic (Rosenlicht (1983a)). When P and Q have coefficients in $\mathbb{R}[x]$, Hardy showed that any such solution y satisfies either $y \sim ax^b e^{p(x)}$ or $y \sim ax^b(\log x)^{1/c}$ where b is a real number, $p(x)$ a polynomial and c an integer (Hardy (1910), Bellman (1969)).

Problem 12. Find an analogue of Hardy's result in the general case of P and Q having real liouvillian functions as coefficients.

Formal methods involving asymptotics have been very useful in perturbation theory. Here we are given a differential equation that depends on a parameter ϵ and we wish to find series in ϵ that represent quantities associated with this equation (e.g. solutions, limit cycles, Poincaré maps). This usually is done by substituting the power series in ϵ into an equation, equating powers of ϵ, deriving new equations for the coefficients and solving these new equations. Computer algebra systems such as MACSYMA have been successfully used in this problem. There is an enormous literature on this subject and the reader is referred to Rand (1984) and Rand & Armbruster (1987) for details and a large bibliography.

VII. Difference Equations

The general problem here is: Consider the questions raised in I–VI above in the context of difference equations. Aside from heuristics (Cohen & Katcoff (1977), Hayden & Lamagna (1986), Ivie (1977) and Moenck (1977)), there are a few recent algorithmic results. In 1977, Gosper (Gosper (1977), (1978)) gave an algorithm which gives a closed form expression for $S(n) = \sum_{x=1}^{n} f(x)$ when $S(n)/S(n-1)$ is a rational function. This algorithm has been successfully used to generate and generalize some very interesting formulas. Problems of this kind can be given a formal setting using difference fields. A difference field is a field F with an automorphism σ (Cohn (1966)). If $F = \mathbb{C}(x)$ the automorphism one usually has in mind is $\sigma(f(x)) = f(x+1)$. We can define the usual difference operator by $\Delta f = \sigma(f) - f$. The problem of finding a closed form expression for $S(n) = \sum_{x=1}^{n} f(x)$, then becomes: Given a difference field F and $f \in F$, compute, if it exists, an element g in a suitable extension of F such that $\Delta g = f$. Karr has investigated this problem in Karr (1981), (1985). He rigorously defines what is meant by "summation in finite terms" in terms of towers of difference fields. These towers are called $\prod \Sigma$ fields and are the analogue of elementary extensions in the theory of integration in finite terms. Karr shows how to solve an arbitrary first order linear difference equation in a given $\prod \Sigma$ field and how to make a judicious choice of such an extension. He also gives a liouville type theorem for summation in finite terms. An exposition of some aspects of Gosper's and Karr's work can be found in Lafon (1982).

Problem 13. Generalize Karr's work to nth order linear difference equations.

Recently, Zeilberger (Zeilberger (1989)) uses a setting similar to that in his work on integrals to give an algorithm for evaluating sums of the form $a(n) = \sum_{k=1}^{n} F(n,k)$ where $F(n+1,k)/F(n,k)$ and $F(n,k+1)/F(n,k)$ are rational functions of n and k.

Della Dora, Tournier and Wazner have considered the problem of finding power series solutions of linear difference equations $L(y) = \sum_{i=1}^{n} a_i \delta^i y = 0$, where $a_i \in \mathbb{C}(x)$ and $\delta(f(x)) = f(x-1)$. In Della Dora & Tournier (1984) they look for solutions of the form $y(x) = \mu^x \left(\sum_{j=0}^{\infty} a_j(x)_{\nu+j} \right)$, where $(x)_\lambda = \Gamma(x+1)/\Gamma(x-\lambda+1)$. They pursue the method of Boole, a method similar to the Frobenius method for solving linear differential equations. This method only works under certain regularity condition imposed on the coefficients of $L(y)$. In Della Dora & Wazner (1985), they pursue a Newton polygon method that handles a more general case. In Barkatou (1989), Barkatou considers systems of linear difference equations and gives an algorithm (along the lines of Moser's algorithm for differential equations) to reduce such a system and decide if it has a regular singularity.

Another approach to difference equations is discussed in Della Dora & Tournier (1986) and Tournier (1987) based on ideas of Pincherle and recent improvements of J. P. Ramis and A. Duval. The idea is to use the transform $P[\varphi] = \int_\gamma t^{-x-1}\varphi(t)dt$, where γ is a suitably chosen path, to transform the difference equation into a linear differential equation, use the techniques developed to understand the solutions of this new differential equation, and then transform back. The original motivation for Della Dora *et al*'s interest in difference equations was to understand the growth properties of the coefficients appearing in the formal expansions of solutions of a linear differential equation at irregular points. These coefficients satisfy difference equations. The Pincherle-Ramis method converts this problem back to a more tractible problem again involving linear differential equations, gives a remarkable and very pretty circle of ideas.

In Maeda (1987), Maeda discusses Lie method for difference equations.

Final Comments

In the previous sections, I have mentioned how techniques for finding formal solutions have been implemented in computer algebra systems. Besides solving differential equations, computer algebra can be used to generate differential equations and manipulate differential equations (of course, generating, manipulating and solving are not mutually exclusive). In Wang (1986) and Tan (1989), the authors describe the symbolic software FINGER that automatically generates the element equations for the finite element method (see also Roache & Steinberg (1985), (1988)). Another example of using symbol manipulation packages to generate equations is in Hirschberg & Schramm (1989), where the authors describe a package that generates the equations of motion of certain robot systems given the masses, moments of inertia, position of mass centers and connection joint locations. A good example of using a computer algebra systems to manipulate differential equations can be found in Davenport, Siret & Tournier (1988), p. 29, where the authors show how to use MACSYMA to obtain successive derivatives of y with respect to x, starting from $g(x,y) = 0$. They get expressions containing partial derivatives of g and are then able to specialize this to a particular g. Other examples can be found in Rand (1984) and Rand & Armbruster (1987). Another example of manipulation is given in Grossman & Larson (1989), where the authors give an efficient algorithm for evaluating higher order differential operators (such as $E_3E_2E_1 - E_3E_1E_2 - E_2E_1E_3 + E_1E_2E_3$, where $E_i = \Sigma a_i^j \frac{\partial}{\partial x_i}$).

All the problems discussed here have their roots in the 19th century and many of them have effective solutions that were outlined at that time. With the rise of symbolic computation systems, these solutions take on a new relevance. I have included the following textbooks and guides to the old literature in the references:

Bieberbach (1935), Gray (1986), Hilb (1915a), (1915b), Hille (1976), Ince (1944), Kamke (1971), Poole (1960), Schlesinger (1895), (1909), Vessiot (1910), Zwillinger (1989).

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