The Inverse Problem in Differential Galois Theory*

Claude Mitschi
Institut de Recherche Mathématique Avancée
Université Louis Pasteur et C.N.R.S.
7 rue René Descartes
67084 Strasbourg Cedex, France.
E-mail: mitschi@math.u-strasbg.fr

and

Michael F. Singer
Department of Mathematics
North Carolina State University
Raleigh, NC 27695-8205, USA
E-mail: singer@math.ncsu.edu

1. Galois Theory

The usual Galois theory of polynomial equations allows one to associate a group to a polynomial in such a way that the algebraic properties of the roots of the polynomial are reflected in properties of the group. More specifically, given a field k and a polynomial $p(x)$ with coefficients in the field, one forms the *splitting field* K of $p(x)$ by formally adjoining all the roots of $p(x)$ to k. The *Galois group* G is defined to be the group of all automorphisms of K that leave each element of k fixed.

A similar construction can be made for linear differential equations. One starts with a *differential field* k. This is a field[1] together with a map $' : k \to k$ (called a *derivation*) satisfying the two properties: $(a+b)' = a' + b'$ and $(ab)' = a'b + ab'$ for all $a, b \in k$. Examples of differential fields are the field of complex numbers $\mathbf{C}$, the field of rational functions $\mathbf{C}(x)$, the quotient field $\mathbf{C}((x))$ of the ring of formal power series and the quotient field $\mathbf{C}(\{x\})$ of the ring of convergent power series. The subfield $C_k = \{c \in k \mid c' = 0\}$ is called the *subfield of constants of* k (when the context is clear we shall drop

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[1]All fields in this paper will be assumed to be of characteristic zero.

the subscript k and refer to C). A linear differential equation is an equation

$$Y' = A\ Y$$

where $A \in M_n(k)$, the set of $n \times n$ matrices with coefficients in k. The differential version of the notion of a splitting field of a polynomial is the *Picard-Vessiot extension* associated with $Y' = AY$. This is defined to be a differential extension $K = k(z_{11}, \ldots, z_{nn})$ where

- the z_{ij} are the entries of a matrix Z satisfying $\det(Z) \neq 0$ and $Z' = AZ$
- the constants of K coincide with the constants of k, *i.e.*, $C_K = C_k$.

We refer to Z as a *fundamental solution matrix* and shall also denote K by $k(Z)$. The second condition above is a technical (but crucial) condition that is used to show that there is a large supply of automorphisms of such fields. If $k = \mathbf{C}(x)$, $x' = 1$, then one can easily show the existence of Picard-Vessiot extensions. To do this, let α be a nonsingular point of $Y' = AY$, that is, a point on the complex plane where the denominators of the entries of A do not vanish. The usual existence theorem then guarantees that, in a neighborhood of α there exist n linearly independent analytic solutions $\mathbf{z_1} = (z_{11}, \ldots, z_{n1})^T, \ldots, \mathbf{z_n} = (z_{n1}, \ldots, z_{nn})^T$. We then have that $k(Z)$ is a Picard-Vessiot extension, where $Z = (z_{ij})$. In general, one can show that if C_k is algebraically closed, there exists a Picard-Vessiot extension of k corresponding to $Y' = AY$ and that this extension is unique up to k-differential isomorphism (*cf.* [7],[10]).

Let K be the Picard-Vessiot extension associated with $Y' = AY$ and let $K = k(Z)$ where Z is a fundamental solution matrix. If $\mathbf{w} \in K^n$ satisfies $\mathbf{w}' = A\mathbf{w}$, then one can show (in the same way as in the classical case) that $\mathbf{w} = \sum c_i \mathbf{z_i}$ where the $\mathbf{z_i}$ are the columns of Z and the c_i are constants. We define the *Galois group* $\mathrm{Gal}(K/k)$ of K over k (or of $Y' = AY$) to be $\{\sigma : K \to K \mid \sigma$ is a field isomorphism $; \sigma(u') = (\sigma(u))'$ for all $u \in K; \sigma \mid_k =$ identity$\}$. If $K = k(Z)$ and $\mathbf{z_j}$ is a column of Z, then the above remarks imply that for $\sigma \in \mathrm{Gal}(K/k)$, $\sigma(\mathbf{z_i}) = \sum_j c_{ij} \mathbf{z_j}$ with the $c_{ij} \in C$. This yields a representation $\mathrm{Gal}(K/k) \to \mathrm{GL}_n(C)$ given by $\sigma \mapsto (c_{ij}) = [\sigma]_Z$. We then have that $\sigma(Z) = Z \cdot [\sigma]_Z$. Note that if we take a different basis of the solution space of $Y' = AY$, then we will get a conjugate representation.

There is a well developed Galois theory (*cf.* [5], [7], [10]). Three important facts of this theory are:

- $\mathrm{Gal}(K/k)$ has the structure of a linear algebraic group, i.e., the image $\mathrm{Gal}(K/k)$ in $\mathrm{GL}_n(C)$ is Zariski closed. The algebraic group structure can be established independent of this embedding (*cf.* [2, 3], [7], [10]).

- There is an inclusion reversing correspondence between Zariski closed subgroups of $\mathrm{Gal}(K/k)$ and differential subfields of K containing k. A Zariski closed subgroup H is associated to the field of elements that are left fixed by H. A differential subfield E is associated with the subgroup of elements of $\mathrm{Gal}(K/k)$ that leave each element of E fixed. Furthermore, Zariski closed *normal* subgroups N correspond to subfields E that are Picard-Vessiot extensions of k and for these, $\mathrm{Gal}(E/k) \simeq \mathrm{Gal}(K/k)/N$.

- One can characterize solvability in terms of certain classes of special functions via properties of the Galois group (*cf.* [5], [7], [14],[15],[16]). For example, the solutions of $Y' = AY$ can be expressed via composition in terms of $\exp, \int$, and algebraic functions if and only if the connected component of the identity in $\mathrm{Gal}(K/k)$ is solvable.

Example 1: Consider the equation $y' = \frac{\alpha}{x}y$ where $\alpha \in \mathbf{C}$ and $k = \mathbf{C}(x)$. The associated Picard-Vessiot extension is $K = k(x^\alpha)$. We therefore have that $\mathrm{Gal}(K/k)$ is a Zariski closed subgroup of $\mathrm{GL}_1(\mathbf{C}) = \mathbf{C}^*$. In fact, one can show that

$$\mathrm{Gal}(K/k) = \begin{cases} \mathbf{C}^* & \text{if } \alpha \notin \mathbf{Q} \\ \mathbf{Z}/q\mathbf{Z} & \text{if } \alpha = \frac{p}{q}, \quad p \wedge q = 1 \end{cases}$$

Note that the Galois group does not just depend *algebraically* on the coefficients of the differential equation but on the arithmetic properties of these coefficients as well (*cf.* [17]).

Example 2: Consider the equation

$$Y' = \begin{pmatrix} 0 & \frac{1}{x} \\ 0 & 0 \end{pmatrix} Y$$

over $k = \mathbf{C}(x)$. This has a fundamental solution matrix of the form

$$Z = \begin{pmatrix} 1 & \log(x) \\ 0 & 1 \end{pmatrix}$$

and so the Picard-Vessiot extension is of the form $K = k(\log(x))$. If $\sigma \in \mathrm{Gal}(K/k)$ then $(\sigma(\log(x)) - \log(x))' = \sigma(\frac{1}{x}) - \frac{1}{x} = 0$ so $\sigma(\log(x)) = \log(x) + c_\sigma$ for some $c_\sigma \in \mathbf{C}$. One can further show that for any $c \in \mathbf{C}$, $\sigma(\log(x)) = \log(x) + c$ defines an element of the Galois group. Therefore the Galois group is of the form

$$\left\{ \begin{pmatrix} 1 & c \\ 0 & 1 \end{pmatrix} \mid c \in \mathbf{C} \right\} \simeq (\mathbf{C}, +)$$

Note that Z is a K-point of the Galois group and that the matrix $Z'Z^{-1} \in \mathrm{Lie}_k(\mathrm{Gal}(K/k))$, where $\mathrm{Lie}_k(\mathrm{Gal}(K/k)) = \mathrm{Lie}(\mathrm{Gal}(K/k)) \otimes_{\mathbf{C}} k$ and $\mathrm{Lie}(\mathrm{Gal}(K/k))$ is the Lie algebra of $\mathrm{Gal}(K/k)$ (that is the set of 2×2 matrices with zero on and below the diagonal). The following example shows that this is not true in general but, as we shall see, it is *almost* true.

Example 3: Consider the equation

$$Y' = \begin{pmatrix} \frac{1}{x} & 1 \\ 0 & 0 \end{pmatrix} Y$$

over $k = \mathbf{C}(x)$. One has a fundamental solution matrix of the form

$$Z = \begin{pmatrix} x & x\log(x) \\ 0 & 1 \end{pmatrix}$$

and so the Picard-Vessiot extension is again of the form $K = k(\log(x))$. We have the same Galois group as in Example 2 yet Z is not a point of the Galois group. If we let

$$U = \begin{pmatrix} \frac{1}{x} & 0 \\ 0 & 1 \end{pmatrix}$$

then

$$\overline{Z} = U \cdot Z = \begin{pmatrix} 1 & \log(x) \\ 0 & 1 \end{pmatrix} \in \mathrm{Gal}(K/k)$$

Furthermore, denoting by A the matrix satisfying $Z' = AZ$, we have

$$\begin{aligned} \overline{Z}' &= U'Z + UZ' \\ &= (U'U^{-1} + UAU^{-1})\overline{Z} \end{aligned}$$

Since $\overline{Z}'\overline{Z}^{-1} \in \mathrm{Lie}_k(\mathrm{Gal}(K/k))$, we have $U'U^{-1} + UAU^{-1} \in \mathrm{Lie}_k(\mathrm{Gal}(K/k))$.

This last example illustrates the following fact:

> Let $K = C$ or $C(x)$ or $C((x))$ or $C(\{x\})$, C algebraically closed and $Y' = AY$ with $A \in M_n(k)$. If the Picard-Vessiot extension $K = k(Z)$ associated to this equation has a *connected* Galois group, then there exists $U \in \mathrm{GL}_n(k)$ such that $UZ \in \mathrm{Gal}(K/k)$ and $U'U^{-1} + UAU^{-1} \in \mathrm{Lie}_k(\mathrm{Gal}(K/k))$.

This fact is a manifestation of a more general truth. Recall that a k-principal homogeneous space for G is a variety V defined over k and an action of G on V (also defined over k) which is transitive and free. For example, G is a principal homogeneous space over itself where the action is

defined by right multiplication. Kolchin [7] was the first to show that if K is a Picard-Vessiot extension of k with Galois group G, then there exists a k-principal homogeneous space V such that K is k-isomorphic to $k(V)$, the function field of the group over k. This isomorphism is furthermore $G(C)$-equivariant, where the action of $G(C)$ (the C-points of G) on K is the Galois action and the action on $k(V)$ is induced by the action of G on V. The fields mentioned above are all cohomologically trivial and it is known [18] that in this case a k-principal homogeneous space for a *connected* group G must contain a k-point and so be isomorphic to G. Therefore, for these fields and connected groups G, we have that a Picard-Vessiot extension $k(Z)$ is of the form $k(\Gamma)$, where Γ is a generic point of G over k. To verify the displayed fact, one shows that $U = \Gamma Z^{-1}$ is left invariant by the group and so must lie in k. We refer to [8] for a proof of the statement concerning the Lie algebra of G.

There is one more important fact that we will need from [8].

> If $G \subset GL_n$ is a linear algebraic group defined over $C \subset k$ and $A \in \mathrm{Lie}_k(G) \subset M_n(k)$, then there exists an element $g \in G$ such that $k(g)$ is a Picard-Vessiot extension of k with $g' = Ag$ and with Galois group of a subgroup of G.

In geometric terms, differential equations can be thought of as connections on GL_n-bundles. In these terms, the two facts mentioned above say that, for equations with connected Galois group G, we can reduce the group of the bundle to G, but to no smaller subgroup.

2. The Inverse Problem

The inverse problem we consider is:

> Given a differential field k, what groups are Galois groups of Picard-Vessiot extensions of k?

To start, let us consider the following.

Example 4: Let $k = \mathbf{C}$ and let $Y' = AY$ be a differential equation with $A \in \mathrm{GL}_n(\mathbf{C})$. Classically, it is known that by reducing the matrix A to rational normal form one can reduce the problem of finding solutions of this latter equation to finding solutions of scalar linear differential equations $L(y) = 0$ of order possibly greater than one. All solutions of a linear differential equation with constant coefficients are of the form $y = \sum p_i(x)e^{\alpha_i x}$ where the $p_i \in \mathbf{C}[x]$ and the $\alpha_i \in \mathbf{C}$. Therefore the Picard-Vessiot extension $K = \mathbf{C}(Z)$

associated to $Y' = AY$ lies in a field of the form $F = \mathbf{C}(x, e^{\alpha_1 x}, \ldots, e^{\alpha_t x})$. One can show that F is a Picard-Vessiot extension of $\mathbf{C}$ with Galois group $G = \mathbf{C} \times \mathbf{C}^* \times \ldots \times \mathbf{C}^*$. Therefore the Galois group of K over $\mathbf{C}$ is a quotient of such a group. Any quotient of such a group can be shown to be of the form $\mathbf{C}^s \times \mathbf{C}^{*t}$ where s and t are nonnegative integers and $0 \leq s \leq 1$. Conversely, one can adjoin appropriate elements to $\mathbf{C}$ and show that any such group is a Galois group. Group theoretically, one can characterize such groups as linear algebraic groups that are (i) connected, (ii) abelian, and (iii) whose largest unipotent subgroup has dimension at most one (a linear algebraic group is unipotent if all of its elements are unipotent).

We now turn to fields of finite nonzero transcendence degree over C. Building on the work of Kovacic [8, 9], we showed the following theorem [11].

Theorem 1. *Let C be an algebraically closed field of characteristic zero, G a connected linear algebraic group defined over C and k a differential field containing C as its field of constants and of finite, nonzero transcendence degree over C, then G can be realized as the Galois group of a Picard-Vessiot extension of k.*

Using an argument of Kovacic, one can reduce the above theorem to the case when $k = C(x)$, $x' = 1$. In this case one can say much more. Let G be a connected linear algebraic group and let R_u be its unipotent radical (the largest normal unipotent subgroup). It is known that there exists a reductive group P, called a *Levi factor* [12], and a decomposition $G \simeq R_u \rtimes P$ as a semidirect product. The group $R_u/(R_u, R_u)$, where (R_u, R_u) is the (closed) subgroup generated by commutators $ghg^{-1}h^{-1}$, $g, h \in R_u$ is a commutative unipotent group and so isomorphic to a vector group C^n. The group P acts on R_u by conjugation and this action factors to an action on $R_u/(R_u, R_u)$. Therefore we may write $R_u/(R_u, R_u) = U_1^{n_1} \oplus \ldots \oplus U_s^{n_s}$, where each U_i is an irreducible P-module. We shall assume that U_1 is the trivial one dimensional P-module (and so allow the possibility that $n_1 = 0$). We may write $P = T \cdot H$, where T is a torus and H is a semisimple group. Let $m_i = n_i$ if the action of H on U_i is trivial and let $m_i = n_i + 1$ if the action of H on U_i is nontrivial. Let $N = 0$ if H is trivial and $N = 1$ if H is nontrivial. We define the *defect* $d(G)$ of G to be the number n_1 and the *excess* $e(G)$ of G to be $\max\{N, m_2, \ldots, m_s\}$. For example, if G is reductive then $d(G) = 0$ and $e(G) = 0$ or 1 depending on whether G is a torus or not. We note that any two Levi factors are conjugate so that these numbers are independent of the choice of P. Furthermore , one can show that $d(G)$ is the dimension of $R_u/(G, R_u)$.

Theorem 2: *Let G be a connected linear algebraic group defined over an algebraically closed field C of characteristic zero. Then G is the Galois group of a Picard-Vessiot extension of $C(x)$ corresponding to a system of the form*

$$Y' = (\frac{A_1}{x-\alpha_1} + \ldots + \frac{A_{d(G)}}{x-\alpha_{d(G)}} + A_\infty)Y$$

where A_i, $i = 1, \ldots, d(G)$ are constant matrices and A_∞ is a matrix with polynomial entries of degree at most $e(G)$. In particular, the only possible singularities of this system are $d(G)$ regular singular points in the finite plane and a (possibly irregular) singular point at infinity.

For example, this theorem implies that if a connected group G is reductive, then G can be realized by an equation of the form $Y' = (A + Bx)Y$ where A and B are constant matrices.

One can show that the bound on the number of singular points is sharp. In fact we have:

Lemma 3: *If $G = C^r$, then $d(G) = r$ and G cannot be realized by a differential equation with fewer than $r + 1$ singular points.*

Proof: Assume not and let $k = C(x)$ and K be a Picard-Vessiot extension corresponding to the system. We know that $K = k(G)$ so we can write $k(G) = k(t_1, \ldots, t_r)$ where the action of the Galois group is given by $\sigma(t_i) = t_i + c_{ig}$ for $\sigma \in G$ and $c_{ig} \in C$. This implies that each t_i' is left invariant by the Galois group G and so must be in k. Therefore each t_i is of the form $t_i = \sum_j c_{ij} \log(f_j) + g_i$. where $f_j, g_i \in k$. The assumption on the singularities implies that any element in $k(G)$ has singularities (except for poles) in some fixed set $\{p_1, \ldots, p_{r-1}, p_r\}$. Making a change of variables we can assume that $p_r = \infty$. Therefore we may write $\log(f_j) = \sum_{i=1}^{r-1} n_{ij} \log(x - p_i)$. This gives us r equations in the $r - 1$ logarithms and so the t_i will be algebraically dependent over $C(x)$, a contradiction. ∎

We do not know the full extent to which the bound on the degrees of the entries of A_∞ is sharp.

To prove Theorem 2, we follow the program laid out by Kovacic in [8, 9]. Kovacic reduced the inverse problem to the case of powers of a simple connected linear algebraic group. Our contribution is to show that one is able to realize any connected semisimple group as a Galois group and, when $k = C(x)$, $x' = 1$ to control the number and types of singularities when

one constructs a system $Y' = AY$, $A \in M_n(C(x))$ realizing an arbitrary connected linear algebraic group as its Galois group.

We mention here some other results concerning the inverse problem in differential Galois theory. Tretkoff and Tretkoff [19] have shown that any linear algebraic group is a Galois group over $\mathbf{C}(x)$. Their result depends on the solution of a weak version of the 21^{st} Hilbert Problem, that is, the fact that given a finitely generated group H of matrices, there exists a homomorphism of the fundamental group of the sphere minus a finite set of points onto this group and that this representation is the monodromy representation of an equation with regular singular points. For arbitrary C, Singer [15] showed that a class of linear algebraic groups (including all connected groups and large classes of nonconnected linear algebraic groups) are Galois groups over $C(x)$. Singer's proof used the result of Tretkoff and Tretkoff and a transfer principle to go from $\mathbf{C}$ to any algebraically closed field of characteristic zero. In both of these proofs the number of (nonapparent) singularities is bounded by one more than the number of topological generators of the group. Ramis [13] has also shown that any linear algebraic group defined over $\mathbf{C}$ is the Galois group of a Picard-Vessiot extension of $\mathbf{C}(x)$. Ramis's proof relies on his solution of the inverse problem for the local field $\mathbf{C}(\{x\})$ and a technique for glueing solutions of the local problem to form a solution of the global problem. Ramis is also able to give sharp bounds on the number and type of singularities.

We now give a brief taste of the ideas involved in our proof of Theorem 2. We shall only consider examples of tori and connected semisimple goups. The plan is the following:

Let $k = C(x)$ as before and let G be a connected linear algebraic group defined over C. If K is a Picard-Vessiot extension of k whose Galois group is G, then we have already remarked that $K = k(\Gamma)$ for some k-generic point $\Gamma \in G$ and that $\Gamma'\Gamma^{-1} = A \in \mathrm{Lie}_k(G)$. We have also remarked that for any $A \in \mathrm{Lie}_k(G)$ there exists a $g \in G$ such that $k(g)$ is a Picard-Vessiot extension of k with Galois group a subgroup of G. Therefore we will try to find an $A \in \mathrm{Lie}_k(G)$ such that the Galois group is the *full* group G.

To do this we will first select an element $A \in \mathrm{Lie}_k(G)$ such that the Galois group is connected. We will use the following result of Gabber ([6], Proposition 1.2.5). Let G_{mono} be the monodromy group of a differential equation $Y' = AY$, let G be the Galois group and let G^0 be the connected component of the identity of G. There is a natural inclusion $G_{mono} \to G$. Gabber's Lemma says that the induced map $G_{mono} \to G/G^0$ is surjective. Therefore if the monodromy group is trivial, the Galois group will be connected. If we

select A with entries in $C[x]$, then the equation $Y' = AY$ has only one singular point (at infinity) and so has trivial monodromy group and connected Galois group.

We now must insure that for such a choice of A, the Galois group is not a proper connected subgroup H of G. Let us assume that for some $\gamma \in G$ with $\gamma' = A\gamma$, the Galois group of $k(\gamma)$ over k is a connected subgroup H of G. There will then exist an element $U \in \mathrm{GL}_n(k)$ such that $U\gamma \in H$ (so $U \in G$) and $U'U^{-1} + UAU^{-1} \in \mathrm{Lie}_k(H)$. Therefore to realize the group G as a Galois group, we need to

> find an element $A \in \mathrm{Lie}(G) \otimes_C C[x]$ such that there is no $U \in G(C(x))$ and no connected subgroup H of G with $U'U^{-1}+UAU^{-1} \in \mathrm{Lie}_k(H)$.

Example 5: Consider the torus $G = C^* \times C^* =$

$$\left\{ \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix} \mid ab \neq 0 \right\}$$

For this group we have that $\mathrm{Lie}(G) \otimes_C C[x]$

$$\left\{ \begin{pmatrix} f_1 & 0 \\ 0 & f_2 \end{pmatrix} \mid f_1, f_2 \in C[x] \right\}$$

The proper closed subgroups of G are of the form

$$G_{m,n} = [\left\{ \begin{pmatrix} c & 0 \\ 0 & d \end{pmatrix} \mid c^m d^n = 1 \right\}$$

where m and n are integers, not both zero [4]. The connected subgroups are those $G_{m,n}$ where m and n are relatively prime. For these, the Lie algebras $\mathrm{Lie}_k(G_{m,n})$ are

$$\mathrm{Lie}_k(G_{m,n}) = \left\{ \begin{pmatrix} g_1 & 0 \\ 0 & g_2 \end{pmatrix} \mid mg_1 + ng_2 = 0 \right\}$$

Therefore to represent G as a Galois group, we must find an element

$$A = \begin{pmatrix} f_1 & 0 \\ 0 & f_2 \end{pmatrix}$$

with $f_1, f_2 \in C[x]$ such that there does not exist an invertible matrix

$$U = \begin{pmatrix} u & 0 \\ 0 & v \end{pmatrix}$$

$u, v \in C(x)$ and integers m, n with $U'U^{-1} + UAU^{-1} \in \mathrm{Lie}(G_{m,n}) \otimes_C C(x)$. That is we want to make sure that if u and v are rational functions and m and n are integers such that

$$m(\frac{u'}{u} + \frac{f_1'}{f_1}) + n(\frac{v'}{v} + \frac{f_2'}{f_2}) = 0$$

then $m = n = 0$. It is sufficient to find f_1, f_2 such that if

$$mf_1 + nf_2 = \frac{h'}{h}$$

for some $h \in C(x)$ then $m = n = 0$. For example, $f_1 = 1, f_2 = \sqrt{2}$ will satisfy this requirement so

$$Y' = \begin{pmatrix} 1 & 0 \\ 0 & \sqrt{2} \end{pmatrix} Y$$

has Galois group G. This clearly generalizes to any torus.

Example 6: Consider the groups $G = \mathrm{SL}_2$. Its Lie algebra is the set of matrices of trace zero. Let

$$A = A_0 + xA_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} + x \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

As noted before, $Y' = AY$ has connected Galois group. Any proper Lie subalgebra of $\mathrm{Lie}(\mathrm{SL}_2)$ has dimension at most 2 and so must be solvable. Lie's theorem implies that any such algebra leaves a one dimensional subgroup invariant. A calculation (*cf.* [11]) shows that for any $U \in \mathrm{SL}_2(C(x))$ the matrix $U'U^{-1} + UAU^{-1}$ leaves no one dimensional subspace of $(C(x))^2$ invariant. Therefore the Galois group must be SL_2.

This example generalizes. Note that A_0 is the sum of generators of the root spaces of the Lie algebra and the element A_1 is a regular element of the Cartan subalgebra and these two elements generate SL_2. In general ([1], Ch. 8, Sec. 2, Ex. 8, p. 221), for any semisimple Lie algebra $\mathcal{L}$, letting A_0 be the sum of generators of the root spaces and A_1 an appropriately selected regular element of a Cartan subalgebra, we have that A_0 and A_1 generate the Lie algebra. One can furthermore always find a representation of $\mathcal{L}$ so that any proper algebraic subalgebra has an invariant line while $\mathcal{L}$ itself does not. The calculation used in the case of SL_2 also generalizes to arbitrary semisimple $\mathcal{L}$. The reader is refered to [11] for the details of this as well as a complete description of the proof of Theorem 2.

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