Asymptotic Behavior of Solutions of Differential Equations & Hardy Fields: Preliminary Report

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The general aim of these investigations is to describe the asymptotic behavior of solutions of differential equations. In this note we shall consider an approach originated by G. H. Hardy ([5]). Briefly, this technique involves two steps. First, one shows that any solution of the differential equation under consideration has the property that any rational function $\frac{P(y,y',\dots)}{Q(y,y',\dots)}$ in y and its derivatives is eventually monotonic. This, then allows one to compare such expressions and write down asymptotic expressions for y.

§1 HARDY FIELDS:

We consider real valued ~~fcns~~ functions defined on some semi-infinite interval on the positive real line. We say such a function f is ultimately positive (ultimately differentiable, ultimately monotonic, etc.) if $\exists\, a>0$ s.t. $f(t)$ is positive (differentiable, monotonic, etc.) for all $t>a$. We can define an equivalence relation on functions as follows. We say $f_1 \simeq f_2$ if

$f_1 - f_2$ is ultimately zero and we let $\bar{f}_1$ denote the equivalence class of f_1. If $f_1 \approx f_2$ and $g_1 \approx g_2$ then $f_1 + g_1 \approx f_2 + g_2$, so we can define addition on equivalence classes as $\bar{f}_1 + \bar{g}_1 = \overline{f_1 + g_1}$. Similarly, we can define $\bar{f}_1 \cdot \bar{g}_1 = \overline{f_1 \cdot g_1}$. In this way the set of equivalence classes forms a ring, which we will denote by $\mathcal{H}$. If f is ultimately differentiable, we will define $D\bar{f} = \bar{f'}$. A subring $\mathcal{F}$ of $\mathcal{H}$ is said to be a <u>HARDY FIELD</u> if (i) $\mathcal{F}$ is a field and (ii) if $\bar{f} \in \mathcal{F}$, then f is ultimately differentiable and $D\bar{f} \in \mathcal{F}$. For example, $\mathbb{R}(\bar{x})$ where $x' = 1$ is a Hardy field. If $\mathcal{F}$ is a Hardy field and $\bar{f} \in \mathcal{F}$, $\bar{f} \neq 0$ then f is ultimately positive or ultimately negative. To see this, note that since $\mathcal{F}$ is a field, $\exists$ a func g s.t. $f \cdot g$ is ultimately equal to 1. Therefore f is ultimately non-zero. Since f is assumed to be ultimately differentiable it is therefore ultimately continuous and either ultimately positive or ultimately negative. Using this fact, we can order $\mathcal{F}$ by saying $\bar{f} > 0$ iff f is ultimately positive. In this way we can consider any Hardy field $\mathcal{F}$ as an ordered field. As an abstract ordered field, we can form its real closure $\hat{\mathcal{F}}$ and ask if $\hat{\mathcal{F}}$ can be realized as a Hardy field (Recall, that the real closure of an ordered field $\mathcal{F}$ is an ordered field $\hat{\mathcal{F}}$ algebraic over $\mathcal{F}$ which has no proper ordered algebraic extensions, ~~for~~ see [10]). The following result of A. Robinson says that this is the case.

Thm: Suppose that $\tau : \mathcal{F} \to \hat{\mathcal{F}}$ injects a Hardy field into its real closure $\hat{\mathcal{F}}$. Then there exist a Hardy field $\mathcal{G}$ which is an extension of $\mathcal{F}$ such that $\mathcal{G}$ is isomorphic to $\hat{\mathcal{F}}$ by an isomorphism $\lambda : \mathcal{G} \to \hat{\mathcal{F}}$ which extends τ.

We just mention here that ~~if~~ f lying in a Hardy field says quite a bit about f. For instance, f must be ultimately monotonic, since Df is also in a Hardy field so f' is either ultimately zero or ultimately non-zero. The fact that a Hardy field is a nice place to live was exploited by several authors ([4], [6], [3]), to get results about rates of growth of functions. In §2 we give some criteria for a function to lie in a Hardy field and in §3 we give an example of what being in a Hardy field can give you.

§2 THE DIFFERENTIAL EQUATION $y' = \frac{P(y)}{Q(y)}$.

Prop: Let $\mathcal{F}$ be a Hardy field and $P(u) = \sum_i^n a_i u^i$, $Q(u) = \sum_j^n b_j u^j$, and $R(u) = \sum_{k=1}^n c_k u^k$ be polynomials with all $\bar{a}_i, \bar{b}_j, \bar{c}_k$ in $\mathcal{F}$. Let y be an ultimately continuously differentiable solution of $y' = \frac{P(y)}{Q(y)}$. Then $R(y)$ is ~~either~~ ultimately continuous and either ultimately zero or ultimately non-zero.

Pf: Since $\bar{c}_k \in \mathcal{F}$ we know that $R(y)$ is ultimately continuous. By Theorem 1, we can extend $\mathcal{F}$ to its real closure, which is again a Hardy field. We can therefore assume $\mathcal{F}$ is real closed. We will now show that $R(y)$ is ultimately zero or ultimately non-zero. Assume that $R(y)$ is ~~either~~ not ultimately non-zero, i.e. that there are real numbers $t_i \to \infty$ s.t. $R(y)(t_i) = \sum c_i(t_i)(y(t_i))^i = 0$. We will show that $R(y)$ is ultimately zero. Since we are assuming $\mathcal{F}$ to be real closed, the polynomial $\sum \bar{c}_i u^i$ factors into a product of irreducible quadratic polynomials $Q_i(u)$ and linear polynomials $L_i(u)$ over $\mathcal{F}$. I claim that for one of the linear factors $L_j(u) = \bar{a}u + \bar{b}$, we can find a subsequence $\{s_i\}$ of $\{t_i\}$ s.t. $a(s_i)y(s_i) + b(s_i) = 0$. If not, then for one of the quadratic factors $\bar{c}u^2 + \bar{d}u + \bar{e}$, we could find a subsequence $\{s_i\}$ of $\{t_i\}$ s.t. $c(s_i)(y(s_i))^2 + d(s_i)y(s_i) + e(s_i) = 0$. This implies that $(d(s_i))^2 - 4c(s_i)e(s_i) \geq 0$. Since $\bar{d}^2 - 4\bar{c}\bar{e} \in \mathcal{F}$, we must have that $\bar{d}^2 - 4\bar{c}\bar{e} \geq$. This would mean that the quadratic $\bar{c}u^2 + \bar{d}u + \bar{e}$ is not irreducible. This contradiction verifies the claim. Therefore, for $\frac{-\bar{b}}{\bar{a}} \in \mathcal{F}$ we have a sequence $s_i \to \infty$ s.t. $y(s_i) + \frac{b}{a}(s_i) = 0$. So to show $R(y)$ is ultimately zero, it is enough to show $y + \frac{b}{a}$ is ultimately zero. Let $y^* = y + \frac{b}{a}$, then $y^{*\prime} = \frac{P^*(y^*)}{Q^*(y^*)}$ where P^* and Q^* are polynomials satisfying the same conditions as P & Q. We are now reduced to showing: (we drop the asterisks for convenience) If y is an ultimately differentiable solution of $y' = \frac{P(u)}{Q(u)}$ where P & Q are as described in the ~~the~~ proposition, and if there is a sequence $s_i \to \infty$ s.t. $y(s_i) = 0$ then y is ultimately zero. To see this, let $y' = \frac{\sum a_i y^i}{\sum b_i y^i}$. Assume y is not ultimately zero and in particular that the zeroes of y are not ultimately dense. For large enough zeroes x of y we have $y'(x) = \frac{a_0(x)}{b_0(x)}$. If a_0 is not ultimately zero, it must either be ultimately positive or ultimately negative. Therefore $y'(x)$ is either ultimately positive or ultimately negative for zeroes x of y. Since the zeroes of y are not ultimately ~~not~~ ultimately dense, we have, for large enough consecutive zeroes x_1, x_2 we have $y'(x_1)$ and $y'(x_2)$ are both positive or both negative. This is ~~impossible~~ impossible so a_0 must be ultimately zero. Therefore, for some $j \neq 0$, we ultimately have $y' = \frac{y^j \sum a_i y^i}{\sum b_i y^i}$ where $a_0 b_0 \neq 0$. For large zeroes x of y we then have $y(x) = y'(x) = 0$. Since $u \equiv 0$ is also a solution of the equation $u' = \frac{u^j \sum a_i u^i}{\sum b_i u^i}$, uniqueness of solutions of differential equations gives us that y is ultimately zero. ∎

<u>Theorem 2</u>: Let $\mathcal{F}$ be a Hardy field and $P(u) = \sum a_i u^i$ and $Q(u) = \sum b_i u^i$ polynomials where $\bar{a}_i$ and $\bar{b}_i$ are all in $\mathcal{F}$. Let y be an ultimately continuously differentiable solution of

$y' = \frac{P(y)}{Q(y)}$. Then the ring $\mathcal{F}[\bar{y}]$ is an integral domain and its quotient field is a Hardy field.

Pf: Note that any element of $\mathcal{F}[\bar{y}]$ can be expressed as a polynomial in $\bar{y}$ with coefficients in $\mathcal{F}$. The first assertion follows from the proposition by noting that for $\overline{R(y)}, \overline{S(y)} \in \mathcal{F}[\bar{y}]$ if $\overline{R(y)S(y)} = 0$ then either $R(y)$ is ultimately zero or $S(y)$ is ultimately zero. To prove the second assertion we need only note that 1) if f is a function which is ultimately non-zero then $\frac{1}{f}$ is ultimately defined and 2) the quotient field of $\mathcal{F}[\bar{y}]$ is closed under D since for any $\overline{T(y)}$ in this field, $D(\overline{T(y)}) = \overline{\frac{\partial T}{\partial x} + \frac{\partial T}{\partial y}\frac{P}{Q}}$ which is again in the field. ∎

We remark that for $\mathcal{F} = \mathbb{R}(x)$ where $x' = 1$, this theorem was originally proven by Hardy [5]. Using this result Hardy was able to show that such a y must satisfy one of the relations y is asymptotic to $ax^b e^{P(x)}$ or y is asymptotic to $ax^b(\log x)^{1/c}$ where a, b are real numbers, c is an integer and P is a polynomial in x with coefficients in $\mathbb{R}$. For the general case, Marić [7] has some weaker results along these lines.

We shall now give some consequences of this theorem.

<u>Cor 1:</u> Let $\mathcal{F}$ be a Hardy field and $L(u) = u'' - a u' - b u = 0$ be a second order linear differential equation with $\bar{a}$ and $\bar{b}$ in $\mathcal{F}$. Let y be an ultimately continuously twice differentiable and ultimately non-zero solution of $L(u) = 0$. Then ~~the~~ $\mathcal{F}[\bar{y}]$ is an integral domain whose quotient field is a Hardy field.

<u>Pf:</u> Let $v = \frac{y'}{y}$. v is ultimately continuously differentiable and satisfies $v' = av + b - v^2$. Therefore by Theorem 2, the quotient field K of $\mathcal{F}[\bar{v}]$ is a Hardy field. y is a solution of $y' = vy$, so apply Theorem 2 again to get that $K[\bar{y}]$ is an integral domain whose quotient field is a Hardy field. Since ~~the~~ $\mathcal{F}[\bar{y}]$ is closed under D & $\mathcal{F}[\bar{y}] \subset K[\bar{y}]$ we are done.

<u>Cor 2:</u> There is a Hardy field $\mathcal{F}$, containing $\mathbb{R}(\bar{x})$ s.t. if $\bar{y} \in \mathcal{F}$ then $\overline{\int y} \in \mathcal{F}$ and $\overline{e^{\int y}} \in \mathcal{F}$.

<u>Pf:</u> Note that $\int y$ satisfies $(\int y)' = y$ and $e^{\int y}$ satisfies $(e^{\int y})' = y e^{\int y}$. If K is a Hardy field <u>with</u> $y \in K$, we can use Theorem 2 to adjoin either $\overline{\int y}$ or $\overline{e^{\int y}}$

and still get a Hardy field. The corollary then follows by transfinite induction. ■

This corollary is also proven in [3]. Hardy proved a weaker form [6], when he showed that any real elementary function is ultimately monotonic. One can use the above techniques to show:

Theorem 2': Let $\mathcal{F}$ be a Hardy field and $P(u) = \Sigma a_i u^i$ and $Q(u) = \Sigma b_i u^i$ polynomials with $\bar{a}_i, \bar{b}_i$ in $\mathcal{F}$. Let r be an odd integer and y an ultimately continuously differentiable solution of $(y')^r = \frac{P(y)}{Q(y)}$. Then $\mathcal{F}[\bar{y}, \bar{y}']$ is an integral domain whose quotient field is a Hardy field.

There are two related problems I want to mention:

① Find other classes of equations whose solutions lie in a Hardy field.

② Cor. 2 says that any real elementary function has only a finite number of zeroes. Given such a function can one effectively get a bound on the size of these zeroes?

§3 Bounds on the Rates of Growth of ~~[illegible]~~ Solutions of ODE's.

In what follows, $f = O(g)$ means there is a positive real number A s.t. we ultimately have $|f| < A|g|$, and $f = o(g)$ means $\lim_{x \to \infty} \frac{f}{g} = 0$, and $f \sim g$ means $\lim_{x \to \infty} \frac{f}{g} = 1$.

Let $f(x, y, y', \dots, y^{(n)})$ be a polynomial in $x, y, y', \dots, y^{(n)}$ with coefficients in $\mathbb{R}$. Borel [2] and Hardy [5] showed that for $n=1$, any solution of $f(x, y, y', \dots, y^{(n)}) = 0$ satisfies $y = O(e^{x^N})$ for some N. Borel claimed to have outlined a proof that, for any given n, a solution of $f(x, y, \dots, y^{(n)}) = 0$ satisfies $y = o(e_{n+1}(x))$ where $e_{k+1}(x) = \exp(e_k(x))$ & $e_0(x) = x$. There are gaps in his outline and it is now known that this statement is false, even for $n=2$. In fact, given any function $\varphi(t)$, there is an irrational number α s.t. $u(t) = \frac{1}{2 - \cos t - \cos \alpha t}$ is continuous, satisfies a second order differential equation & satisfies $\overline{\lim_{t \to \infty}} \frac{u(t)}{\varphi(t)} \geq 1$. ([1], p 97). We show that if y lies in a Hardy field this can't happen.

Theorem 3: Let $f(x, y, \dots, y^{(n)})$ be a polynomial in $x, y, y', \dots, y^{(n)}$ with real coefficients and let N be the largest power of x appearing in f. Let φ be a real valued function s.t. $\lim_{x \to \infty} \varphi = 0$. If y lies in a Hardy field and satisfies $f(x, y, y', \dots, y^{(n)}) = \varphi$ then

$y = o\left(e_n\left(\frac{x^{N+1+\varepsilon}}{N+1+\varepsilon}\right)\right)$ for all $\varepsilon > 0$.

Before starting the proof, some facts:

① Given any Hardy field $\mathcal{F}$, we can assume $\mathcal{F}$ contains $\overline{x}$ and $\overline{e_n(x)}$ for all n; if not, theorem 2 allows us to adjoin these.

② If $\mathcal{F}$ is a Hardy field ~~a~~ and $\overline{y} \in \mathcal{F}$ and $\lim_{x \to \infty} y = \infty$, then for any $\varepsilon > 0$ we have ~~[illegible]~~ y' is ultimately smaller than $y^{1+\varepsilon}$. (If not, then since we are working with Hardy fields, ~~so~~ there would be ~~[illegible]~~ positive numbers a & ε s.t. $y'(t) > y(t)^{1+\varepsilon}$ for $t > a$, so $\frac{y'(t)}{y(t)^{1+\varepsilon}} > 1$ for $t > a$. This would imply $x - a < \int_a^x \frac{y'}{y^{1+\varepsilon}} = \frac{y^{-\varepsilon}}{\varepsilon} \to 0$ as $x \to \infty$, a contradiction.

③ By ②, we have: If k is a Hardy field and $\overline{y} \in k$ and ~~[illegible]~~ $\lim_{x \to \infty} y = \infty$ then for all $n > 0$ & $\varepsilon > 0$, $y^{(n)} = o(y^{1+\varepsilon})$.

④ For $n \geq 2$, and $\mu > 1$, we have $\int e_{n-1}\left(\frac{x^{\mu}}{\mu}\right) dx \leq e_{n-1}\left(\frac{x^{\mu}}{\mu}\right)$, since
$$\int e_{n-1}\left(\frac{x^{\mu}}{\mu}\right) \sim x\, e_{n-1}\left(\frac{x^{\mu}}{\mu}\right) \prod_{i=0}^{n-2} \frac{1}{e_i\left(\frac{x^{\mu}}{\mu}\right)}$$ (See [3], p115).

<u>Proof of Theorem 3</u>: This idea seems to go back to Borel. By induction on n. We can assume $\overline{y} > 0$

<u>$n=0$:</u> $f(x,y) = \varphi$. In fact we will show $y = o(x^{N+\varepsilon})$ for every $\varepsilon > 0$. If not, then for some $c > 0$ & $\varepsilon > 0$, y would be

ultimately bigger than $cx^{N+\varepsilon}$. Let $f(x,y) = a_k y^k + \cdots + a_0 = \varphi$. and divide by $a_k y^k$. We get $1 + \left(\frac{a_{k-1}}{y}\right)\frac{1}{a_k} + \cdots + \left(\frac{a_0}{y^k}\right)\frac{1}{a_k} = \frac{\varphi}{y^k}$ (*) Since $y \geq cx^{N+\varepsilon}$, we get $\lim_{x\to\infty} \frac{a_{k-i}}{y^i} = 0$, so the left hand side of (*) approaches 1, while the right hand side approaches 0, a contradiction.

__Induction step:__ Assume $n \geq 1$ and that the theorem is true for $n-1$. Write $f = f_M + f_{M-1} + \cdots + f_0$ where f_i is a polynomial homogeneous in $y, y', \ldots, y^{(n)}$ of total degree i, with coefficients in $\mathbb{R}[x]$. We can assume that for some positive real numbers c and ε, that y is eventually larger than $c e_1 x^{N+\varepsilon}$ (since we are in a Hardy field). Using this and fact ③ above, we see that for $i < M$, $\lim_{x\to\infty} \frac{f_i}{y^M} \to 0$. Dividing $f(x,y,y',\ldots,y^{(n)}) = f_M + \cdots + f_0 = \varphi$ by y^M and manipulating, we get:

$$\frac{f_M(x,y,\ldots,y^{(n)})}{y^M} = \frac{\varphi - f_{M-1} - \cdots - f_0}{y^M} \qquad (**)$$

By what we have just seen, the right hand side of (**) approaches 0 as $x \to \infty$. Note that $f_M(x,y,\ldots,y^{(n)}) = f_M\left(x, 1, \frac{y'}{y}, \ldots, \frac{y^{(n)}}{y}\right)$. If we let $u = \frac{y'}{y}$, then we can write $\frac{y^{(i)}}{y}$ as a polynomial in $u, u', \ldots, u^{(i-1)}$ with coefficients in $\mathbb{Q}$ (since $\left(\frac{y^{(n)}}{y}\right)' = \frac{y^{(n+1)}}{y} - \frac{y^{(n)}y'}{y^2}$). This means that the left hand of $(**)$ can be written as $F(x, u, u', .., u^{(n-1)})$ where F is a polynomial in $x, u, u', .., u^{(n-1)}$ with real coefficients. Note that the highest power of x in F is less than or equal to N — the highest power of x in f. Since u is a solution of $F(x, u, u', .., u^{(n-1)}) = \frac{e - f_{M-1} - \cdots - f_0}{y^M}$, where both sides satisfy the hypothesis of the theorem, we can apply induction. If $n=1$, then we have by the previous step that $u = o(x^{N+\varepsilon})$ for any $\varepsilon > 0$ so $\log y = o(\int x^{N+\varepsilon}) = o\left(\frac{x^{N+1+\varepsilon}}{N+1+\varepsilon}\right)$ so $y = o\left(e_1\left(\frac{x^{N+1+\varepsilon}}{N+1+\varepsilon}\right)\right)$. If $n \geq 2$, then by induction we know $u = o\left(e_{n-1}\left(\frac{x^{N+1+\varepsilon}}{N+1+\varepsilon}\right)\right)$, so $\log y = o\left(\int e_{n-1}\left(\frac{x^{N+1+\varepsilon}}{N+1+\varepsilon}\right)\right)$. Using fact ④ we get $y = o\left(e_n\left(\frac{x^{N+1+\varepsilon}}{N+1+\varepsilon}\right)\right)$. ∎

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