Biography

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Chapter 1

Introduction

Let k be a differential field of characteristic zero with algebraically closed field of constants C. Let $L(y) = 0$ be a homogeneous linear differential equation of order n with coefficients in this field. There is a Galois theory associated with such equations. In this paper we construct a list of those groups which are possible Galois groups of irreducible homogeneous linear differential equations of order four. In particular we construct and characterize the irreducible linear algebraic subgroups of $\mathrm{SL}(4, \mathrm{C})$ where C is an algebraically closed field of characteristic zero. Given this list we use the representation theory of algebraic groups to construct sets of necessary and sufficient conditions for determining if a given unimodular group is a monomial group, an imprimitive nonmonomial group, or a finite or infinite primitive group. Note, the definitions of monomial, primitive and imprimitive groups are given in Chapter 2. These conditions are given in the following example result.

Theorem 1.1 *Let G be an irreducible subgroup of $SL(4, C)$. Let $V \cong C^4$. Either*

1. *one of the following conditions is satisfied in which case G is an imprimitive monomial group.*

    - *$Sym^2 V$ has a 2 dimensional $G-$ invariant subspace.*
    - *$\bigwedge^2 V$ and $Sym^2 V$ both have an irreducible $G-$ invariant subpace of dimension 6.*

or,

2. *condition 1 does not hold in which case G is either an imprimitive nonmonomial group or a primitive group. Furthermore, G is a primitive group if and only if one of the following is true.*

   - $\bigwedge^2 V$ *has no $G-$ invariant subspace of dimension 2.*
   - $\bigwedge^2 V$ *has two 1 dimensional $G-$ invariant subspaces and $Sym^3 V$ has a $G-$ invariant subspace of dimension 6.*

We also state results which allow one to distinguish finite groups from infinite groups, for example,

Theorem 1.2 *Let G be an primitive subgroup of $SL(4, C)$. Let $V \cong C^4$. Then G is a finite group if and only if one of the following conditions holds.*

1. *$Sym^3 V$ has a $G-$ invariant subspace of dimension 1.*
2. *$Sym^3 V$ has a $G-$ invariant subspace of dimension 8 and $Sym^6 V$ has a $G-$ invariant subspace of dimension 2,3, or 4.*
3. *$Sym^3 V$ is $G-$ irreducible and $Sym^4 V$ is $G-$ reducible.*
4. *$Sym^3 V$ is $G-$ reducible and $Sym^2 V$ is $G-$ irreducible.*
5. *None of the above hold and there exists a $k \neq 5, 7,$ or 21 such that $Sym^6 V$ has more than one $G-$ invariant subspace of dimension k.*

One may also use the representation theory to determine very specific groups. For example,

Theorem 1.3 *Let G be an irreducible subgroup of $SL(4, C)$. Let $V \cong C^4$ and denote by π the canonical projection from $GL(4, C)$ onto $PGL(4, C)$. Then $\pi(G) = G_{168}$ if and only if $\bigwedge^2 V$ is irreducible and $Sym^2 V$ has an irreducible $G-$ invariant subspace of dimension 7.*

There is much information that can be gained about the solutions of a linear differential equation given the knowledge of its Galois group. Each of the results concerning the subgroups of SL(4, C) can be restated to yield information about a given differential equation and its Galois group. For example,

Theorem 1.4 *Let $L(y) = 0$ be an irreducible order four linear homogeneous differential equation with coefficients in k and unimodular differential Galois group $Gal(L)$. Then $L(y) = 0$ is* **not** *solvable in terms of solutions of lower order equations if and only if $L_{Sym^4}(y)$ is irreducible.*

Theorem 1.5 *Let $L(y) = 0$ be an irreducible order four linear homogeneous differential equation with coefficients in k and unimodular differential Galois group $Gal(L)$. Then $L(y) = 0$ has only liouvillian solutions if and only if one of the following conditions is satisfied.*

1. *$L_{Sym^4}(y)$ has an irreducible factor of even order k where $k \neq 6, 10,$ or 16.*
2. *$L_{Sym^4}(y)$ has more than one order 1 factor.*
3. *For some $k \neq 5, 10$, $L_{Sym^4}(y)$ has an even number of irreducible factors of order k.*
4. *For each k, $L_{Sym^4}(y)$ has at most one factor of order k and there is an $m \neq 7$ such that $L_{Sym^6}(y)$ has more than one irreducible factor of order m.*
5. *None of the above hold and $L_{Sym^6}(y)$ has no irreducible factor of order equal to a multiple of 7.*

The definitions of a liouvillian solution and an equation that is solvable in terms of lower order equations are given in Chapter 2. We may also use the results concerning the groups to determine if the Galois group is a specific group.

Theorem 1.6 *Let $L(y) = 0$ be an irreducible order four linear homogeneous differential equation with coefficients in k and unimodular differential Galois group $Gal(L)$.*

Then $Gal(L)$ projects onto G_{168} if and only if $L_{\bigwedge^2}(y)$ is irreducible and $L_{Sym^2}(y)$ has an irreducible factor of order 7.

This manuscript will be organized as follows. In chapter 2 we will cover the basic definitions and results from differential algebra. The group theory needed will also be covered in this chapter. In Chapter 3 we will summarize the results of J.Kovacic, M.F.Singer and F.Ulmer concerning Galois groups of linear differential equations of orders two and three. One will also find short descriptions of the processes used to prove these results. In Chapter 4 we construct and list the irreducible subgroups of $\mathrm{SL}(4, \mathrm{C})$. This serves as a list of possible Galois groups G of an irreducible fourth order linear differential equation. Chapter 5 contains the information about the $\mathrm{G}-$ modules and their invariant subspaces that is needed for the decision procedure. Also included in this chapter are general procedures for decomposing certain types of $\mathrm{G}-$ modules for both monomial and nonmonomial imprimitive groups. Finally, Chapter 6 contains the decision procedure for characterizing or determining a given irreducible subgroup of $\mathrm{SL}(4, \mathrm{C})$ as well as many other useful corollaries and results.

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Chapter 2

Preliminaries : Differential Algebra and Group Theory

In this section we will simply define the basic terms and state the theorems that are needed for our research. For a more detailed exposition see [KA], [KOL2], [SI1], and [SI2]. The following definitions are standard in the literature.

A **derivation** δ is an additive operation on a ring R satisfying $\delta(ab) = a\delta(b) + \delta(a)b$ for all $a, b \in R$. Note that any derivation can be naturally extended to the field of quotients of a ring. A **differential field (k,**δ**)** is a field k together with a derivation δ on k. A **differential field extension** $(K, \tilde{\delta})$ is a differential field $(K, \tilde{\delta})$ such that K is a field extension of k and $\tilde{\delta}$ is an extension of δ to a derivation of K.

For notational convenience, we write $y^{(n)}$ instead of $\delta^n(y)$ and y', y'' for $\delta(y), \delta^2(y)$, etc. Throughout this manuscript, whenever we refer to a differential equation, $L(y) = 0$, we will mean an ordinary linear differential equation $L(y) = y^{(n)} + a_{n-1}y^{(n-1)} + \ldots + a_1y' + a_0y$ over a differential field, k, of characteristic zero whose field of constants, $\mathrm{C} = \ker_k(\delta)$, is algebraically closed. An example of such a field is $(\mathrm{C}(x), \frac{d}{dx})$.

A differential field extension K of k is called a **liouvillian extension** if there is a tower of fields

$$k = K_0 \subset K_1 \subset \ldots \subset K_m = K,$$

where K_{i+1} is a simple field extension $K_i(\eta_i)$ of K_i, such that one of the following holds:

1. η_i is algebraic over K_i, or

2. $\delta(\eta_i) \in K_i$, ie. K_{i+1} is an extension of K_i by an integral of an element of K_i, or

3. $\delta(\eta_i)/\eta_i \in K_i$, ie. K_{i+1} is an extension of K_i by the exponential of an integral of an element of K_i.

A **differential k-isomorphism** between two differential field extensions, K_1 and K_2, of k is a field isomorphism that leaves k fixed and commutes with δ. The **differential Galois group**, $G(K/k)$, of a differential field extension K of k is the set of all differential k-automorphisms of K.

Let $L(y) = 0$ be a homogeneous linear differential equation of degree n with coefficients in a differential field k. A differential field extension K of k is called a **Picard-Vessiot extension** of k for $L(y) = 0$ if the following holds:

1. $K = k < y_1, y_2, \ldots, y_n >$, the differential field generated by k and $y_1, y_2, \ldots, y_n$, where $\{y_1, y_2, \ldots, y_n\}$ is a fundamental set of solutions of $L(y) = 0$.

2. K and k have the same field of constants.

Now, let $L(y) = 0$ be a linear differential equation of order n over k, and let K be a Picard-Vessiot extension of k for $L(y) = 0$. We denote by $Gal(L)$ the Galois group $\mathrm{G}(K/k)$. If $\{y_1, \ldots, y_n\}$ is a fundamental set of solutions of $L(y) = 0$ in K, then for each automorphism $\sigma \in Gal(L)$ we have that $\sigma(y_i) = \sum_{j=1}^{n} c_{ij} y_j$ where $c_{ij} \in \mathrm{C}$. This gives a faithfull representation of $Gal(L)$. Hence, given any linear differential equation $L(y) = 0$ we may always consider $Gal(L)$ as a subgroup of $\mathrm{GL}(n, \mathrm{C})$. As a matter of fact in [KOL1], Kolchin shows that if $L(y) = 0$ is any linear differential equation of order n over k and K is an associated Picard-Vessiot extension of k, then the Galois group $Gal(L) = \mathrm{G}(K/k)$ is a linear algebraic subgroup of $\mathrm{GL}(n, \mathrm{C})$.

The connection between the Galois group of $L(y) = 0$ and the existence of liouvillian solutions is given in the following result of Kolchin [KOL1].

Theorem 2.1 *A linear differential equation $L(y) = 0$ with coefficients in k has*

1. *only solutions which are algebraic over k if and only if $Gal(L)$ is a finite group.*

2. *only liouvillian solutions over k if and only if the component of the identity $Gal(L)^0$ of $Gal(L)$ is solvable. In this case $L(y) = 0$ has a solution whose logarithmic derivative is algebraic over k.*

A linear differential equation defined over k is said to be **solvable in terms of linear differential equations of lower order** if the associated Picard-Vessiot extension K of k lies in a tower of fields $k = k_0 \subset k_1 \subset \ldots \subset k_n$, where for each i, $k_i = k_{i-1}(t_i)$ where t_i is algebraic over k_{i-1}, or satisfies a linear differential equation of lower order defined over k_{i-1}.

We have the following theorem of Singer ([SI2] Theorem 5.1) giving the correspondence between an equation that is not solvable in terms of lower order equations and its Galois group, G.

Theorem 2.2 *Let k be a differential field with algebraically closed field of constants C and let $L(y) = y^{(n)} + a_{n-1}y^{(n-1)} + \ldots + a_0 y$ with $a_i \in k$. $L(y) = 0$ is* **not** *solvable in terms of equations of lower order if and only if the associated Picard-Vessiot extension has Galois group G whose connected component G^0 of the identity is*

1. *simple, and*

2. *there does not exist a linear algebraic group H with finite-to-one homomorphism $\alpha : H \to G^0$ such that H has a nontrivial representation of dimension less than n.*

In developing the algorithm for determining the Galois group of a differential equation it is necessary to construct a list of all possible Galois groups. That is, a list of all linear algebraic groups of a particular degree. The following group theoretic definitions give a partial classification of such groups that will serve as a starting point for constructing a detailed list.

A subgroup G of GL(V) is said to act **irreducibly** if the only G$-$ invariant subspaces of V are $\{0\}$ and V. When it is clear what the vector space V is, we may refer to such a group simply as an irreducible group. The G-module V is **completely reducible** if there are minimal G$-$ invariant subspaces $V_1, \ldots, V_k$ such that $V = V_1 \oplus \cdots \oplus V_k$.

Let G be a subgroup of GL(n, C) acting irreducibly, i.e. G is a linear group acting irreducibly on the vector space V of dimension n over C. Then G is called **imprimitive** if, for $k > 1$, there exist subspaces $V_1, \ldots, V_k$ such that $V = V_1 \oplus \ldots \oplus V_k$ and, for each $g \in$ G, the mapping $\mathrm{V}_i \rightarrow g(\mathrm{V}_i)$ is a permutation of the set $S = \{\mathrm{V}_1, \ldots, \mathrm{V}_k\}$. The set S is called a **system of imprimitivity of G**. If all the subspaces V_i are 1 dimensional, then G is said to be **monomial**. An irreducible group G $\subset$ GL(n, C) which is not imprimitive is called **primitive.**

When constructing lists of the primitive groups it is useful to first consider primitive subgroups of GL(n, C)/Z(GL(n, C)), which is denoted by PGL(n, C). Subgroups of PGL(n, C) will be called **projective groups**.

Algorithms for bounding the degree of the minimum polynomial of an algebraic solution or that of its logarithmic derivative in the order two and three cases can be found in [SU2] and [SU3]. These algorithms rely on the existence of a 1-reducible subgroup H of the Galois group G. A group G $\subset$ GL(n, C) whose elements have a common eigenvector is called **1-reducible**.

The algorithm for determining the Galois group of a given equation relies heavily on being able to construct and factor new equations which can be associated with different representation spaces of $Gal(L)$. In particular, we consider equations which correspond to the vector spaces which are constructed from the solution space, V, via tensors, duals, and quotients. We will refer to such spaces as simply constructions of V. The following theorem and definition are an example of such an equation and its association with a constructed vector space. This particular type of equation is used in much of the work of Singer and Ulmer.

Theorem 2.3 *[SU1] Let $L_1(y) = 0$ and $L_2(y) = 0$ be linear differential equations of degrees respectively n_1 and n_2 and fundamental systems respectively $S_1 = \{u_1, \ldots, u_{n_1}\}$ and $S_2 = \{v_1, \ldots, v_{n_2}\}$. Then one can construct a differential equation $L(y) = L_1(y) \circledS L_2(y)$ $= 0$ of degree $n_3 \leq n_1 n_2$, whose solution space is spanned by $S = \{u_1 v_1, \ldots, u_{n_1} v_1, \ldots, u_{n_1} v_{n_2}\}$.*

The linear differential equation $L^{\circledS m}(y) = L(y) \circledS L(y) \circledS \cdots \circledS L(y) = 0$ is called the **symmetric power of order** m of $L(y) = 0$. Note that in [SI1] and [SU1] an algorithm for constructing these symmetric powers is given.

Once one has constructed a symmetric power of a given differential equation, one may use the following map and its properties. Let $L(y) = 0$ be a linear differential equation with coefficients in k and let V be the solution space of $L(y) = 0$ in some Picard-Vessiot extension K of k. Then one can define a morphism of $Gal(L)-$ modules, $\phi_m : Sym^m(V) \rightarrow K$, taking $z_1 \circledS \cdots \circledS z_m$ to $z_1 \cdots z_m$ such that the image of $Sym^m(V)$ is the solution space of $L^{\circledS m}(y) = 0$. In [SU2] the authors show that for any polynomial invariant I of $Gal(L)$, $\phi_m(I)$ is a rational solution of $L^{\circledS m}(y) = 0$.

Chapter 3

Known Results for the Order Two and Three Cases

The problem of finding liouvillian solutions of a linear differential equation of arbitrary order has been completely solved (see for ex. [SI1]). In this chapter we will see how J. Kovacic, M.F. Singer, and F. Ulmer have used the theory or linear algebraic groups and their representations to greatly improve the efficiency of this well known process for equations of low order. In particular, we will summarize the results of J. Kovacic presented in [KOV] as well as those of M.F. Singer and F. Ulmer that are given in [SU1], [SU2], and [SU3].

An algorithm for finding liouvillian solutions of a differential equation of order two is very clearly presented in [KOV]. In this paper, Kovacic focuses of the differential equation, $y'' = ry$, where $r \in C(x)$, $C = \mathbb{C}$. This equation is denoted "the DE." One should note that under a suitable variable transformation any second order linear homogeneous differential equation can be put into this form; and more importantly we have the fact that this equation has liouvillian solutions if and only if the original one does (see [SU2]). Kovacic's approach to the problem of finding liouvillian solutions to this equation is as follows. He first states the following four exhaustive cases for the Galois group of this equation.

Lemma 3.1 *([KOV] sec. 1.4) Let G be an algebraic subgroup of $SL(2, C)$. Then one of four cases can occur.*

1. *G is triangulizable.*
2. *G is conjugate to a subgroup of*
$$D^{+} = \left\{ \begin{pmatrix} c & 0 \\ 0 & c^{-1} \end{pmatrix} : c \in C, c \neq 0 \right\} \cup \left\{ \begin{pmatrix} 0 & c \\ -c^{-1} & 0 \end{pmatrix} : c \in C, c \neq 0 \right\}$$
and case 1 does not hold.
3. *G is finite and cases 1 and 2 do not hold.*
4. *$G = SL(2, C)$.*

He then determines the following four cases corresponding to each of the types of groups listed above.

Theorem 3.1 *([KOV] sec. 1.2) Given the differential equation $y'' = ry$, where $r \in C(x)$. There are precisely four cases that can occur.*

1. *The DE has a solution of the form $e^{\int \omega}$ where $\omega \in C(x)$.*
2. *The DE has a solution of the form $e^{\int \omega}$ where ω is algebraic over $C(x)$ of degree 2, and case 1 does not hold.*
3. *All solutions of the DE are algebraic over $C(x)$ and cases 1 and 2 do not hold.*
4. *The DE has no liouvillian solution.*

The first list is used to prove Theorem 3.1 above. For example, note that case two in the list of groups corresponds exactly to the second case for the differential equation. Indeed, if we let $\{\eta, \xi\}$ be a fundamental set of solutions for the DE such that the Galois group $Gal(L)$ is a subgroup of D^{+} and let $\omega = \eta'/\eta$ and $\phi = \xi'/\xi$, then for any $\sigma \in Gal(L)$ we have either $\sigma(\omega) = \omega$ and $\sigma(\phi) = \phi$ or $\sigma(\omega) = \phi$ and $\sigma(\phi) = \omega$. Therefore there are two distinct roots of the minimal polynomial for $\omega \in Gal(L)$ over C(x), which implies that ω is algebraic of degree two over C(x).

This list of groups is used to develop the following set of necessary conditions for determining the existence of liouvillian solutions.

Theorem 3.2 *([KOV]Theorem 1.) The following conditions are necessary for the respective cases to hold.*

1. *Every pole of r must have even order of else have order 1. The order of r at ∞ must be even or else be greater than 2.*

2. *r must have at least one pole that either has odd order greater than 2 or else has order 2.*

3. *The order of a pole of r cannot exceed 2 and the order of r at ∞ must be at least 2. If the partial fraction expansion of r is*

$$r = \sum_i \left(\frac{\alpha_i}{(x - c_i)^2}\right) + \sum_j \left(\frac{\beta_j}{x - d_j}\right).$$

then $\sqrt{1 + 4\alpha_i} \in \mathbb{Q}$, for each i, $\sum_j \beta_j = 0$, and if

$$\gamma = \sum_i \alpha_i + \sum_j \beta_j d_j,$$

then $\sqrt{1 + 4\gamma} \in \mathbb{Q}$.

Case 4 holds if none of the above cases hold.

The proof of the necessary conditions relies on what information is available and useful in each particular case. In case 1, Kovacic uses the Ricatti equation $\omega' + \omega^2 = r$ associated to the DE. Recall that ω is defined by the solution $\eta = e^{\int \omega}$ of the DE where $\omega \in \mathrm{C}(x)$. Since they are elements of $\mathrm{C}(x)$, both ω and r have Laurent series expansions about any point in the complex plane. Assume that c_i is any pole of r. Expand both ω and r about c_i. Substitute these expressions into the Ricatti equation. Compare the coefficients of the leading terms. By leading term here one is referring to the term involving $(x - c_i)^p$ where p is the smallest power appearing. In this way, Kovacic is able to impose the necessary conditions of the poles of r.

In case 2, the form of the Galois group of the DE is enough to impose conditions on r. Again by Lemma 3.1 the Galois group of the DE must be conjugate to a subgroup of D^+. Assume that $\{\eta, \xi\}$ is a fundamental set of solutions such that $Gal(L)$ is actually contained in D^+. Then for each $\sigma \in Gal(L)$ we have either $\sigma(\eta) = c_\sigma \eta$ and $\sigma(\xi) = c_\sigma^{-1} \xi$

or $\sigma(\eta) = -c_\sigma^{-1}\xi$ and $\sigma(\xi) = c_\sigma \eta$. This implies that $\eta^2\xi^2$ is invariant under $Gal(L)$ and hence $\eta^2\xi^2 \in \mathrm{C}(x)$. Again, $\eta^2\xi^2$ has a Laurent series expansion about any point c in the complex plane. Let θ be the logarithmic derivative of $\eta\xi$. Then θ is a solution to the Ricatti equation, and as in case 1, upon substitution of both θ and r into this equation Kovacic is able to obtain the given list of necessary conditions.

Finally, in case 3, it is assumed that the DE has a solution, η, that is algebraic over $\mathrm{C}(x)$. Hence, η has a Puisseaux expansion about any point in the complex plane. Without loss of generality, we may expand η at zero. Let $\eta = ax^\mu + \ldots$ $a \in \mathrm{C}$, $\mu \in \mathbb{Q}$ be this expansion. Note that the coefficient r of the DE is still in $\mathrm{C}(x)$ and hence has a Laurent series expansion about zero. Let $r = \alpha x^\nu + \ldots$, $\nu \in \mathbb{Z}$, $\alpha \neq 0$ be this expansion. One can observe that if we substitute these expansions of r and η into the DE, $y'' = ry$, then the exponent ν must be greater than or equal to -2. Now, assuming that $(\mu - 1) \neq 0$ ie. $\nu = -2$ then the leading coefficients $\mu(\mu-1)$ and α must be equal. If one solves this equation for $\mu \in \mathbb{Q}$ one obtains the condition $\sqrt{1+4\alpha} \in \mathbb{Q}$. This process may be repeated at each singularity of the DE, hence we have that r has a partial fraction expansion of the form

$$r = \sum \frac{\alpha_i}{(x-x_i)^2} + \sum \frac{\beta_i}{(x-d_i)} + P(x)$$

where $P(x) \in \mathrm{C}[x]$ and the above condition is on α_i. In a similar manner, if we consider the expansion of η about ∞ then upon substitution of this expansion of η and the partial fraction expansion of r into the DE we see that at ∞ the exponent $\nu \leq -2$. This implies that the polynomial $P = 0$. Then if we expand the resulting partial fractions expansion of r, substitute this as well as the solution, η, back into the DE, and equate coefficients again, we obtain the rest of the conditions in this case.

Once Kovacic derives this set of necessary conditions, he is able to develop an algorithm that uses them. This algorithm allows one to determine which case one is in while simultaneously constructing the liouvillian solution η if one exists. He does the latter by imposing the necessary conditions on the partial fractions expansion of r in order to obtain a list of candidates for the element $\omega \in \mathrm{C}(x)$. A modified approach due to Singer and Ulmer is described later in this section.

In [SU1] M.F. Singer and F. Ulmer actually calculate the Galois group of a given differential equation of order two or three. Properties of this group are then used to extract information about the solutions, in particular, the existence and form of liouvillian solutions. In addition to the group theory that is used in [KOV] the authors exploit the representation theory of linear algebraic groups and are able to state sets of necessary and sufficient conditions for second and third order equations to have liouvillian solutions or, more specifically, to have particular Galois groups.

Throughout this paper the Galois group of the given differential equation is assumed to be unimodular (ie. the matrix representation of $Gal(L)$ on the solution space of $L(y) = 0$ is contained in $\mathrm{SL}(n, \mathrm{C})$). As noted in [KOV] this has little affect on questions concerning liouvillian solutions. Recall that linear algebraic groups can be characterized via their action on vector spaces. In particular, a subgroup of $\mathrm{GL}(n, \mathrm{C})$ can be characterized via its action on the module $\mathrm{V} = \mathrm{C}^n$. The group acts either reducibly, primitively, or imprimitively on this space V. Given a linear differential equation with Galois group $Gal(L) \subseteq \mathrm{GL}(n, \mathrm{C})$, we have the following useful correspondence between the factorization of this equation and the action of $Gal(L)$ on the solution space $\mathrm{V} \cong \mathrm{C}^n$.

Theorem 3.3 *[KOL1][SI2][SU1] The differential equation $L(y) = 0$ of order n over k factors as a differential operator over k if and only if the Galois group of $L(y) = 0$ acts reducibly on the solution space. Let V be the solution space of $L(y) = 0$. $Gal(L)$ leaves an m dimensional subspace of V invariant if and only if $L(y) = L_{n-m}(L_m(y))$ where $L_{n-m}(y)$ and $L_m(y)$ have coefficients in k and are of order $n - m$ and m.*

Note that Theorem 3.3 implies that if the Galois group of an order 2 differential equation acts reducibly, then the equation has a liouvillian solution. Indeed, if $Gal(L)$ acts reducibly on the vector space, V, of degree two, then V has a 1 dimensional invariant subspace. Therefore by Theorem 3.3 $L(y) = (L_1 \circ L_2)(y)$ where both L_1 and L_2 have order 1. Therefore, $L(y) = 0$ has a solution η such that $\eta'/\eta \in k$.

If an order three differential equation has a reducible Galois group, then one either finds a liouvillian solution from this factorization of $L(y)$ or the problem is reduced to the order two case. This also follows from Theorem 3.3 since the only two possible factorizations of $L(y)$ are of the form $L_1(L_2(y))$ or $L_2(L_1(y))$ where $L_1(y)$ and $L_2(y)$ are differential equations of orders one and two respectively. Theorem 3.3 is also effective in determining the Galois group of an irreducible differential equation. Although an irreducible differential equation has an irreducible Galois group, one may consider the factorization properties of equations of the form $L^{\circledS m}(y) = 0$. The factorization properties of $L^{\circledS m}(y) = 0$ correspond to the action of $Gal(L)$ on the space Sym^mV. The authors use this correspondence to determine Galois groups of irreducible second and third order differential equations.

They first consider the imprimitive case. If $L(y) = 0$ is an order two or three equation with imprimitive Galois group, then this group is monomial and hence all solutions are liouvillian. In particular, the logarithmic derivative of any solution is algebraic of degree at most two or three respectively. This can be seen via the fact that there is a fundamental solution set for $L(y) = 0$ that gives a system of imprimitivity for $Gal(L)$ where each subspace has dimension one.

In order to decide if the Galois group of the given differential equation is imprimitive, the authors use the following facts.

1. Any imprimitive group of degree two or three is monomial and hence contains a 1-reducible subgroup of index two or three respectively.

2. **Theorem 3.4** *([SU1] Proposition 3.6) If an irreducible linear differential equation $L(y) = 0$ of order n with coefficients in k has a monomial differential Galois group $Gal(L) \subset SL(n, C)$, then the n^{th} symmetric power $L^{\circledS n}(y) = 0$ of $L(y) = 0$ has a solution which is the square root of an element of k.*

In the order 2 case these facts lead to the next result.

Theorem 3.5 *([SU1] Theorem 4.1 (ii)) Let $L(y) = 0$ be a second order irreducible linear differential equation over k with unimodular differential Galois group. Then $Gal(L)$ is imprimitive if and only if $L^{\circledS 2}(y) = 0$ is reducible. In this case $L^{\circledS 2}(y) = 0$ has a solution $y \neq 0$ such that $y^2 \in k$.*

Assuming that $Gal(L)$ is imprimitive, Proposition 3.6 from fact 2 above implies the result. The converse can be proven in the following way. Assume that $L^{ⓢ2}(y) = 0$ is reducible. In the order two case, one has that the solution space of this equation is $Gal(L)-$ isomorphic to $Sym^2(\mathrm{V})$ where V is the solution space of $L(y) = 0$. Therefore $Sym^2(\mathrm{V})$ is a reducible $Gal(L)-$ module. One can look at the character tables for the second symmetric power of an irreducible 2 dimensional G$-$ module V for all of the finite primitive subgroups G of $\mathrm{SL}(2, \mathrm{C})$ and observe that none of these acts reducibly on $Sym^2(\mathrm{V})$. Therefore either $Gal(L)$ is imprimitive or it is all of $\mathrm{SL}(2, \mathrm{C})$. Since $\mathrm{SL}(2, \mathrm{C})$ acts reducibly on C^2 and all symmetric products of C^2, and $V \cong \mathrm{C}^2$, $Gal(L)$ must be imprimitive.

Lastly, Singer and Ulmer consider the case where the Galois group of $L(y) = 0$ is primitive. One would know one is in this case given that the first two cases do not hold. Since one is able to list all of the primitive subgroups of both $\mathrm{SL}(2, \mathrm{C})$ and $\mathrm{SL}(3, \mathrm{C})$ (see for ex. [BL3]), one is able to completely determine the Galois group associated with $L(y)$. The key to this identification is in applying Theorem 3.3 to the equation $L^{ⓢm}(y) = 0$ given that its solution space is $Gal(L)-$ isomorphic to a direct summand of $Sym^m\mathrm{V}$. Given the irreducible decompositions of several symmetric powers of characters of each of the finite primitive groups of degree two and three, the authors are able to state very effective results. In the following example result $A_4^{SL_2}, S_4^{SL_2}$ and $A_5^{SL_2}$ denote the three finite primitive subgroups of $\mathrm{SL}(2, \mathrm{C})$.

Theorem 3.6 *([SU1] Proposition 4.3) Let $L(y) = 0$ be a second order homogeneous linear differential equation with coefficeints in k and unimodular differential Galois group. Assume that the Galois group, $Gal(L)$, is primitive.*

1. *$L^{ⓢ3}(y)$ factors over k if and only if $Gal(L) \cong A_4^{SL_2}$. In this case, $L^{ⓢ3}(y) = L_1(L_2(y))$ where $L_1(y)$ and $L_2(y)$ have order 2.*

2. *Assuming $L^{ⓢ3}(y)$ is irreducible, then $L^{ⓢ4}(y)$ factors over k if and only if $Gal(L) \cong S_4^{SL_2}$. In this case, $L^{ⓢ4}(y) = L_1(L_2(y))$ where $L_1(y)$ and $L_2(y)$ have orders 3 and 2.*

3. *Assuming $L^{\circledS 4}(y)$ is irreducible, then $L^{\circledS 6}(y)$ factors over k if and only if $Gal(L) \cong A_5^{SL_2}$. In this case, $L^{\circledS 6}(y) = L_1(L_2(y))$ where $L_1(y)$ and $L_2(y)$ have orders 4 and 3.*

4. *$Gal(L) \cong SL(2, C)$ if and only if $L^{\circledS 6}(y)$ is irreducible over k.*

A combination of the results from the reducible and imprimitive cases yields the next result, which is especially useful if one needs information only about the existence of liouvillian solutions.

Theorem 3.7 *([SU1] Proposition 4.4) Let $L(y) = 0$ be a second order homogeneous linear differential equation with coefficients in k and unimodular differential Galois group. $Gal(L) \cong SL(2, C)$ if and only if $L^{\circledS 6}(y)$ is irreducible over k. Therefore, $L(y) = 0$ has liouvillian solutions if and only if $L^{\circledS 6}(y)$ is reducible.*

Singer and Ulmer are able to obtain similar results in the order three case. For example,

Theorem 3.8 *([SU1] Corollary 4.9) Let $L(y) = 0$ be a third order linear differential equation with coefficients in a differential field k with algebraically closed field of constants and whose differential Galois group $G(L)$ is unimolular. $L(y) = 0$ has liouvillian solutions if and only if*

1. *$L^{\circledS 4}(y)$ has order less than 15 or factors, and*

2. *one of the following holds:*

   (a) *$L^{\circledS 2}(y)$ has order 6 and is irreducible, or*

   (b) *$L^{\circledS 3}(y)$ has a factor of order 4.*

The second paper by M. Singer and F. Ulmer that we have studied deals with the problem of finding the minimal polynomial of the logarithmic derivative of a liouvillian solution assuming that one is known to exist. As a matter of fact, it is known that if the Galois group is primitive, then the solution itself is algebraic over k. One would like to be able to determine the degree of the associated minimal polynomial. The authors show that the index of a 1-reducible subgroup H of $Gal(L)$ gives a bound for the degree of this minimal polynomial. This is based on the following theorem.

Theorem 3.9 *[UL] If an irreducible linear differential equation has a liouvillian solution, then $Gal(L) \subset GL(n, C)$ has a 1-reducible subgroup H of finite index and there is a solution, z, of $L(y) = 0$ such that the algebraic degree of $u = z'/z$ over k is $\leq [Gal(L) : H]$.*

Once this degree is bounded Singer and Ulmer show that in the primitive case, one can use character theory combined with a knowledge of the space of semi-invariants of the Galois group $Gal(L)$ in order to compute the coefficients of this minimal polynomial. When the Galois group is imprimitive, the method presented in [SI1] is used. As in the method of Fuchs, this process uses the partial fraction expansions of each of the coefficients of the differential equation as well as the fact that any solution whose logarithmic derivative is algebraic over k has the property that, this derivative is an algebraic solution of the associated Ricatti equation. One should note that this is currently the only algorithm known for computing liouvillian solutions of linear differential equations whose Galois group is an imprimitive group of degree ≥ 3.

The paper [SU2] proceeds as follows. It is first shown that if $L(y) = 0$ has a liouvillian solution, then it has a solution y that is algebraic or whose logarithmic derivative $\frac{y'}{y}$ is algebraic. In this case the minimal polynomial for y or that of its logarithmic derivative will have a certain form. It is noted that, while the process of determining the minimum polynomial of the logarithmic derivative of a solution is less complicated than that for an algebraic solution itself, the number of coefficients for each of these polynomials is the same. More specifically, if the degree of the logarithmic derivative of an algebraic solution y of $L(y) = 0$ is m, then there exists a positive integer d such that the minimal polynomial of y is of the form $P(Y) = Y^{dm} + a_{m-1}Y^{d(m-1)} + \ldots + a_1 Y^d + a_0$. This integer d is the order of some cyclic factor of the 1-reducible subgroup H of $Gal(L)$. Furthermore, this form for $P(Y)$ can also be written in terms of a product of polynomials over a set of cosets of this 1-reducible subgroup H of $Gal(L)$ that fixes y'/y. These forms will be the starting point for determining the exact minimal polynomial for y or that for y'/y.

The next step is to determine a set of possible degrees of this polynomial. The authors first get a bound on the degree m of the minimal polynomial of the algebraic derivative of a solution. Theorem 3.9 above is used to define a function $\mathcal{F}(n)$ which Ulmer shows plays a large part in bounding this degree. The function $\mathcal{F}(n)$ is defined as the minimum value such that for all finite primitive projective algebraic groups G, any primitive representation of degree n of a Schur representation group of G has a 1-reducible subgroup of index $\leq \mathcal{F}(n)$.

Ulmer shows that if $Gal(L)$ is a primitive group and if $L(y) = 0$ has a liouvillian solution, then $L(y) = 0$ has a solution z such that $u = z'/z$ is algebraic over k of degree $\leq \mathcal{F}(n)$ ([UL]).

Now, since one can obtain a list of the finite primitive projective algebraic subgroups of degree 2 or 3, the authors are able to use the characters of the Schur representation groups of each of these groups in order to determine the value of $\mathcal{F}(n)$. They need the following facts.

1. Let G be a finite primitive projective algebraic group of degree n. Let Γ_{G} be a Schur representation group of G. Let $\rho(\Gamma_{\mathrm{G}})$ be an irreducible representation of degree n with character ζ. The restriction of $\rho(\Gamma_{\mathrm{G}})$ to H is 1-reducible if and only if there is a one dimensional character ϕ of H such that the scalar product $(\phi, \zeta \mid_{\mathrm{H}}) \neq 0$. ([CR])

2. When $n = 3$, if a subgroup H of Γ_{G} has no irreducible character of degree 3, then $\rho(H)$ must be 1-reducible.

Note that 2 follows from the fact that if any degree 3 representation is reducible, then one may write the vector space C^3 as a direct sum of either a $\rho(\mathrm{H})-$ invariant vector space of dimension 2 and one of dimension 1, or three $\rho(H)-$ invariant one dimensional vector spaces. In either case, we have a one dimensional $\rho(H)-$ invariant subspace of C^3. That is, $\rho(H)$ is 1-reducible.

These facts in combination with a good amount of group theory are used to compute $\mathcal{F}(3)$ in the following way.

Let G $\cong \Gamma/Z(\Gamma)$ be one of the primitive subgroups of PGL(3, C). There is a finite list of possible indexes for a subgroup H of G. Construct a character table for all of the subgroups of G. By inspecting the character tables, determine the largest subgroup of G that has no irreducible character of degree 3. Any such subgroup is 1-reducible. One can show that it is the largest by applying the above facts and more character theory in order to exclude the possibility of any larger subgroup from being 1-reducible. Repeat this process for each finite primitive subgroup of PGL(3, C).

If we recall the fact that any imprimitive group of degree 3 has a 1-reducible subgroup of index 3, we see how the authors obtained the following result.

Theorem 3.10 *[SU2] If an irreducible third order linear differential equation $L(y) = 0$ with coefficients in k has a liouvillian solution, then $L(y) = 0$ has a solution z such that for the logarithmic derivative $u = z'/z$ of z one of the following holds:*

1. *u is algebraic of degree 36 over k and $G(L)/Z(G(L)) \cong A_6$.*

2. *u is algebraic of degree 21 over k and $G(L)/Z(G(L)) \cong G_{168}$.*

3. *u is algebraic of degree 9 over k and $G(L)/Z(G(L))$ is isomorphic to H_{72} or H_{216}.*

4. *u is algebraic of degree 6 over k and $G(L)/Z(G(L))$ is isomorphic to F_{36} or A_5.*

5. *u is algebraic of degree 3 over k and $G(L) \subset GL(3, C)$ is an imprimitive group.*

For each group $G(L) \subset GL(3, C)$ the numbers given above are best possible.

Next one would like to determine the degree of the minimum polynomial for an algebraic solution. Again we have that the general form for this polynomial is $P(Y) = Y^{dm} + a_{m-1}Y^{d(m-1)} + \ldots + a_1 Y^d + a_0$ where $a_i \in k$. Recall that given a possible degree, m, for the logarithmic derivative of a solution, one has that this integer d must be the order of a cyclic factor of the 1-reducible subgroup H. Using a computer algebra system one is able to list all of the possible values of d for each of the possible groups H. Also, as a result of work done by Ulmer in [UL] one has that $Gal(L)$ must have a faithful representation as a transitive permutation group of degree dm. Given this information one is able to obtain a finite list of possibilities for the value of d. The resulting theorem is as follows.

Theorem 3.11 *([SU2] Theorem 4.1) Let $L(y) = 0$ be an irreducible third order linear differential equation with Galois group $Gal(L) \subseteq SL(3, C)$. If $L(y) = 0$ has a liouvillian solution then all solutions are algebraic and there is a solution z whose minimal polynomial $P(Y)$ is of the form*

$$Y^{dm} + a_{m-1}Y^{d(m-1)} + \ldots + a_1 Y^d + a_0 \quad a_i \in k$$

such that one of the following holds:

1. *If $Gal(L) \cong A_6^{SL_3}$, $m = 36$ and $d = 6$.*
2. *If $Gal(L) \cong A_5$, $m = 6$ and $d = 2$.*
3. *If $Gal(L) \cong A_5 \times C_3$, $m = 6$ and $d = 6$.*
4. *If $Gal(L) \cong G_{168}$, $m = 21$ and $d = 2$.*
5. *If $Gal(L) \cong G_{168} \times c_3$ $m = 21$ and $d = 6$.*
6. *If $Gal(L) \cong H_{216}^{SL_3}$ $m = 9$ and $d = 9$.*
7. *If $Gal(L) \cong H_{72}^{SL_3}$ $m = 9$ and $d = 3$.*
8. *If $Gal(L) \cong F_{36}^{SL_3}$ $m = 6$ and $d = 6$.*

The above numbers are also the minimal degree of an algebraic solution, except for the group $F_{36}^{SL_3}$, where there also exists a solution whose minimal polynomial is of degree 27 instead of 36.

The final step in computing the minimal polynomial of a liouvillian solution or that of its logarithmic derivative is to calculate the coefficients. It is assumed at this point that one has been able to compute the Galois group of $L(y) = 0$. Also one has the general form of the minimal polynomial as well as a bound on its degree . Recall that if τ is a set of coset representatives for H in G, then we may write this polynomial $P(Y) = \prod_{\sigma \in \tau}(Y^d - \sigma(y)^d)$. Note that σ takes any solution to a linear combination of the set of solutions that form a basis for K over the field of constants, C.

Using this fact one can show that the coefficients a_i of $P(Y)$ must be invariants of the Galois group $Gal(L)$. Using a Groebner basis for the ring of (semi) invariants of $Gal(L)$ (see [CLO]) and a fixed irreducible representation of $Gal(L)$, one is able to obtain expressions for the coefficents of $P(Y)$ in terms of the basic invariants of the group.

The final paper that we will mention here was again written jointly by M.F. Singer and F. Ulmer [SU3]. The point of this paper is to develop sets of necessary conditions that one may impose directly on the coefficients of the given differential equation in order to determine if it has liouvillian solutions of a particular form. We need an additional definition. The differential equation $L(y) = y^{(n)} + a_{n-1}y^{(n-1)} + \ldots + a_1y' + a_0y = 0$ is of **Fuchsian type** if its coefficents are rational and if its solutions have only regular singular points. The following theorem tells us if a differential equation is of Fuchsian type.

Theorem 3.12 *[KU] The linear differential equation $L(y) = y^{(n)} + a_{n-1}y^{(n-1)} + \ldots + a_1y' + a_0y = 0$ is of Fuchsian type if and only if each coefficeint a_i is a rational function and each pole of a_i is of order $\leq n - i$.*

The results of this paper use the coefficients of the differential equation $L(y) = 0$ to put conditions on the exponents of $L(y) = 0$. For a formal definition of an exponent of $L(y) = 0$ see [SU3] p.2. For the purpose of this report it is enough to know that if $L(y) = 0$ has a solution of the form $y = x^\rho \sum_{j \geq 0} c_j x^j$, where $c_0 \neq 0$, then ρ satisfies a polynomial equation called the **indicial equation** in one indeterminant of degree at most n. One is able to obtain this polynomial equation from the partial fractions expansions of the rational coefficients of the differential equation (see [KU] sec. 19). The roots of this polynomial are called the **exponents of** $L(y) = 0$. It should be noted that since we may expand a solution y about any singular point, we have an indicial equation for, and hence exponents of $L(y) = 0$ at, each such point. Also, using the transformation $\{x = \frac{1}{t}, \frac{d}{dx} = -t^2\frac{d}{dt}\}$, one is able to consider ∞ as a singular point of $L(y) = 0$. Thus we may construct an indicial polynomial at the singular point ∞, and hence we are able to define a set of exponents at ∞.

Once again, the authors consider three mutually exclusive cases for the Galois group of a linear differential equation of order three which may have a liouvillian solution. This group must act reducibly, imprimitively, or primitively and in the primitive case the group must be finite. In the first two cases the authors put conditions on the exponents in order for $L(y) = 0$ to have a Galois group of this form. Then, assuming one is aware of the existence of at least one liouvillian solution, once one has used the necessary conditions to rule out the first two cases the third case is the only possibility. In this third case one may use the results in [SU2] to decide exactly which group is the Galois group.

In all three cases, the basic process is to assume we have a solution of the desired form and substitute its available power series expansion into the differential equation or the Ricatti equation associated to $L(y) = 0$. One then compares coefficients, specifically those of the leading term since this is where one finds possible exponents, and uses the results to impose the conditions. Assuming one is able to list the exponents of $L(y) = 0$, for example, when $L(y) = 0$ is of Fuchsian type, the authors prove the following facts which are used in developing the necessary conditions.

1. ([SU3] Lemma 2.2) Let $L(y) = 0$ be a Fuchsian equation of order n and $a_1, \ldots, a_n$ exponents at some point c. Then the m^{th} symmetric power $L^{\circledS m}(y) = 0$ is Fuchsian and the exponents at c are in the set $\{(\sum_{j=1}^{m} a_{i_j}) + t \;: i_j \in \{1, \ldots, n\}; t \in \mathbb{Z}\,; t \geq 0\}$.

2. ([SU3] Lemma 2.3) Let $L(y) = 0$ be a linear differential equation with coefficients in $\mathrm{C}(x)$. Assume all solutions of $L(y) = 0$ are algebraic over $\mathrm{C}(x)$. If a is any exponent at any point c, then $Na \in \mathbb{Z}$ for some integer N that is the order of an element of $Gal(L)$.

3. ([SU3] Lemma 3.1) If a linear differential equation $L(y) = 0$ of order n has a

solution of the form $P(x)\prod_i(x-c_i)^{a_i}$, where $P(x)\in \mathrm{C}[x]$, $c_i\neq\infty$ are the singular points of $L(y)=0$, a_i are exponents at x_i and $P(x)$ is a polynomial, then there is an exponent e_∞ at ∞ such that the sum $(\sum_i a_i)+e_\infty$ is a non-positive integer.

4. If $L(y)=0$ is Fuchsian, then any solution y such that y'/y is rational is of the form $P(x)\prod_i(x-c_i)^{a_i}$ where c_i is a singular point of $L(y)=0$ and each a_i is an exponent of $L(y)=0$ at c_i.

Consider the reducible case. Here the result gives information about the reducibility of the differential equation based on assumed knowledge of the existence of solutions of the above form.

If the equation $L(y)=0$ is reducible then either $L(y)$ or its adjoint equation $L^*(y)$ has a right factor of order 1. Therefore one of these two equations has a solution y such that its logarithmic derivative is rational. If this equation is Fuchsian, then fact 4 listed above tells us that $y=P(x)\prod_i(x-c_i)^{a_i}$ where c_i is a singular point of $L(y)=0$ and each a_i is an exponent of $L(y)=0$ at c_i. Applying Lemma 3.1 from fact 3 above, one obtains the following corollary which helps one to determine irreducibility in the Fuchsian case.

Corollary 3.1 *[SU3] Let $L(y)=0$ be a third order differential equation which is of Fuchsian type. If $L(y)=0$ is reducible, then for either $L(y)=0$ or $L^*(y)=0$ at each finite singular point c_i there are exponents a_i such that for some exponent e_∞ at ∞, the sum $(\sum_i a_i)+e_\infty$ is a non-positive integer.*

If the equation $L(y)=0$ is not Fuchsian yet one is able to show that the equation has no solution of a particular form, then one may apply the following necessary conditions, also based on Lemma 3.1, in order to determine if $L(y)=0$ is irreducible.

Theorem 3.13 *[SU3] Let $L(y)=y'''+Ay''+By'+Cy$ be a third order linear differential equation with coefficients in $C(x)$ such that $L(y)=0$ and its adjoint $L^*(y)=0$ have no solutions of the form $P(x)\prod_i(x-c_i)^{a_i}$, where $P(x)\in C[x]$, $c_i\neq\infty$ are regular singular points of $L(y)=0$ and a_i are exponents of $L(y)=0$ at c_i. If $L(y)$ factors over $C(x)$, then one of the following holds:*

1. *$C = 0$ and*
$$L(y) = \left(\frac{d^2}{dx^2} + A\frac{d}{dx} + B\right)\left(\left(\frac{d}{dx}\right)(y)\right)$$

2. *$C \neq 0$ and at some finite singular pont of $L(y) = 0$ the coefficients A, B, and C have exponents a, b, and c such that one of the following holds:*

   (a) *$A \neq 0, a \leq -2, 3a \leq c$ and (if $B \neq 0$ then $2a \leq b$).*

   (b) *$B \neq 0, b \leq -4, b \in 2\mathbb{Z}, b \leq \frac{2}{3}c$ and (if $A \neq 0$ then $b \leq 2a$).*

   (c) *$c \leq -6, c \in 3\mathbb{Z}$, (if $A \neq 0$ then $c \leq 3a$) and (if $B \neq 0$ then $c \leq \frac{3}{2}b$).*

   (d) *$AB \neq 0, a < -2, b < -4, b - a < -1, 2a < b$ and $2b - a \leq c$.*

   (e) *$A \neq 0, a < -2, c < -6, c - a < -4, (c - a) \in 2\mathbb{Z}, 3a < c$ and (if $b \neq 0$ then $a + c \leq 2b$).*

   (f) *$B \neq 0, c < -6, b < -4, c \leq b - 2, 3b < 2c$ and (if $A \neq 0$ then $2b < a + c$).*

3. *$C \neq 0$ and for A, B, and C, one of the conditions above holds for the exponents a_∞, b_∞ and c_∞ at the singular point ∞.*

The proof of the conditions can be summarized as follows. Assume that neither $L(y) = 0$ nor its adjoint has solutions of the form $P(x)\prod_i (x - c_i)^{a_i}$, where c_i is a singular point of $L(y)$ and the a_i's are the exponents of $L(y)$ at c_i. Suppose that the given third order differential equation is reducible. Then either $L(y)$ itself or its adjoint has a right factor of order 1. Suppose $L(y) = 0$ has a right factor of order 1. (The proof of the results for the other case is similar.) Then the Ricatti equation $R(u) = 0$ associated with $L(y) = 0$ has a solution $u \in \mathrm{C}(x)$. Therefore u has a Laurent series expansion. Either u has a pole of order bigger than one or not. If it does we expand u about that singular point, substitute this expansion into the equation $R(u) = 0$ and equate the corresponding coefficients. In this way the authors obtain the conditions on the exponents in this first case.

Now suppose that u does not have a pole of order bigger than one. Then we may write $u = \sum(\frac{\gamma_i}{x - c_i}) + Q(x)$, where $Q(x) \in \mathrm{C}[x]$ and $Q(x) \neq 0$. This implies that ∞ is a singular point of $L(y) = 0$. Again, by substituting u into $R(u) = 0$ and equating coefficients one is able to derive the stated conditions on the exponents of the rational coefficients of $L(y) = 0$.

Next the authors develop a list of necessary conditions in order for the Galois group of the given differential equation to be monomial. Note that in the order two and three cases, this is equivalent to the Galois group being imprimitive. One would check to see if the Galois group is reducible first. Hence in this case we assume that case 1 does not hold, ie., we assume the group is irreducible. We know from [SU1] that given an irreducible n-th order linear differential equation $L(y) = 0$, the Galois group $Gal(L)$ is a monomial group if and only if the associated equation $L^{\circledS n}(y) = 0$ has a solution which is quadratic over $C(x)$. This fact is used in the following form to prove the conditions necessary for an equation to be monomial.

Theorem 3.14 *([SU3]) If an irreducible linear differential equation $L(y) = 0$ of order n with coefficients in $C(x)$ has a monomial differential Galois group $Gal(L) \subseteq SL(n, C)$, then the n-th symmetric power $L^{\circledS n}(y) = 0$ of $L(y) = 0$ has a solution which is the square root of an element of $C(x)$. At any singularity, $L^{\circledS n}(y) = 0$ must have an exponent of the form $a/2$, where $a \in \mathbb{Z}$.*

The list of conditions are as follows.

Theorem 3.15 *Let $L(y) = 0$ be an irreducible linear differential equation of order n over $C(x)$ with monomial differential Galois group $Gal(L) \subseteq SL(n, C)$. The n-th symmetric power $L^{\circledS n}(y) = 0$ of $L(y) = 0$ must have a nontrivial solution of the form $P(x)\prod_i (x - c_i)^{a_i}$, where $P(x) \in C[x], c_i \neq \infty$ are the singular points of $L^{\circledS n}(y) = 0$ with exponents $\alpha_i \in \frac{1}{2}\mathbb{Z}$. In particular, any singularity of $L^{\circledS n}(y) = 0$ must have an exponent of the form $\frac{b}{2}$, where $b \in \mathbb{Z}$.*

If, in addition, $L(y) = 0$ is of Fuchsian type, then at each of the m singular points a_i of $L(y) = 0$ on the Riemann sphere there must be exponents $\alpha_{i,1}, \alpha_{i,2}, \ldots, \alpha_{i,n}$ of $L(y) = 0$ at a_i such that

- $(\sum_{j=1}^{n} \alpha_{i,j}) \in \frac{1}{2}\mathbb{Z}$ *, and*
- $(\sum_{i=1}^{m} \sum_{j=1}^{n} \alpha_{i,j}) \in \mathbb{Z}$ *and in non positive.*

Note that the necessary conditions for the Fuchsian case are more detailed. While we still have that the symmetric product has a solution of the given form, we also have a relationship between the exponents of $L(y) = 0$ and $L^{\circledS n}(y) = 0$ given by Lemma 2.2 [SU3]. Using this along with various relationships between the exponents many of which can be found in [KU] section 19, for example, $e_\infty = \sum_i e_i$, the conditions can be obtained.

In the finite primitive case, we know that all solutions of the differential equation are algebraic over k and hence $L(y) = 0$ is of Fuchsian type. Using Lemma 2.3 of [SU3] listed as fact 2 above, and the basic properties of each of the eight finite primitive subgroups of SL(3, C), the authors impose conditions on the exponents of $L(y) = 0$. An example of one of these conditions is as follows.

Theorem 3.16 *[SU3] Let L(y)=0 be an irreducible third order linear differential equation with Galois group a finite primitive group $Gal(L) \subset SL(3, C)$. Then $L(y) = 0$ must be a differential equation of Fuchsian type having 3 distinct rational exponents at any singularity. Furthermore, if c is any singularity of $L(y) = 0$ and e_1, e_2, e_3 are the exponents at any c, then one to the following holds:*

- *(Example) If $Gal(L) \approx A_6^{SL_3}$, then*
  1. *each e_i is of the form a_i/m_i, where $(a_i, m_i) = 1$, $a_i, m_i \in \mathbb{Z}$ and $lcm(m_1, m_2, ,_3) \in \{1, 2, 3, 4, 5, 6, 12, 15\}$ and*
  2. *there exists non negative integers n_1, n_2, n_3, such that $\sum_{i=1}^3 n_i = 6$ and $\sum_{i=1}^3 n_i e_i \in \mathbb{Z}$.*

Chapter 4

The Irreducible Linear Algebraic Groups of Degree 4

For the convenience of the reader, we have tabulated these groups at the end of this section (see Summary of Group Notation, pp. 71).

4.1 The Imprimitive Algebraic Groups of Degree Four

4.1.1 The Infinite Monomial Subgroups of $\mathbf{SL}(4, \mathbf{C})$

Definition: For any matrix, M, define the coordinate projections π_i by $\pi_i(M) = M_{i,i}$.

Lemma 4.1 *Let G be an algebraic subgroup of $GL(4, C)$ which acts monomially on C^4. Then the following hold.*

1. *G has a normal subgroup $H \subsetneq diag(GL(4, C))$ such that G/H is isomorphic to a transitive subgroup of S_4.*

2. *$\pi_1(G) = \pi_2(G) = \pi_3(G) = \pi_4(G)$*

Proof: Since G is imprimitive and monomial there exist vector subspaces $V_1, \ldots, V_4$ of C^4 such that $C^4 = V_1 \oplus V_2 \oplus V_3 \oplus V_4$ where the dimension of each V_i is one and each $g \in \mathrm{G}$ permutes the set $\{\mathrm{V}_1, \mathrm{V}_2, \mathrm{V}_3, \mathrm{V}_4\}$. Let $\alpha : \mathrm{G} \to \mathrm{S}_4$ be the homomorphism defined by the action of $g \in \mathrm{G}$ on the set $\{\mathrm{V}_1, \mathrm{V}_2, \mathrm{V}_3, \mathrm{V}_4\}$. Let $\mathrm{H} = ker(\alpha)$.

(Proof of 1) In any basis $\{v_1, v_2, v_3, v_4\}$ of V where $v_i \in \mathrm{V}_i$, H is in diagonal form. By the First Homomorphism Theorem, G/H is isomorphic to $Im(\alpha)$ which is a subgroup of S_4. Since G acts irreducibly, $Im(\alpha)$ must be transitive.

(Proof of 2) One can show that for any $g \in \mathrm{G}$ conjugation by g permutes the diagonal entries of $h \in \mathrm{H}$ according to $\alpha(g) \in \mathrm{S}_4$. Since H is a normal subgroup of G, and $Im(\alpha)$ is transitive we must have $\pi_1(\mathrm{G}) = \pi_2(\mathrm{G}) = \pi_3(\mathrm{G}) = \pi_4(\mathrm{G})$.

∎

Notation:

- We will denote by $\mathcal{P}_{\mathrm{G}}$ the image of G under the homomorphism α of Lemma 4.1.
- Let $\sigma \in \mathrm{S}_4$.
  - We will denote by P_σ the permutation matrix in $\mathrm{GL}(4, \mathrm{C})$ associated with σ.
  - We will denote by $\widetilde{\mathrm{P}}_\sigma$ any element in $\mathrm{P}_\sigma \mathrm{diag}(\mathrm{GL}(4, \mathbb{Z}_2))$ of determinant one.
  - We will denote by m_σ any element in $\mathrm{P}_\sigma \mathrm{diag}(\mathrm{GL}(4, \mathrm{C}))$.
- Let G be a monomial group. Let α be the map given in Lemma 4.1. Given any $h \in ker(\alpha)$ and any $\sigma \in \mathcal{P}_{\mathrm{G}}$ we will denote by h^σ the element resulting from permuting the diagonal entries of h according to the permutation σ. Note this action is equivalent to conjugating h by any element $m_\sigma \in \mathrm{P}_\sigma \mathrm{diag}(\mathrm{GL}(4, \mathrm{C}))$.

Remark: Note that this diagonal subgroup H of G is an algebraic group and hence is an algebraic variety. One can associate a Lie algebra with the connected component H^0 of H. The dimension of the variety H is equal to the dimension of the Lie algebra associated with H^0. We will refer to this number simply as the dimension of H and will use both of its interpretations.

Lemma 4.2 *Suppose that H is a connected one dimensional diagonal algebraic subgroup of $GL(4, C)$ such that $\pi_i(H) = C^*$ for $i = 1, 2, 3, 4$. Then there are integers, $n_1, \ldots, n_4$, such that H is equal to*

$$\left\{ \begin{bmatrix} a^{n_1} & 0 & 0 & 0 \\ 0 & a^{n_2} & 0 & 0 \\ 0 & 0 & a^{n_3} & 0 \\ 0 & 0 & 0 & a^{n_4} \end{bmatrix} : a \in U_n \right\}.$$

Proof: Let H be such a group. H $\cong$ C^* since it is a one dimensional torus. Let f be this isomorphism. Then each of the maps $\alpha_i = \pi_i \circ f^{-1}$ is an endomorphism of C^*. Therefore each of these maps is of the form $\sum c_i x^{n_i}$ with $c_i \in C^*$ and $n_i \in \mathbb{Z}$. Since H is a group we must have that each α_i is actually an integer power map. ∎

Lemma 4.3 *Suppose that H is a connected two dimensional diagonal algebraic subgroup of $SL(4, C)$ such that $\pi_i(H) = C^*$ for $i = 1, 2, 3, 4$. Suppose also that H is closed under conjugation by P_σ for each σ in any order four subgroup of S_4. Then H is conjugate to*

$$\left\{ \begin{bmatrix} a & 0 & 0 & 0 \\ 0 & b & 0 & 0 \\ 0 & 0 & a^{-1} & 0 \\ 0 & 0 & 0 & b^{-1} \end{bmatrix} : a, b \in C^* \right\}.$$

Proof: Consider Lie(H), the Lie algebra associated with H. Note that Theorem 5 in [ROS] implies that H can be defined via polynomials of the form $x^a y^b z^c w^d = 1$ where (a, b, c, d) is in some subset S of $\mathbb{Z}^4$. Therefore Lie(H) is contained in the solution space of $\{a\alpha + b\beta + c\gamma + d\delta = 0 \ : \ (a, b, c, d) \in \mathrm{S}\}$. Without loss of generality we may allow the powers a, b, c and d to be in $\mathbb{Q}$. Since H $\subset$ SL(4, C), $(1, 1, 1, 1) \in$ S. Therefore when looking at an arbitrary element of S other than $(1, 1, 1, 1)$, without loss of generality we may assume that $d = 0$. We may also assume that one of the powers is nonzero, since H is a proper subgroup of $(\mathrm{C}^*)^4$. Hence, given such an equation which vanishes on H, we may assume that it is in the form $x^a y^b z = 1$. That is, $(a, b, 1, 0) \in \mathrm{S}$. Now let $\sigma \in \mathrm{S}_4$ act on elements of S by permuting the entries. Recall that given any $h \in \mathrm{H}$ and $\sigma \in \mathcal{P}_{\mathrm{G}}$, $h^\sigma \in \mathrm{H}$ also. This implies that if $(a, b, 1, 0) \in \mathrm{S}$ then $\sigma(a, b, 1, 0) \in \mathrm{S}$ for each $\sigma \in \mathcal{P}_{\mathrm{G}}$.

Case 1. Suppose that $\mathcal{P}_{\mathrm{G}}$ is cyclic of order four. For now assume that $\mathcal{P}_{\mathrm{G}} =< (1234) >$. Then $(a, b, 1, 0) \in \mathrm{S}$ implies $(0, a, b, 1), (1, 0, a, b)$, and $(b, 1, 0, a)$ are also in S. Therefore $Lie(\mathrm{H})$ is contained in the nullspace of the following matrix.

$$M = \begin{bmatrix} 1 & 1 & 1 & 1 \\ a & b & 1 & 0 \\ 0 & a & b & 1 \\ 1 & 0 & a & b \\ b & 1 & 0 & a \end{bmatrix}$$

Since $Lie(\mathrm{H})$ has dimension two, the nullspace of this matrix has dimension ≥ 2. This implies that all 3 by 3 cofactors are zero. Upon choosing various submatrices we have that (a, b, c) is a solution of the following system of equations.

$$(1)\ (a^2 + 1)b = 0 \quad (2)\ 1 - b + b^2 - a = 0$$

Therefore $b = 0$ and $a = 1$ which implies that the equations defining H in this case have the form $xz = yw = 1$. Therefore $\mathrm{H} = \left\{ \begin{bmatrix} a & 0 & 0 & 0 \\ 0 & b & 0 & 0 \\ 0 & 0 & a^{-1} & 0 \\ 0 & 0 & 0 & b^{-1} \end{bmatrix} : a, b \in \mathrm{C}^* \right\}$.

Similarly, if $\mathcal{P}_{\mathrm{G}}$ contains either of the permutations (1243) or (1324) then applying the same argument to the matrices corresponding to M above yields defining equations for H of the form $xw = yz = 1$ and $xy = zw = 1$ respectively. Note that the group H defined by $xw = yz = 1$ is conjugate to $\mathrm{H} = \left\{ \begin{bmatrix} a & 0 & 0 & 0 \\ 0 & b & 0 & 0 \\ 0 & 0 & a^{-1} & 0 \\ 0 & 0 & 0 & b^{-1} \end{bmatrix} : a, b \in \mathrm{C}^* \right\}$ via $\mathrm{P}_{(34)}$ and the group H defined by $xy = zw = 1$ is conjugate to $\mathrm{H} = \left\{ \begin{bmatrix} a & 0 & 0 & 0 \\ 0 & b & 0 & 0 \\ 0 & 0 & a^{-1} & 0 \\ 0 & 0 & 0 & b^{-1} \end{bmatrix} : a, b \in \mathrm{C}^* \right\}$ via $\mathrm{P}_{(23)}$.

Case 2. Assuming that the Klein four group is in $\mathcal{P}_{\mathrm{G}}$ and using the same argument as above we can show that the exponents of a defining equation for H must satisfy the following equations.

$$(1)\ -2ab = 0 \quad (2)\ 1 - b - a = 0$$

Therefore either $b = 0$ and $a = 1$ or $a = 0$ and $b = 1$. Hence, the defining equations for such a group are of the form $xz = yw = 1$, $xy = zw = 1$, or $xw = yz = 1$. Each of the resulting groups is conjugate to $\left\{ \begin{bmatrix} a & 0 & 0 & 0 \\ 0 & b & 0 & 0 \\ 0 & 0 & a^{-1} & 0 \\ 0 & 0 & 0 & b^{-1} \end{bmatrix} : a, b \in \mathrm{C}^* \right\}$ via a permutation matrix in $\mathrm{GL}(4, \mathrm{C})$. ∎

Lemma 4.4 *Suppose that $H \subseteq diag(SL(4, C))$ is a finite extension of one of the following two groups.*

$$H_1 = \left\{ \begin{bmatrix} a & 0 & 0 & 0 \\ 0 & a^{-1} & 0 & 0 \\ 0 & 0 & a & 0 \\ 0 & 0 & 0 & a^{-1} \end{bmatrix} : a \in C^* \right\} \ H_2 = \left\{ \begin{bmatrix} a & 0 & 0 & 0 \\ 0 & b & 0 & 0 \\ 0 & 0 & a^{-1} & 0 \\ 0 & 0 & 0 & b^{-1} \end{bmatrix} : a, b \in C^* \right\}.$$

1. *Then there exists a positive integer, n, such that H is equal to either*

$$H_1^n = \left\{ \begin{bmatrix} \xi a & 0 & 0 & 0 \\ 0 & a^{-1} & 0 & 0 \\ 0 & 0 & \xi^{-1} a & 0 \\ 0 & 0 & 0 & a^{-1} \end{bmatrix} : a \in C^*, \xi \in U_n \right\}$$

   or

$$H_2^n = \left\{ \begin{bmatrix} \xi a & 0 & 0 & 0 \\ 0 & \xi^{-1} b & 0 & 0 \\ 0 & 0 & a^{-1} & 0 \\ 0 & 0 & 0 & b^{-1} \end{bmatrix} : a, b \in C^*, \xi \in U_n \right\}$$

   respectively, where U_n is the group of n^{th} roots of unity.

2. *Suppose furthermore that $H = ker(\alpha)$ where α is the map from a degree four monomial group, G, to S_4 as given in Lemma 4.1 and that $Im(\alpha)$ contains either the cyclic group $< (1234) >$ or K, the degree four permutation representation of the Klein four group. Then H is equal to either H_1^1, H_1^2 or H_2^n for some positive integer n.*

Proof: **(Proof of 1)** Let f_1 be the map from $\mathrm{diag}(\mathrm{GL}(4, \mathrm{C}))$ into $\mathrm{GL}(2, \mathrm{C})$ defined by $f\left(\begin{bmatrix} a & 0 & 0 & 0 \\ 0 & b & 0 & 0 \\ 0 & 0 & c & 0 \\ 0 & 0 & 0 & d \end{bmatrix} \right) = \begin{bmatrix} ab & 0 \\ 0 & cd \end{bmatrix}$. Let f_2 be the map from $\mathrm{diag}(\mathrm{GL}(4, \mathrm{C}))$ into

GL(2, C) defined by $f\left(\begin{bmatrix} a & 0 & 0 & 0 \\ 0 & b & 0 & 0 \\ 0 & 0 & c & 0 \\ 0 & 0 & 0 & d \end{bmatrix}\right) = \begin{bmatrix} ac & 0 \\ 0 & bd \end{bmatrix}$. Note that each f_i is a group homomorphism with $\mathrm{H}_i = ker(f_i)$ and $f_i(\mathrm{H}_i) \subseteq \mathrm{SL}(2, \mathrm{C})$. In each case since H_i is a normal subgroup of finite index in H, $f_i(\mathrm{H}_i)$ is a normal subgroup of finite index in $f_i(\mathrm{H})$. Therefore given $\begin{bmatrix} ab & 0 \\ 0 & cd \end{bmatrix} \in f_1(\mathrm{H})$ there exists a positive integer p such that $(ab)^p = (cd)^p = 1$ if $\mathrm{H}_1 \subseteq \mathrm{H}$, and similarly given $\begin{bmatrix} ac & 0 \\ 0 & bd \end{bmatrix} \in f_2(\mathrm{H})$ there exists a positive integer p such that $(ac)^p = (bd)^p = 1$ if $\mathrm{H}_2 \subseteq \mathrm{H}$. Therefore each $f_i(\mathrm{H})$ is contained in a group of the form $\left\{\begin{bmatrix} \xi & 0 \\ 0 & \eta \end{bmatrix} : \xi^p = \eta^p = 1\right\} \bigcap \mathrm{SL}(2, \mathrm{C})$. That is each $f_i(\mathrm{H}) \subseteq \left\{\begin{bmatrix} \xi & 0 \\ 0 & \xi^{-1} \end{bmatrix} : \xi^p = 1\right\}$ which is isomorphic to the finite cyclic group U_p. Since each f_i is a group homomorphism, $f_i(\mathrm{H})$ must be isomorphic to a subgroup of U_p. These are of the form U_n for some n dividing p. Therefore H is equal to one of the groups

$$\left\{\begin{bmatrix} \xi a & 0 & 0 & 0 \\ 0 & a^{-1} & 0 & 0 \\ 0 & 0 & \xi^{-1}a & 0 \\ 0 & 0 & 0 & a^{-1} \end{bmatrix} : a \in \mathrm{C}^*, \xi \in \mathrm{U}_n\right\} \text{ or}$$

$$\left\{\begin{bmatrix} \xi a & 0 & 0 & 0 \\ 0 & \xi^{-1}b & 0 & 0 \\ 0 & 0 & a^{-1} & 0 \\ 0 & 0 & 0 & b^{-1} \end{bmatrix} : a, b \in \mathrm{C}^*, \xi \in \mathrm{U}_n\right\}.$$

(Proof of 2) Now suppose further that $\mathrm{H} = ker(\alpha)$ where α is the map from an infinite monomial group, G, to S_4 as given in Lemma 4.1. Let $Im(\alpha) = \mathcal{P}_{\mathrm{G}}$. Now since $\mathrm{H} \lhd \mathrm{G}$, $h^\sigma \in \mathrm{H}$ for each $\sigma \in \mathcal{P}_{\mathrm{G}}$.

Case 1. H is a finite extension of H_1.

Case 1.(a) Assume that $\mathcal{P}_{\mathrm{G}}$ contains $<(1234)>$. Let $\sigma = (1234)$.

Then $h = \begin{bmatrix} \xi a & 0 & 0 & 0 \\ 0 & a^{-1} & 0 & 0 \\ 0 & 0 & \xi^{-1}a & 0 \\ 0 & 0 & 0 & a^{-1} \end{bmatrix} \in \mathrm{H}$ implies that $h^\sigma \in \mathrm{H}$. Therefore there exists a $c \in \mathrm{C}^*$ and an $\eta \in \mathrm{U}_n$ such that $\begin{bmatrix} a^{-1} & 0 & 0 & 0 \\ 0 & \xi a & 0 & 0 \\ 0 & 0 & a^{-1} & 0 \\ 0 & 0 & 0 & \xi^{-1}a \end{bmatrix} =$

$\begin{bmatrix} \eta c & 0 & 0 & 0 \\ 0 & c^{-1} & 0 & 0 \\ 0 & 0 & \eta^{-1}c & 0 \\ 0 & 0 & 0 & c^{-1} \end{bmatrix}$. A computation shows that this implies $\eta = \xi = \pm 1$.

Case 1.(b) If we assume that $\mathrm{K} \subseteq \mathcal{P}_{\mathrm{G}}$ then we have that $h \in \mathrm{H}$ implies that $h^\sigma \in \mathrm{H}$ where $\sigma = (12)(34)$. This implies exactly as in case 1 that $\eta = \xi = \pm 1$. The result follows.

Case 2. H is a finite extension of H_2. In this case one can show that given any $h \in \mathrm{H}$ $h^\sigma \in \mathrm{H}$ for each σ in either $<(1234)>$ or K. ∎

Notation: In addition to H_1^n and H_2^n of Lemma 4.4, we define H_3 to be diag(SL(4, C)).

Lemma 4.5 *Let G be an infinite algebraic subgroup of SL(4, C) which acts monomially on C^4. Let α be the map defined in Lemma 4.1. Then $H = ker(\alpha)$ is conjugate, via a permutation matrix in GL(4, C), to H_1^1, H_1^2, H_2^n for some positive integer n, or diag(SL(4, C)).*

Proof: We will determine the form of H via its dimension. Since G is an infinite algebraic subgroup of SL(4, C), H is an infinite algebraic subgroup of diag(SL(4, C)). Therefore the dimension of H is either 1,2, or 3. Since G is monomial, its image in S_4 is transitive and hence must contain a four cycle or the degree four permutation representation of the Klein four group.

Case 1. Suppose dim(H) = 1. Note that for an algebraic variety, H, dim(H) is defined to be $\dim(\mathrm{H}^0)$. Lemma 4.2 implies that $\mathrm{H}^0 = \left\{ \begin{bmatrix} a^{n_1} & 0 & 0 & 0 \\ 0 & a^{n_2} & 0 & 0 \\ 0 & 0 & a^{n_3} & 0 \\ 0 & 0 & 0 & a^{n_4} \end{bmatrix} : a \in \mathrm{U}_n \right\}$ for some set of integers, $\{n_1, \ldots, n_4\}$.

Case 1.(a) Suppose that $\mathcal{P}_{\mathrm{G}}$ contains a four cycle. Note that the groups $<(1243)>$ and $<(1324)>$ are conjugate to $<(1234)>$ via the permutations (34) and (23) respectively. For now assume that $\sigma = (1234) \in \mathcal{P}_{\mathrm{G}}$. Let $h \in \mathrm{H}^0$. Since H^0 is a normal subgroup of G, $ghg^{-1} \in \mathrm{H}^0$ for any $g \in \mathrm{G}$. Since $\sigma \in \mathcal{P}_{\mathrm{G}}$, there exist elements in G which map to σ, σ^2, and σ^3. Conjugation of h by these elements implies that h^{σ}, h^{σ^2}, and h^{σ^3} are in H^0. Therefore the following equations hold $\forall\ a \in \mathrm{C}^*$.

$$(1)\ a^{n_1^2} = a^{n_3^2} = a^{n_2 n_4} \quad (2)\ a^{n_2^2} = a^{n_4^2} = a^{n_1 n_3} \quad (3)\ a^{n_1 n_2} = a^{n_3 n_4}$$

Hence the following equations hold for $n_1, n_2, n_3, n_4 \in \mathbb{Z}$.

$$(1)\ n_1^2 = n_3^2 = n_2 n_4 \quad (2)\ n_2^2 = n_4^2 = n_1 n_3 \quad (3)\ n_1 n_2 = n_3 n_4$$

Any integer solution to such a system is of the form $n_1 = \pm n_2 = \pm n_3 = \pm n_4$. Since $\mathrm{H}^0 \subseteq \mathrm{SL}(4, \mathrm{C})$, we have that H^0 is conjugate to $\left\{ \begin{bmatrix} a^n & 0 & 0 & 0 \\ 0 & a^{-n} & 0 & 0 \\ 0 & 0 & a^n & 0 \\ 0 & 0 & 0 & a^{-n} \end{bmatrix} : a \in \mathrm{C}^* \right\}$ for some $n \in \mathbb{Z}$. Since C^* is algebraically closed $\left\{ \begin{bmatrix} a^n & 0 & 0 & 0 \\ 0 & a^{-n} & 0 & 0 \\ 0 & 0 & a^n & 0 \\ 0 & 0 & 0 & a^{-n} \end{bmatrix} : a \in \mathrm{C}^* \right\}$ $= \left\{ \begin{bmatrix} a & 0 & 0 & 0 \\ 0 & a^{-1} & 0 & 0 \\ 0 & 0 & a & 0 \\ 0 & 0 & 0 & a^{-1} \end{bmatrix} : a \in \mathrm{C}^* \right\}$. One can show in exactly the same manner that if a generator of either $< (1243) >$ or $< (1324) >$ is in $\mathcal{P}_{\mathrm{G}}$, then the resulting group H^0 is conjugate to $\left\{ \begin{bmatrix} a & 0 & 0 & 0 \\ 0 & a^{-1} & 0 & 0 \\ 0 & 0 & a & 0 \\ 0 & 0 & 0 & a^{-1} \end{bmatrix} : a \in \mathrm{C}^* \right\}$ via the permutation matrix $\mathrm{P}_{(34)}$ or $\mathrm{P}_{(23)}$ respectively.

Case 1.(b) Now assume that the Klein four group is in $\mathcal{P}_{\mathrm{G}}$. Let $\sigma_1 = (12)(34)$, $\sigma_2 = (13)(24)$, and $\sigma_3 = (14)(23)$. Again $h \in \mathrm{H}^0$ implies that $h^{\sigma_1}, h^{\sigma_2}$ and h^{σ_3} are all in H^0. Therefore the following hold $\forall a \in \mathrm{C}^*$.

$$(1)\ a^{n_1^2} = a^{n_2^2} = a^{n_3^2} = a^{n_4^2} \quad (2)\ a^{n_1 n_2} = a^{n_3 n_4} \quad (3)\ a^{n_1 n_3} = a^{n_2 n_4} \quad (4)\ a^{n_1 n_4} = a^{n_2 n_3}$$

Hence the following equations hold for $n_1, n_2, n_3, n_4 \in \mathbb{Z}$.

$$(1)\ n_1^2 = n_2^2 = n_3^2 = n_4^2 \quad (2)\ n_1 n_2 = n_3 n_4 \quad (3)\ n_1 n_3 = n_2 n_4 \quad (4)\ n_1 n_4 = n_2 n_3$$

Again, upon restriction to $\mathrm{SL}(4, \mathrm{C})$ we have that H^0 is conjugate in $\mathrm{GL}(4, \mathrm{C})$ to either

$$\left\{\begin{bmatrix} a^n & 0 & 0 & 0 \\ 0 & a^{-n} & 0 & 0 \\ 0 & 0 & a^n & 0 \\ 0 & 0 & 0 & a^{-n} \end{bmatrix} : a \in \mathrm{C}^*\right\}, \left\{\begin{bmatrix} a^n & 0 & 0 & 0 \\ 0 & a^{n} & 0 & 0 \\ 0 & 0 & a^{-n} & 0 \\ 0 & 0 & 0 & a^{-n} \end{bmatrix} : a \in \mathrm{C}^*\right\}$$

or

$$\left\{\begin{bmatrix} a^n & 0 & 0 & 0 \\ 0 & a^{-n} & 0 & 0 \\ 0 & 0 & a^{-n} & 0 \\ 0 & 0 & 0 & a^{n} \end{bmatrix} : a \in \mathrm{C}^*\right\}.$$

The latter two are conjugate to the first via $\mathrm{P}_{(23)}$ and $\mathrm{P}_{(34)}$ respectively. Therefore we have that H^0 is conjugate to $\left\{\begin{bmatrix} a & 0 & 0 & 0 \\ 0 & a^{-1} & 0 & 0 \\ 0 & 0 & a & 0 \\ 0 & 0 & 0 & a^{-1} \end{bmatrix} : a \in \mathrm{C}^*\right\}$ via a permutation matrix in $\mathrm{GL}(4, \mathrm{C})$.

Now since H is an algebraic group, $[\mathrm{H} : \mathrm{H}^0]$ is finite. Therefore Lemma 4.4 implies that H is conjugate to one of the first two groups listed in this result.

Case 2. Suppose $\dim(\mathrm{H}) = 2$. Then $\dim(\mathrm{H}^0) = 2$. Lemma 4.3 implies that H^0 is conjugate to $\left\{\begin{bmatrix} a & 0 & 0 & 0 \\ 0 & b & 0 & 0 \\ 0 & 0 & a^{-1} & 0 \\ 0 & 0 & 0 & b^{-1} \end{bmatrix} : a, b \in \mathrm{C}^*\right\}$ via a permutation matrix in $\mathrm{GL}(4, \mathrm{C})$. Since H is an algebraic group $[\mathrm{H} : \mathrm{H}^0]$ is finite. Lemma 4.4 then implies that there exists a positive integer m such that H is conjugate to $\left\{\begin{bmatrix} \xi a & 0 & 0 & 0 \\ 0 & \xi^{-1} b & 0 & 0 \\ 0 & 0 & a^{-1} & 0 \\ 0 & 0 & 0 & b^{-1} \end{bmatrix} : a, b \in \mathrm{C}^*, \xi \in \mathrm{U}_n\right\}$.

In the proof of Lemma 4.4 we find that this conjugation is via a permutation matrix.

Case 3. Suppose that $\dim(\mathrm{H}) = 3$. Then H is all of $\mathrm{diag}(\mathrm{SL}(4, \mathrm{C}))$ which is clearly closed under conjugation by any element in $\mathrm{P}_\sigma \mathrm{diag}(\mathrm{SL}(4, \mathrm{C}))$ for any $\sigma \in \mathrm{S}_4$. ∎

Lemma 4.6 *Let G be an infinite algebraic subgroup of SL(4, C) which acts monomially on C^4. Suppose $\mathcal{P}_{\mathrm{G}} = A_4$ or S_4. Then $H = ker(\alpha) = diag(SL(4, C))$.*

Proof: By Lemma 4.5 H is conjugate to H_1^1, H_1^2, H_2^n for some integer n, or H_3. Recall that $\mathcal{P}_{\mathrm{G}} = Im(\alpha)$ acts on H via conjugation by permuting the diagonal elements. Note that both A_4 and S_4 contain the three cycle, (123). It is clear that the action of this element does not normalize H_1^m or H_2^n for any positive integer n but does normalize H_3. ∎

Theorem 4.1 *Let G be an infinite algebraic subgroup of $SL(4,C)$ which acts monomially on C^4. Then G is conjugate to $\bigcup_{\tilde{m_\sigma}\in\mathcal{S}} \tilde{m_\sigma} H$ where one of the following holds:*

1. *$H = H_1^1, H_1^2, H_2^n$ or H_3, and $\mathcal{S}$ is one of the following sets:*

$$\mathcal{S}_1 = \left\{ \begin{bmatrix} 1&0&0&0\\0&1&0&0\\0&0&1&0\\0&0&0&1 \end{bmatrix}, \begin{bmatrix} 0&0&0&-1\\1&0&0&0\\0&1&0&0\\0&0&1&0 \end{bmatrix}, \begin{bmatrix} 0&0&-1&0\\0&0&0&-1\\1&0&0&0\\0&1&0&0 \end{bmatrix}, \begin{bmatrix} 0&1&0&0\\0&0&-1&0\\0&0&0&1\\1&0&0&0 \end{bmatrix} \right\}$$

$$\mathcal{S}_2 = \left\{ \begin{bmatrix} 1&0&0&0\\0&1&0&0\\0&0&1&0\\0&0&0&1 \end{bmatrix}, \begin{bmatrix} 0&0&0&-1\\1&0&0&0\\0&1&0&0\\0&0&1&0 \end{bmatrix}, \begin{bmatrix} 0&0&-1&0\\0&0&0&-1\\1&0&0&0\\0&1&0&0 \end{bmatrix}, \begin{bmatrix} 0&1&0&0\\0&0&-1&0\\0&0&0&1\\1&0&0&0 \end{bmatrix}, \right.$$

$$\left. \begin{bmatrix} 0&1&0&0\\1&0&0&0\\0&0&0&-1\\0&0&-1&0 \end{bmatrix}, \begin{bmatrix} 0&0&0&1\\0&0&1&0\\0&1&0&0\\1&0&0&0 \end{bmatrix} \begin{bmatrix} 0&0&1&0\\0&1&0&0\\1&0&0&0\\0&0&0&-1 \end{bmatrix}, \begin{bmatrix} 1&0&0&0\\0&0&0&1\\0&0&-1&0\\0&1&0&0 \end{bmatrix} \right\}$$

$$\mathcal{S}_3 = \left\{ \begin{bmatrix} 1&0&0&0\\0&1&0&0\\0&0&1&0\\0&0&0&1 \end{bmatrix}, \begin{bmatrix} 0&1&0&0\\1&0&0&0\\0&0&0&1\\0&0&1&0 \end{bmatrix}, \begin{bmatrix} 0&0&1&0\\0&0&0&1\\1&0&0&0\\0&1&0&0 \end{bmatrix}, \begin{bmatrix} 0&0&0&1\\0&0&1&0\\0&1&0&0\\1&0&0&0 \end{bmatrix} \right\}$$

$$\mathcal{S}_4 = \left\{ \begin{bmatrix} 1&0&0&0\\0&1&0&0\\0&0&1&0\\0&0&0&1 \end{bmatrix}, \begin{bmatrix} 0&-1&0&0\\1&0&0&0\\0&0&0&-1\\0&0&1&0 \end{bmatrix}, \begin{bmatrix} 0&0&1&0\\0&0&0&-1\\1&0&0&0\\0&-1&0&0 \end{bmatrix}, \begin{bmatrix} 0&0&0&-1\\0&0&1&0\\0&-1&0&0\\1&0&0&0 \end{bmatrix} \right\}$$

2. *$H = H_3$ and $\mathcal{S} = \{\widetilde{P}_\sigma : \sigma \in \mathcal{P}_{\mathrm{G}}\}$ where $\mathcal{P}_{\mathrm{G}} = A_4$ or S_4, and each $\widetilde{P}_\sigma$ is any element in $P_\sigma diag(GL(4,\mathbb{Z}_2))$ of determinant one.*

Proof: By Lemma 4.1 G has a normal subgroup $\mathrm{H} \subseteq \mathrm{diag}(\mathrm{SL}(4,\mathrm{C}))$ such that G/H is isomorphic to a transitive subgroup of S_4 which we will denote by $\mathcal{P}_{\mathrm{G}}$. The transitive subgroups of S_4 are the three order four cyclic groups, the three order eight groups each of which contains one of the order four cyclic groups, the Klein four group, A_4 and S_4. Lemma 4.5 implies that H must be conjugate to $\mathrm{H}_1^1, \mathrm{H}_1^2, \mathrm{H}_2^n$ for some positive integer n or H_3. Hence, given a pair H and $\mathcal{P}_{\mathrm{G}}$ we can write G as $\bigcup_{\sigma\in\mathcal{P}_{\mathrm{G}}} m_\sigma \mathrm{H}$ where $m_\sigma \in \mathrm{P}_\sigma \mathrm{diag}(\mathrm{GL}(4,\mathrm{C}))$.

Remark: In the proof of Lemma 4.5 we show that the groups $<(1243)>$ and $<(1324)>$ are conjugate to $<(1234)>$ via (34) and (23) respectively. We also show for each possible dimension of this diagonal subgroup H of G, the subgroups H associated with $\mathcal{P}_{\mathrm{G}} =<(1243)>$ and $<(1324)>$ are conjugate to those associated with $\mathcal{P}_{\mathrm{G}} =<$

$(1234)>$ via $\mathrm{P}_{(34)}$ and $\mathrm{P}_{(23)}$ respectively. Hence any monomial group $\mathrm{G} \subset \mathrm{SL}(4,\mathrm{C})$ where $\mathcal{P}_\mathrm{G}$ is an order four cyclic group will be conjugate in $\mathrm{GL}(4,\mathrm{C})$ to one where $\mathcal{P}_\mathrm{G} =<(1234)>$. Similarly one can show that any monomial subgroup of $\mathrm{SL}(4,\mathrm{C})$ with permutation action an order eight subgroup of S_4 is conjugate to one with permutation action given by $<(1234),(12)(34)>$. Hence we need only look at the cases where $\mathcal{P}_\mathrm{G} =<(1234)>,\ <(1234),(12)(34)>,\ \mathrm{K}, \mathrm{A}_4$ or S_4.

In each case, using the fact that $\mathrm{G}/\mathrm{H} = \mathcal{P}_\mathrm{G}$, we wil find a diagonal matrix, D, such that $\mathrm{D} m_\sigma \mathrm{D}^{-1} \in \tilde{m_\sigma}\mathrm{H}$ where $\tilde{m_\sigma}$ is in the given set $\mathcal{S}$. Furthermore, we will show that such a matrix will actually fix H and this will imply that G is conjugate to $\bigcup_{\tilde{m_\sigma}\in\mathcal{S}} \tilde{m_\sigma}\mathrm{H}$

Case 1. Suppose $\mathcal{P}_\mathrm{G} =<(1234)>$. Let $\sigma_1 = (1234)$, $\sigma_2 = (13)(24)$ and $\sigma_3 = (1432)$. For $i = 1,2,3$ let $m_{\sigma_i} \in \widetilde{\mathrm{P}}_{\sigma_i}\mathrm{diag}(\mathrm{SL}(4,\mathrm{C}))$. Since $\mathcal{P}_\mathrm{G} =<(1234)>$, in each of the following subcases the matrices m_{σ_i} must satisfy the following conditions: $(1) m_{\sigma_1}^2 \in m_{\sigma_2}\mathrm{H}$, and $(2) m_{\sigma_1}m_{\sigma_2} \in m_{\sigma_3}\mathrm{H}$.

Case 1.(a) Suppose that $\mathrm{H} = \mathrm{H}_1^n = \left\{ \begin{bmatrix} \xi a & 0 & 0 & 0 \\ 0 & a^{-1} & 0 & 0 \\ 0 & 0 & \xi^{-1}a & 0 \\ 0 & 0 & 0 & a^{-1} \end{bmatrix} : a \in \mathrm{C}^*, \xi \in \mathrm{U}_n \right\}$ where $n = 1$ or 2. Since the coordinate projection, π_1, of H is onto C^*, we may assume that $m_{\sigma_1} = \begin{bmatrix} 0 & 0 & 0 & -(a_1a_2)^{-1} \\ 1 & 0 & 0 & 0 \\ 0 & a_1 & 0 & 0 \\ 0 & 0 & a_2 & 0 \end{bmatrix}$, $m_{\sigma_2} = \begin{bmatrix} 0 & 0 & b_2 & 0 \\ 0 & 0 & 0 & (b_1b_2)^{-1} \\ 1 & 0 & 0 & 0 \\ 0 & b_1 & 0 & 0 \end{bmatrix}$, and $m_{\sigma_3} = \begin{bmatrix} 0 & c_1 & 0 & 0 \\ 0 & 0 & c_2 & 0 \\ 0 & 0 & 0 & -(c_1c_2)^{-1} \\ 1 & 0 & 0 & 0 \end{bmatrix}$, where each a_i, b_i and c_i is in C^*. A computation shows that conditions (1) and (2) imply that if we choose $m_{\sigma_1} = \begin{bmatrix} 0 & 0 & 0 & -(a_1a_2)^{-1} \\ 1 & 0 & 0 & 0 \\ 0 & a_1 & 0 & 0 \\ 0 & 0 & a_2 & 0 \end{bmatrix}$, then there exist $\eta, \xi \in \mathrm{U}_n$ such that

$$m_{\sigma_2} = \begin{bmatrix} 0 & 0 & -\eta a_1^{-2} & 0 \\ 0 & 0 & 0 & -a_2^{-1} \\ \eta^{-1} & 0 & 0 & 0 \\ 0 & a_1^2a_2 & 0 & 0 \end{bmatrix} \text{ and } m_{\sigma_3} = \begin{bmatrix} 0 & \xi a_1a_2 & 0 & 0 \\ 0 & 0 & -a_1^{-2}a_2^{-1} & 0 \\ 0 & 0 & 0 & \xi^{-1}a_1 \\ 1 & 0 & 0 & 0 \end{bmatrix}.$$

Let $\mathrm{D} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & a_1^{-1} & 0 \\ 0 & 0 & 0 & (a_1a_2)^{-1} \end{bmatrix}$. Then $\mathrm{D}m_{\sigma_1}\mathrm{D}^{-1} = \begin{bmatrix} 0 & 0 & 0 & -1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}$, which is clearly in $\begin{bmatrix} 0 & 0 & 0 & -1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}$ H. $\mathrm{D}m_{\sigma_2}\mathrm{D}^{-1} = \begin{bmatrix} 0 & 0 & -\eta a_1^{-1} & 0 \\ 0 & 0 & 0 & -a_1 \\ \eta^{-1}a_1^{-1} & 0 & 0 & 0 \\ 0 & a_1 & 0 & 0 \end{bmatrix}$,

which is in $\begin{bmatrix} 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}$ H, and finally,

$\mathrm{D}m_{\sigma_3}\mathrm{D}^{-1} = \begin{bmatrix} 0 & \xi a_1 a_2 & 0 & 0 \\ 0 & 0 & -(a_1a_2)^{-1} & 0 \\ 0 & 0 & 0 & \xi^{-1}a_1a_2 \\ (a_1a_2)^{-1} & 0 & 0 & 0 \end{bmatrix}$ which is in

$\begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \end{bmatrix}$ H. Hence $\mathrm{G} = \bigcup_{\sigma \in <(1234)>} m_\sigma \mathrm{H}$

is conjugate to $\cup_{\tilde{m_\sigma} \in \mathcal{S}_1} \tilde{m_\sigma}\mathrm{H}$.

Case 1.(b) $\mathrm{H} = \mathrm{H}_2^n = \left\{ \begin{bmatrix} \xi a & 0 & 0 & 0 \\ 0 & \xi^{-1}b & 0 & 0 \\ 0 & 0 & a^{-1} & 0 \\ 0 & 0 & 0 & b^{-1} \end{bmatrix} : a, b \in \mathrm{C}^*, \xi \in \mathrm{U}_n \right\}$. Since both of the coordinate projections π_1 and π_2 are onto C^*, we may assume that $m_{\sigma_1} = \begin{bmatrix} 0 & 0 & 0 & -a^{-1} \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & a & 0 \end{bmatrix}$, $m_{\sigma_2} = \begin{bmatrix} 0 & 0 & b & 0 \\ 0 & 0 & 0 & b^{-1} \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}$, and $m_{\sigma_3} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & c & 0 \\ 0 & 0 & 0 & -c^{-1} \\ 1 & 0 & 0 & 0 \end{bmatrix}$.
Then a computation shows that conditions (1) and (2) imply that there exist $\eta, \xi \in \mathrm{U}_n$ such that $m_{\sigma_2} = \begin{bmatrix} 0 & 0 & -\eta & 0 \\ 0 & 0 & 0 & -\eta^{-1} \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}$ and $m_{\sigma_3} = \begin{bmatrix} 0 & \xi & 0 & 0 \\ 0 & 0 & -\xi^{-1}a & 0 \\ 0 & 0 & 0 & a^{-1} \\ 1 & 0 & 0 & 0 \end{bmatrix}$.

Now choose $\mathrm{D} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & a^{-1} \end{bmatrix}$. One can easily show

$\mathrm{D}m_{\sigma_1}\mathrm{D}^{-1} = \begin{bmatrix} 0 & 0 & 0 & -1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}$, $\mathrm{D}m_{\sigma_2}\mathrm{D}^{-1} \in \begin{bmatrix} 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}$ H and $\mathrm{D}m_{\sigma_3}\mathrm{D}^{-1} \in$ $\begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \end{bmatrix}$ H. Therefore G is conjugate to $\bigcup_{\tilde{m_\sigma} \in \mathcal{S}_1} \tilde{m_\sigma}\mathrm{H}$.

Case 1.(c) $\mathrm{H} = \mathrm{H}_3 = \mathrm{diag}(\mathrm{SL}(4,\mathrm{C}))$. In this case any choice of m_{σ_i} of determinant one is already in $\tilde{m_\sigma}\mathrm{H}$ since $\tilde{m_\sigma}\mathrm{H} = \tilde{\mathrm{P}}_\sigma \mathrm{diag}(\mathrm{SL}(4,\mathrm{C}))$. Hence G is actually equal to $\bigcup_{\tilde{m_\sigma}\in\mathcal{S}_1} \tilde{m_\sigma}\mathrm{H}$.

Case 2. Suppose $\mathcal{P}_{\mathrm{G}} = \mathrm{K}$. Let $\sigma_1 = (12)(34)$, $\sigma_2 = (13)(24)$ and $\sigma_3 = (14)(23)$. For $i = 1,2,3$ let $m_{\sigma_i} \in \mathrm{P}_{\sigma_i}\mathrm{diag}(\mathrm{SL}(4,\mathrm{C}))$ have determinant one. Since $\mathcal{P}_{\mathrm{G}} = \mathrm{K}$, in each of the following subcases the matrices m_{σ_i} must satisfy the following conditions:

$$(1) m_{\sigma_1}m_{\sigma_2} \in m_{\sigma_3}\mathrm{H} \ \text{ and } \ (2) m_{\sigma_1}^2, m_{\sigma_2}^2, m_{\sigma_3}^2 \in \mathrm{H}.$$

Case 2.(a) $\mathrm{H} = \mathrm{H}_1^n$ with $n = 1$ or 2. Since the coordinate projection π_1 of H is onto C^*, we may assume that $m_{\sigma_1} = \begin{bmatrix} 0 & a_1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & (a_1a_2)^{-1} \\ 0 & 0 & a_2 & 0 \end{bmatrix}$, $m_{\sigma_2} = \begin{bmatrix} 0 & 0 & b_2 & 0 \\ 0 & 0 & 0 & (b_1b_2)^{-1} \\ 1 & 0 & 0 & 0 \\ 0 & b_1 & 0 & 0 \end{bmatrix}$, and $m_{\sigma_3} = \begin{bmatrix} 0 & 0 & 0 & (c_1c_2)^{-1} \\ 0 & 0 & c_2 & 0 \\ 0 & c_1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}$, where each a_i, b_i and c_i is in C^*. A computation shows that conditions (1) and (2) imply that there are elements $a, c \in \mathrm{C}^*$ and $\eta, \xi \in \mathrm{U}_n$ such that either

$$m_{\sigma_1} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & a^{-1} \\ 0 & 0 & a & 0 \end{bmatrix}, \ m_{\sigma_2} = \begin{bmatrix} 0 & 0 & \eta a c^{-1} & 0 \\ 0 & 0 & 0 & a^{-1} \\ \eta^{-1} & 0 & 0 & 0 \\ 0 & c & 0 & 0 \end{bmatrix}$$
$$\text{and } m_{\sigma_3} = \begin{bmatrix} 0 & 0 & 0 & \xi \\ 0 & 0 & c^{-1} & 0 \\ 0 & \xi^{-1}c & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}$$

or

$$m_{\sigma_1} = \begin{bmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -a^{-1} \\ 0 & 0 & a & 0 \end{bmatrix}, \ m_{\sigma_2} = \begin{bmatrix} 0 & 0 & -\eta c^{-1} a & 0 \\ 0 & 0 & 0 & a^{-1} \\ \eta^{-1} & 0 & 0 & 0 \\ 0 & -c & 0 & 0 \end{bmatrix}$$
$$\text{and } m_{\sigma_3} = \begin{bmatrix} 0 & 0 & 0 & -\xi \\ 0 & 0 & -c^{-1} & 0 \\ 0 & \xi^{-1}c & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}.$$

These two possible sets give rise to the two groups listed.

Let $\mathrm{D} = \begin{bmatrix} a^{\frac{1}{2}}c^{\frac{3}{2}} & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & ac & 0 \\ 0 & 0 & 0 & a^{\frac{-1}{2}}c^{\frac{-1}{2}} \end{bmatrix}$ Then for the first set we have the following.

$$\mathrm{D}m_{\sigma_1}\mathrm{D}^{-1} = \begin{bmatrix} 0 & a^{\frac{1}{2}}c^{\frac{3}{2}} & 0 & 0 \\ a^{\frac{-1}{2}}c^{\frac{-3}{2}} & 0 & 0 & 0 \\ 0 & 0 & 0 & a^{\frac{1}{2}}c^{\frac{3}{2}} \\ 0 & 0 & a^{\frac{-1}{2}}c^{\frac{-3}{2}} & 0 \end{bmatrix} \text{ which is in } \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix} \mathrm{H}.$$

$$\mathrm{D}m_{\sigma_2}\mathrm{D}^{-1} = \begin{bmatrix} 0 & 0 & \eta a^{\frac{1}{2}}c^{\frac{-1}{2}} & 0 \\ 0 & 0 & 0 & a^{\frac{-1}{2}}c^{\frac{1}{2}} \\ \eta^{-1}a^{\frac{1}{2}}c^{\frac{-1}{2}} & 0 & 0 & 0 \\ 0 & a^{\frac{-1}{2}}c^{\frac{1}{2}} & 0 & 0 \end{bmatrix}, \text{ which is in } \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix} \mathrm{H},$$

and finally, $\mathrm{D}m_{\sigma_3}\mathrm{D}^{-1} = \begin{bmatrix} 0 & 0 & 0 & \xi ac^2 \\ 0 & 0 & a^{-1}c^{-2} & 0 \\ 0 & \xi^{-1}ac^2 & 0 & 0 \\ a^{-1}c^{-2} & 0 & 0 & 0 \end{bmatrix}$ is in $\begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}$ H.

Hence $\mathrm{G} = \cup_{\sigma\in\mathrm{K}} m_\sigma \mathrm{H}$ is conjugate to $\bigcup_{\tilde{m_\sigma}\in\mathcal{S}_3} \tilde{m_\sigma}\mathrm{H}$. One can show that conjugation by the same matrix D takes each m_{σ_i} in the second set into $\tilde{m_\sigma}\mathrm{H}$ for some $\tilde{m_\sigma} \in \mathcal{S}_4$. Therefore G may be conjugate to $\bigcup_{\tilde{m_\sigma}\in\mathcal{S}_4} \tilde{m_\sigma}\mathrm{H}$.

Case 2.(b) $\mathrm{H} = \mathrm{H}_2^n$. Since the coordinate projections, π_1 and π_2 of H are onto C^*, we may assume that $m_{\sigma_1} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & a^{-1} \\ 0 & 0 & a & 0 \end{bmatrix}$, $m_{\sigma_2} = \begin{bmatrix} 0 & 0 & b & 0 \\ 0 & 0 & 0 & b^{-1} \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}$,

and $m_{\sigma_3} = \begin{bmatrix} 0 & 0 & 0 & c^{-1} \\ 0 & 0 & c & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}$, where a, b and c are in C^*. A computation then shows that conditions (1) and (2) imply that given m_{σ_1}, as above, we again get two possible pairs for m_{σ_2} and m_{σ_3}. In particular there exist $\eta, \xi \in \mathrm{U}_n$ such that either $m_{\sigma_2} = \begin{bmatrix} 0 & 0 & \eta & 0 \\ 0 & 0 & 0 & \eta^{-1} \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}$ and $m_{\sigma_3} = \begin{bmatrix} 0 & 0 & 0 & \xi a^{-1} \\ 0 & 0 & \xi^{-1}a & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}$

or $m_{\sigma_2} = \begin{bmatrix} 0 & 0 & -\eta & 0 \\ 0 & 0 & 0 & -\eta^{-1} \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}$ and $m_{\sigma_3} = \begin{bmatrix} 0 & 0 & 0 & -\xi a^{-1} \\ 0 & 0 & -\xi^{-1}a & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}$. Let $\mathrm{D} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & a^{-1} \end{bmatrix}$. Then given the first set we have $\mathrm{D}m_{\sigma_1}\mathrm{D}^{-1} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix}$

which is clearly in $\begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix}$H. $\mathrm{D}m_{\sigma_2}\mathrm{D}^{-1} = \begin{bmatrix} 0 & 0 & \eta & 0 \\ 0 & 0 & 0 & \eta^{-1}a \\ 1 & 0 & 0 & 0 \\ 0 & a^{-1} & 0 & 0 \end{bmatrix}$, which is in $\begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}$H, and finally, $\mathrm{D}m_{\sigma_3}\mathrm{D}^{-1} = \begin{bmatrix} 0 & 0 & 0 & \xi \\ 0 & 0 & \xi^{-1}a & 0 \\ 0 & 1 & 0 & 0 \\ a^{-1} & 0 & 0 & 0 \end{bmatrix}$ is in $\begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}$H. Hence $\mathrm{G} = \bigcup_{\sigma \in \mathrm{K}} m_\sigma \mathrm{H}$ is conjugate to $\bigcup_{\tilde{m_\sigma} \in \mathcal{S}_3} \tilde{m_\sigma}\mathrm{H}$. One can show that conjugation by the same matrix D takes each m_{σ_i} in the second set into $\tilde{m_\sigma}\mathrm{H}$ for some $\tilde{m_\sigma} \in \mathcal{S}_4$. Therefore G may be conjugate to $\bigcup_{\tilde{m_\sigma} \in \mathcal{S}_4} \tilde{m_\sigma}\mathrm{H}$.

Case 2.(c) $\mathrm{H} = \mathrm{H}_3$. As in Case 1(c), any choice of m_{σ_i} is already in $\tilde{m_\sigma}\mathrm{H}$ since $\tilde{m_\sigma}\mathrm{H} = \widetilde{\mathrm{P}}\mathrm{diag}(\mathrm{SL}(4,\mathrm{C}))$. Hence G is actually equal to $\bigcup_{\tilde{m_\sigma} \in \mathcal{S}_3} \tilde{m_\sigma}\mathrm{H}$.

Case 3. Suppose $\mathcal{P}_\mathrm{G} =< (1234), (12)(34) >$. As in the cases where $\mathcal{P}_\mathrm{G} =< (1234) >$ or K, given any set of elements $m_{\sigma_i} \in \widetilde{\mathrm{P}}_{\sigma_i}\mathrm{diag}(\mathrm{SL}(4,\mathrm{C}))$ where $\sigma_i \in \mathcal{P}_\mathrm{G}$, in each case for H we are able to restrict the form of these elements using group relations from $\mathcal{P}_\mathrm{G}$. Once this process is complete one can find a diagonal element D which conjugates each m_{σ_i} into $\tilde{m_\sigma}\mathrm{H}$ where $\tilde{m_\sigma}$ is in the set $\mathcal{S}_2$ of the theorem. Since this matrix D fixes H, the group G is conjugate to $\bigcup_{\tilde{m_\sigma} \in \mathcal{S}_2} \tilde{m_\sigma}\mathrm{H}$.

Case 4. Suppose $\mathcal{P}_\mathrm{G}$ is either A_4 or S_4. Lemma 4.6 implies that the only possibility for H in either of these cases is $\mathrm{H} = \mathrm{diag}(\mathrm{SL}(4,\mathrm{C}))$. The proof for this H is then the same as either of the cases 1(c) or 2(c) above. ∎

4.1.2 The Imprimitive Nonmonomial Subgroups of $\mathbf{SL}(4,\mathbf{C})$

Lemma 4.7 *Let G be an imprimitive, nonmonomial algebraic subgroup of $SL(4,C)$. Then the following hold:*

1. *G has a normal subgroup H of index 2 such that $H \subseteq \left\{ \begin{bmatrix} A & 0 \\ 0 & B \end{bmatrix} : A, B \in GL(2,C) \right\}$.*
2. *If we define the coordinate projections π_1 and π_2 of H by $\pi_1\left(\begin{bmatrix} A & 0 \\ 0 & B \end{bmatrix}\right) = A$ and $\pi_2\left(\begin{bmatrix} A & 0 \\ 0 & B \end{bmatrix}\right) = B$, then $\pi_1(H) \cong \pi_2(H) \cong T$ where T is a primitive linear algebraic subgroup of $GL(2,C)$.*
3. *There exists an element $\gamma \in G\backslash H$ of the form $\gamma = \begin{bmatrix} 0 & g_1 \\ g_2 & 0 \end{bmatrix}$, where $g_i \in GL(2,C)$, $g_1 g_2 \in SL(2,C)$.*

Proof: **(Proof of 1)** Since G is imprimitive and nonmonomial there exist vector subspaces, V_1 and V_2, of C^4 such that $\mathrm{C}^4 = \mathrm{V}_1 \oplus \mathrm{V}_2$ where $\dim \mathrm{V}_1 = \dim \mathrm{V}_2 = 2$ and each $g \in \mathrm{G}$ permutes the set $\{\mathrm{V}_1, \mathrm{V}_2\}$. Let $\alpha : \mathrm{G} \to \mathrm{S}_2$ be the homomorphism defined by the action of $g \in \mathrm{G}$ on the set $\{\mathrm{V}_1, \mathrm{V}_2\}$. Let $\mathrm{H} = ker(\alpha)$. Since G is irreducible, $Im(\alpha) = \mathrm{S}_2$. Therefore $\mathrm{G}/\mathrm{H} \cong \mathrm{S}_2$. In particular, $[\mathrm{G} : \mathrm{H}] = 2$. By definition H acts reducibly on C^4. Therefore $\mathrm{H} \subset \left\{ \begin{bmatrix} \mathrm{A} & 0 \\ 0 & \mathrm{B} \end{bmatrix} : \mathrm{A}, \mathrm{B} \in \mathrm{GL}(2,\mathrm{C}) \right\}$.

(Proof of 2) A result by Dixon in [DI] pp.25 implies that $\mathrm{H}\mid_{\mathrm{V}_1}$ and $\mathrm{H}\mid_{\mathrm{V}_2}$ are isomorphic as linear algebraic groups. Therefore if we define the projection maps on H by $\pi_1\left(\begin{bmatrix} \mathrm{A} & 0 \\ 0 & \mathrm{B} \end{bmatrix}\right) = \mathrm{A}$ and $\pi_2\left(\begin{bmatrix} \mathrm{A} & 0 \\ 0 & \mathrm{B} \end{bmatrix}\right) = \mathrm{B}$, then we have that $\pi_1(\mathrm{H}) \cong \pi_2(\mathrm{H}) \subset \mathrm{GL}(2,\mathrm{C})$. Finally since G is nonmonomial H acts primitively on V_1 and V_2. Therefore $\pi_1(\mathrm{H}) \cong \pi_2(\mathrm{H}) \cong \mathrm{T}$, a primitive algebraic subgroup of $\mathrm{GL}(2,\mathrm{C})$.

(Proof of 3) Since G is irreducible there exists some element $\gamma \in \mathrm{G}$ such that γ nontrivially permutes the set $\{\mathrm{V}_1, \mathrm{V}_2\}$. That is, γ is not an element of H, and in the basis $\{\mathrm{V}_1, \mathrm{V}_2\}$, γ has the form $\gamma = \begin{bmatrix} 0 & g_1 \\ g_2 & 0 \end{bmatrix}$ where $g_1, g_2 \in \mathrm{GL}(2,\mathrm{C})$. Note also that since $\mathrm{G} \subset \mathrm{SL}(4,\mathrm{C})$, $g_1 g_2 \in \mathrm{SL}(2,\mathrm{C})$. ∎

The Primitive Subgroups of GL$(2, \mathbf{C})$

Notation and Definitions:

- Recall that any subgroup of $\mathrm{PGL}(n, \mathrm{C}) = \mathrm{GL}(n, \mathrm{C})/Z(\mathrm{GL}(n, \mathrm{C}))$ is called a **projective group**.

- Throughout this section we will denote by π the canonical projection from $\mathrm{GL}(n, \mathrm{C})$ to $\mathrm{PGL}(n, \mathrm{C})$.

- For each integer n we define the following groups.

  (1) $\mathrm{U}_n^{GL_2} = \left\{ \begin{bmatrix} c & 0 \\ 0 & c \end{bmatrix} : c^n = 1 \right\}.$

  (2) $\mathrm{G}_n = \{\mathrm{A} \in \mathrm{GL}(2, \mathrm{C}) : \det(\mathrm{A}^n) = 1\}$. Note that $\mathrm{G}_0 = \mathrm{GL}(2, \mathrm{C})$ and $\mathrm{G}_1 = \mathrm{SL}(2, \mathrm{C})$.

  (3) $\mathrm{P}_n = \tilde{\mathrm{P}} \cdot \mathrm{U}_n^{GL_2}$ where $\tilde{\mathrm{P}}$ is the group generated by $\{p_1 \dots p_k\}$, a set of representatives of determinant one of a projective group, P. We will denote the set of all subgroups of $\mathrm{GL}(2, \mathrm{C})$ which project onto P by $\mathcal{P}^{GL_2}$. We will denote by P_∞ the group $\tilde{\mathrm{P}} Z(\mathrm{GL}(2, \mathrm{C}))$.

- A group T such that $\pi(\mathrm{T})$ is finite will be called **projectively finite**.

- **Remark:** Given any primitive group, T, Schur's Lemma implies that $Z(\mathrm{T})$ is scalar. Therefore if T_1 and T_2 are both elements of $\mathcal{P}^{GL_2}$ with $\pi(\mathrm{T}_1) = \pi(\mathrm{T}_2)$, then $\pi(t_1) = \pi(t_2)$ implies that $\exists\ \xi \in Z(\mathrm{GL}(2, \mathrm{C}))$ such that $t_1 = \xi t_2$. That is $\mathrm{T}_1 \subseteq \mathrm{T}_2 Z(\mathrm{GL}(2, \mathrm{C}))$ and vice versa.

Taking Lemma 4.7 into consideration, it is clear that in order to construct a list of the imprimitive subgroups of $\mathrm{SL}(4, \mathrm{C})$, one must have a list of the primitive subgroups of $\mathrm{GL}(2, \mathrm{C})$. The following lemma gives a classification of the primitive subgroups of $\mathrm{GL}(2, \mathrm{C})$.

Lemma 4.8 *Let T be a primitive subgroup of $GL(2, C)$. Let π be the canonical projection from $GL(2, C)$ to $PGL(2, C)$. Then one of the following holds.*

1. *$T = G_n$ for some nonnegative integer, n.*

2. *T is projectively finite and either there exists a smallest integer, n such that $T \subseteq P_n$ or there is no such n and $T \subseteq P_\infty$, where $P = A_4,\ S_4$ or A_5.*

Proof: First of all since $\mathrm{SL}(2, \mathrm{C})$ is a primitive subgroup of $\mathrm{GL}(2, \mathrm{C})$ and $\mathrm{SL}(2, \mathrm{C}) \subseteq \mathrm{G}_n$ for any nonnegative integer n, each G_n is also a primitive subgroup of $\mathrm{GL}(2, \mathrm{C})$. Similarly, since P_n contains the primitive group, $\tilde{\mathrm{P}}$, each group P_n is a primitive subgroup of $\mathrm{GL}(2, \mathrm{C})$. Let T be a primitive subgroup of $\mathrm{GL}(2, \mathrm{C})$. Then $\pi(\mathrm{T})$ is a primitive subgroup of $\mathrm{PGL}(2, \mathrm{C})$. Therefore either $\pi(\mathrm{T}) = \mathrm{PSL}(2, \mathrm{C})$ or $\pi(\mathrm{T}) \cong \mathrm{A}_4,\ \mathrm{S}_4$ or A_5. Let $d : \mathrm{GL}(2, \mathrm{C}) \to \mathrm{C}^*$ be the determinant map. Consider the following two cases:

Case 1. Suppose that $\pi(\mathrm{T}) = \mathrm{PSL}(2, \mathrm{C})$. We will show that T^0 contains $\mathrm{SL}(2, \mathrm{C})$. Since $\mathrm{PSL}(2, \mathrm{C})$ is connected, we have that $\pi(\mathrm{T}^0) = \mathrm{PSL}(2, \mathrm{C})$. The latter is not solvable. Therefore T^0 is not solvable. The only non-solvable connected subgroups of $\mathrm{GL}(2, \mathrm{C})$ are $\mathrm{GL}(2, \mathrm{C})$ itself and $\mathrm{SL}(2, \mathrm{C})$. Hence $\mathrm{SL}(2, \mathrm{C}) \subseteq \mathrm{T}$. We will use this and the fact that T is an algebraic subgroup of $\mathrm{GL}(2, \mathrm{C})$ to show that T must be equal to G_n for some nonnegative integer n. Assume that T is a proper subgroup of $\mathrm{GL}(2, \mathrm{C})$. Note that if $\mathrm{T} = \mathrm{GL}(2, \mathrm{C})$, then $\mathrm{T} = \mathrm{G}_0$ and the lemma holds. Since d is a homomorphism of algebraic groups, $d(\mathrm{T})$ is a closed subgroup of C^*. Since T is a proper subgroup of $\mathrm{GL}(2, \mathrm{C})$ containing $\mathrm{SL}(2, \mathrm{C})$, we actually have that $d(\mathrm{T})$ is a proper closed subgroup of C^*. Now, any proper subgroup of C^* that is closed in the Zariski topology is finite. The only finite subgroups of C^* are of the form U_n, ie., the cyclic subgroup of the n^{th} roots of unity, for some positive integer n. Therefore $d(\mathrm{T}) = \mathrm{U}_n$ for some n, which implies $\mathrm{T} \subseteq \{\mathrm{A} \in \mathrm{GL}(2, \mathrm{C}) : \det \mathrm{A}^n = 1\}$. Finally since $\mathrm{SL}(2, \mathrm{C}) \subseteq \mathrm{T}$, $\mathrm{T} = \{\mathrm{A} \in \mathrm{GL}(2, \mathrm{C}) : \det(\mathrm{A})^n = 1\} = \mathrm{G}_n$.

Case 2. Suppose $\pi(\mathrm{T}) \cong \mathrm{A}_4,\ \mathrm{S}_4$ or A_5. Let $\{\tilde{t}_1, \ldots, \tilde{t}_k\}$ be a set of generators for $\pi(\mathrm{T})$ with the property that $\det(\tilde{t}_i) = 1$ for each i. Let $\tilde{\mathrm{T}}$ be the subgroup of $\mathrm{GL}(2, \mathrm{C})$ generated by $\{\tilde{t}_1, \ldots, \tilde{t}_k\}$. It is clear that $\mathrm{T} \subseteq \tilde{\mathrm{T}} \cdot Z(\mathrm{GL}(2, \mathrm{C}))$. Consider now the image $d(\mathrm{T})$. $d(\mathrm{T})$ is either C^* or U_m for some m. If $d(\mathrm{T}) = \mathrm{U}_m$ then $\mathrm{T} \subseteq \tilde{\mathrm{T}} \cdot \mathrm{U}_n^{GL_2}$ where $n = \frac{m}{2}$. Note that since any element g in $\mathrm{GL}(2, \mathrm{C})$ can be written as $\pm h \frac{1}{\sqrt{\det(g)}}$ where $h \in \mathrm{SL}(2, \mathrm{C})$, it is clear that n is the smallest integer such that $\mathrm{T} \subseteq \tilde{\mathrm{T}} \cdot \mathrm{U}_n^{GL_2}$. ∎

The next step is to determine all of the possibilities for the normal subgroup H of an imprimitive group G. Note that Lemma 4.7 implies that we may consider H as a subgroup of $\mathrm{T} \times \mathrm{T}$ where T is a primitive subgroup of $\mathrm{GL}(2, \mathrm{C})$.

Subgroups of $\mathbf{T} \times \mathbf{T}$

The following lemma is a generalization of Goursat's Lemma which can be found in [LA] on page 75. Kolchin also states a version of this lemma for simple groups (see [KOL1]).

Lemma 4.9 *Let T be a primitive subgroup of $GL(2, C)$ with center $Z(T)$. Let $\pi : T \to T/Z(T)$ be the canonical projection. Let H be a proper subgroup of $T \times T$ such that the coordinate projections π_1 and π_2 are onto T. Let $ker(\pi_1) = \{I\} \times F$. Then the following hold:*

1. *$F \subseteq \tilde{F} = \pi^{-1}(N)$, where $N \lhd T/Z(T)$ and $\pi(F) = \pi(\tilde{F})$.*

2. *There exists an automorphism, f, of $T/\tilde{F}$ such that $f(A\tilde{F}) = B\tilde{F}$ for each $(A, B) \in H$.*

Proof: (1) Let $ker(\pi_1) = \{\mathrm{I}\} \times \mathrm{F}$. Since both π_1 and π_2 are onto, $\mathrm{F} \lhd \mathrm{T}$. For $\mathrm{N} = \pi(\mathrm{F})$, $\mathrm{F} \subseteq \pi^{-1}(\mathrm{N})$.

(2) Let π_{F} and $\pi_{\tilde{\mathrm{F}}}$ be the canonical projections of T onto T/F and $\mathrm{T}/\tilde{\mathrm{F}}$ respectively. Since $ker(\pi_1) = \{\mathrm{I}\} \times \mathrm{F} \subseteq\ ker\ (\pi_{\mathrm{F}} \circ \pi_2)$, there exists a homomorphism $g : \mathrm{T} \to \mathrm{T}/\mathrm{F}$ such that $g \circ \pi_1 = \pi_{\mathrm{F}} \circ \pi_2$. Since we also have $\mathrm{F} \subseteq \tilde{\mathrm{F}}$, the projection $h : \mathrm{T}/\mathrm{F} \to \mathrm{T}/\tilde{\mathrm{F}}$ is surjective. Let $g' = h \circ g$. Since $ker(\pi_{\tilde{\mathrm{F}}}) = \tilde{\mathrm{F}} \subseteq\ ker(g')$, there exists a homomorphism $f : \mathrm{T}/\tilde{\mathrm{F}} \to \mathrm{T}/\tilde{\mathrm{F}}$ such that $f \circ \pi_{\tilde{\mathrm{F}}} = g' = h \circ g$.

Case 1. Suppose that $\pi(\mathrm{T})$ is finite. Then $\mathrm{T}/\tilde{\mathrm{F}}$ is finite. Therefore since f is surjective, f is an isomorphism.

Case 2. Suppose that $\mathrm{T} = \mathrm{G}_n$. Since $\pi(\mathrm{T}) = \mathrm{PSL}(2, \mathrm{C})$ is a simple group, $\tilde{\mathrm{F}} = Z(\mathrm{G}_n)$ or G_n. The latter implies that f is the trivial map and the lemma clearly holds. Suppose $\tilde{\mathrm{F}} = Z(\mathrm{G}_n)$. Since f is surjective , its kernel is a proper normal subgroup of $\mathrm{T}/Z(\mathrm{T})$ and hence must be the identity. Therefore f is an automorphism. Finally combining the equations $g \circ \pi_1 = \pi_{\mathrm{F}} \circ \pi_2$ and $f \circ \pi_{\tilde{\mathrm{F}}} = g' = h \circ g$ we see that $f \circ \pi_{\tilde{\mathrm{F}}} \circ \pi_1 = h \circ \pi_{\mathrm{F}} \circ \pi_2$, which gives the result. ∎

Lemma 4.10 *Let T be a primitive subgroup of $GL(m, C)$. Suppose that $g \in GL(m, C)$ is such that for each $t \in T$, there exists a $d_t \in C^*$ such that $g^{-1}tg = d_t t$. Then g is a scalar matrix.*

Proof: We will show that each $d_t = 1$. Schur's Lemma then implies that g is scalar. For each $c \in \mathrm{C}^*$, let $\mathrm{V}_c = \{v \in \mathrm{C}^m : (cg - \mathrm{I})^m v = 0\}$. Note that $\mathrm{V}_c \neq \{0\}$ if and only if c^{-1} is an eigenvalue of g. Furthermore, $\mathrm{C}^m = \bigoplus_{c \in \mathrm{C}^*} \mathrm{V}_c$. Since $t(cg - \mathrm{I}) = (cd_t g - \mathrm{I})t$ we have that $t(cg - \mathrm{I})^m = (cd_t g - \mathrm{I})^m t$. Therefore $t(\mathrm{V}_c) \subseteq \mathrm{V}_{cd_t}$. Indeed, let $w = tv$ where $v \in \mathrm{V}_c$. Consider $(cd_t g - \mathrm{I})^m w$. This is equal to $(cd_t g - \mathrm{I})^m tv = t(cg - \mathrm{I})^m v = 0$. Now since $\bigoplus_{c \in \mathrm{C}^*} \mathrm{V}_c = \bigoplus_{c \in \mathrm{C}^*} \mathrm{V}_{cd_t}$ we must have that $t(\mathrm{V}_c) = \mathrm{V}_{cd_t}$ so t permutes the spaces V_c.

By assumption, T acts primitively on $\mathrm{C}^m = \bigoplus \mathrm{V}_c$ so there exists a unique c such that $\mathrm{V}_c = \mathrm{C}^m$. Furthermore, we must then have that $\mathrm{V}_c = \mathrm{V}_{cd_t}$ for any t and so $d_t = 1$. ∎

Corollary 4.1 *Let $\overline{T} \subseteq PGL(m, C)$ be a primitive group. If $\overline{g} \in PGL(m, C)$ commutes with $\overline{T}$ then $\overline{g} = \overline{1}$.*

Proof: Lift $\overline{\mathrm{T}}$ to $\mathrm{T} \subseteq \mathrm{GL}(m, \mathrm{C})$ and $\overline{g}$ to $g \in \mathrm{GL}(m, \mathrm{C})$. Then for any $t \in \mathrm{T}$ there exist a $d_t \in \mathrm{C}^*$ such that $g^{-1}tg = d_t t$. Lemma 4.10 implies that g is constant so $\overline{g} = \overline{1}$. ∎

Lemma 4.11 *Let T be a projectively finite primitive subgroup of $GL(2, C)$ and let $g \in GL(2, C)$ normalize T. Then $< T, g >$ is also a projectively finite primitive subgroup of $GL(2, C)$.*

Proof: The group $< \mathrm{T}, g >$ is clearly primitive since it contains the primitive group T. It is also clearly contained in $\mathrm{GL}(2, \mathrm{C})$. Hence we need only show that it is projectively finite. Let $\overline{\mathrm{T}}$ and $\overline{g}$ denote the images of T and g in $\mathrm{PGL}(2, \mathrm{C})$. Note that conjugation by g is an automorphism of T. Since $\overline{\mathrm{T}}$ is finite, its automorphism group is also finite. Therefore some power $\overline{g}^n$ of $\overline{g}$ commutes with every element of T. By Corollary 4.1 of Lemma 4.10, $\overline{g}^n = \overline{1}$. Since $\overline{g}$ normalizes $\overline{\mathrm{T}}$, the set $\{\overline{g}^i\overline{t} : \ i \in \mathbb{Z}\ , \overline{t} \in \overline{\mathrm{T}}\}$ is a group and by the preceding remarks it is finite. ∎

Lemma 4.12 *Let H be a subgroup of $GL(4, C)$. Let γ be an element such that $\gamma^2 \in H$, $\gamma H\gamma^{-1} = H$, $\gamma \notin H$. Then $H \cup \gamma H$ is a subgroup of $GL(4, C)$.*

Proof: Since γ normalizes H, the set $\{\gamma^i h : h \in \mathrm{H}\}$ forms a group. Since $\gamma^2 \in \mathrm{H}$ and $\gamma \notin \mathrm{H}$, the latter set is $\mathrm{H} \cup \gamma\mathrm{H}$. ∎

Lemma 4.13 *Let T be a primitive subgroup of $GL(2, C)$. Let $H \subseteq GL(4, C)$ be a subgroup of $T \times T$ such that the coordinate projections, π_1 and π_2, of H are onto T. Let $\{I\} \times F$ be the kernel of π_1. Suppose also that $\gamma = \begin{bmatrix} 0 & g_1 \\ g_2 & 0 \end{bmatrix} \in SL(4, C)$ satisfies (1) $\gamma^2 \in H$ and (2) $\gamma H\gamma^{-1} = H$. Then the following hold.*

1. *H is conjugate to a group of the form $\left\{\begin{bmatrix} A & 0 \\ 0 & \xi_{\mathrm{A}} B \end{bmatrix} : A \in T, B \in A \cdot F\right\}$ where ξ_{A} is an element of $Z(T)$ which depends on A.*

2. *$\exists$ an element $y \in GL(4, C)$, and a scalar $\xi \in Z(GL(2, C))$ such that $yHy^{-1} = H$ and one of the following holds.*

   (a) $\pi(T) \cong A_4$, in which case $y\gamma Hy^{-1}$ is equal to either $\begin{bmatrix} 0 & I \\ \xi & 0 \end{bmatrix} H$ or $\begin{bmatrix} 0 & O_{S_4} \\ \xi O_{S_4} & 0 \end{bmatrix} H$, where $O_{S_4} = \begin{bmatrix} 0 & 1 \\ i & 0 \end{bmatrix}$.

   (b) $y\gamma Hy^{-1} = \begin{bmatrix} 0 & I \\ \xi & 0 \end{bmatrix} H$.

Remark: Let F, $\tilde{\mathrm{F}}$ and f be as in Lemma 4.9. Suppose that f can be defined via conjugation by an element $\mathrm{D} \in \mathrm{GL}(2, \mathrm{C})$. Then for each $(\mathrm{A}, \mathrm{B}) \in \mathrm{H}$ there exists a scalar, $\xi_{\mathrm{A}} \in Z(\mathrm{T})$ such that $\mathrm{B} \in \xi_{\mathrm{A}}\mathrm{DAD}^{-1}\mathrm{F}$. Indeed, suppose $\mathrm{DAD}^{-1}\tilde{\mathrm{F}} = \mathrm{B}\tilde{\mathrm{F}}$. Then there exists an element $\tilde{g} \in \tilde{\mathrm{F}}$ depending on A such that $\mathrm{DAD}^{-1} = \mathrm{B}\tilde{g}$. Since $\tilde{\mathrm{F}} \subseteq \mathrm{F}Z(\mathrm{T})$, there exist elements $g \in \mathrm{F}$ and $\alpha_{\mathrm{A}} \in Z(\mathrm{T})$ such that $\mathrm{DAD}^{-1} = \alpha_{\mathrm{A}}\mathrm{B}g$. Let $\xi_{\mathrm{A}} = \alpha_{\mathrm{A}}^{-1}$. Then we have that $\mathrm{B} \in \xi_{\mathrm{A}}\mathrm{DAD}^{-1}\mathrm{F}$.

Proof: Let T be a primitive subgroup of GL(2, C) and let H and γ be as stated in the lemma. By Lemma 4.9 there exists a normal subgroup, F, of T and an automorphism, f, of $\mathrm{T}/\tilde{\mathrm{F}}$ where $\tilde{\mathrm{F}} = \pi^{-1}(\pi(\mathrm{F}))$ such that $f(\mathrm{A}\tilde{\mathrm{F}}) = \mathrm{B}\tilde{\mathrm{F}}$ for each $(\mathrm{A}, \mathrm{B}) \in \mathrm{H}$. In each of the following cases we will show that all automorphisms of $\mathrm{T}/\tilde{\mathrm{F}}$ are either inner or can be realized via conjugation by some element in GL(2, C). Then by the above remark we have that H is conjugate to a group of the form $\left\{\begin{bmatrix} \mathrm{A} & 0 \\ 0 & \xi_{\mathrm{A}}\mathrm{B} \end{bmatrix} : \mathrm{A}, \mathrm{B} \in \mathrm{T}, \mathrm{B} \in \mathrm{AF}\right\}$ where $\mathrm{F} \subseteq \tilde{\mathrm{F}}$ is a normal subgroup of T which projects onto a normal subgroup of $\pi(\mathrm{T})$. We will then use the above conditions on γ in order to construct the element y. We will consider the cases $\mathrm{T} = \mathrm{G}_n$ and T projectively finite separately.

Case 1. $\mathrm{T} = \mathrm{G}_n$. There are two subcases to consider.

Case 1.(a) $\tilde{\mathrm{F}} = Z(\mathrm{G}_n)$. Then $\mathrm{T}/\tilde{\mathrm{F}}$ is isomorphic to PSL(2, C) and hence f is an inner automorphism. This implies that H is conjugate to a group of the form $\left\{\begin{bmatrix} \mathrm{B} & 0 \\ 0 & \xi_{\mathrm{B}}\mathrm{B} \end{bmatrix} : \mathrm{B} \in \mathrm{T}\right\}$. Let $\gamma = \begin{bmatrix} 0 & g_1 \\ g_2 & 0 \end{bmatrix}$ where $g_i \in \mathrm{GL}(2, \mathrm{C})$ and $\det g_1 g_2 = 1$. Let $h \in \mathrm{H}$. Conditions (1) and (2) imply that $\gamma = \begin{bmatrix} 0 & g \\ cg & 0 \end{bmatrix}$ where $c = Z(\mathrm{GL}(2, \mathrm{C}))$. Indeed $\gamma h \gamma^{-1} = \begin{bmatrix} 0 & g_1 \\ g_2 & 0 \end{bmatrix} \begin{bmatrix} \mathrm{B} & 0 \\ 0 & \xi_{\mathrm{B}}\mathrm{B} \end{bmatrix} \begin{bmatrix} 0 & g_2^{-1} \\ g_1^{-1} & 0 \end{bmatrix} =$ $\begin{bmatrix} \xi_{\mathrm{B}} g_1 \mathrm{B} g_1^{-1} & 0 \\ 0 & g_2 \mathrm{B} g_2^{-1} \end{bmatrix}$. $\xi_{\mathrm{B}} g_1 \mathrm{B} g_1^{-1}$ is clearly in T. Therefore this matrix is in H if and only if there exists a scalar $b \in Z(\mathrm{T})$ such that $g_2 \mathrm{B} g_2^{-1} = b\xi_{\mathrm{B}} g_1 \mathrm{B} g_1^{-1}$. This implies that $(g_1^{-1} g_2)\mathrm{B}(g_1^{-1} g_2)^{-1} = b\xi_{\mathrm{B}}\mathrm{B} \ \forall \ \mathrm{B} \in \mathrm{GL}(2, \mathrm{C})$. Since GL(2, C) is a primitive group, Corollary 4.1 of Lemma 4.10 implies that $(g_1^{-1} g_2)$ is a scalar matrix. That is, $g_2 = c g_1$ for some $c \in Z(\mathrm{GL}(2, \mathrm{C}))$. Note that since $\gamma \in \mathrm{SL}(4, \mathrm{C})$, $c \in \{\frac{\pm 1}{\det_{g_1}}\}$.

Let $y = \begin{bmatrix} \mathrm{C} & 0 \\ 0 & \mathrm{D} \end{bmatrix}$ where $\mathrm{C} = (\sqrt{\det g})\mathrm{I}$ and $\mathrm{D} = (\det g)\mathrm{I}$. Since C and D are scalar matrices and y is in block form it is clear that $y\mathrm{H}y^{-1} = \mathrm{H}$. Also $y\gamma y^{-1} = \begin{bmatrix} 0 & \mathrm{I} \\ \mathrm{I} & 0 \end{bmatrix} \begin{bmatrix} \frac{1}{\sqrt{\det_g}} g & 0 \\ 0 & \frac{1}{\sqrt{\det_g}} g \end{bmatrix}$. Note that $\det(\frac{1}{\sqrt{\det_g}} g) = 1$. Therefore $\begin{bmatrix} \frac{1}{\sqrt{\det_g}} g & 0 \\ 0 & \frac{1}{\sqrt{\det_g}} g \end{bmatrix} \in \mathrm{H}$.

Case 1.(b) $\tilde{\mathrm{F}} = \mathrm{G}_n$. Then $\mathrm{T}/\tilde{\mathrm{F}} = \{\mathrm{I}\}$ and hence f is the identity mapping. Therefore there exist scalars $\xi_{\mathrm{A}} \in Z(\mathrm{GL}(2,\mathrm{C}))$ such that H is conjugate to $\left\{\begin{bmatrix} \mathrm{A} & 0 \\ 0 & \xi_{\mathrm{A}}\mathrm{B} \end{bmatrix} : \mathrm{A},\mathrm{B} \in \mathrm{G}_n\right\}$. Let $\gamma = \begin{bmatrix} 0 & g_1 \\ g_2 & 0 \end{bmatrix}$ where $g_i \in \mathrm{GL}(2,\mathrm{C})$ and $\det g_1 g_2 = 1$. γ satisfies conditions (1) and (2). Let $y = \begin{bmatrix} \mathrm{C} & 0 \\ 0 & \mathrm{D} \end{bmatrix}$ where $\mathrm{C} \in \mathrm{G}_n$ and $\mathrm{D} = g_1$. For $h = \begin{bmatrix} \mathrm{A} & 0 \\ 0 & \xi_{\mathrm{A}}\mathrm{B} \end{bmatrix} \in \mathrm{H}$ $yhy^{-1} = \begin{bmatrix} \mathrm{CAC}^{-1} & 0 \\ 0 & g_1\xi_{\mathrm{A}}\mathrm{B}g_1^{-1} \end{bmatrix}$ which is clearly in H so again we have $y\mathrm{H}y^{-1} = \mathrm{H}$. Also $y\gamma y^{-1} = \begin{bmatrix} 0 & \mathrm{I} \\ \mathrm{I} & 0 \end{bmatrix}\begin{bmatrix} g_1g_2\mathrm{C}^{-1} & 0 \\ 0 & \mathrm{C} \end{bmatrix}$. Since $\mathrm{C} \in \mathrm{G}_n$ and $\det(g_1g_2) = 1$, $\begin{bmatrix} g_1g_2\mathrm{C}^{-1} & 0 \\ 0 & \mathrm{C} \end{bmatrix} \in \mathrm{H}$. Therefore $y\gamma y^{-1} \in \begin{bmatrix} 0 & \mathrm{I} \\ \mathrm{I} & 0 \end{bmatrix}\mathrm{H}$.

Case 2. Suppose that T is projectively finite.

Remark: In this case we will use that fact that in order for $\gamma = \begin{bmatrix} 0 & g_1 \\ g_2 & 0 \end{bmatrix}$ to satisfy conditions (1) and (2) of the lemma, γ must satisfy the following condition: Each element g_i normalizes T and furthermore, once we have established part 1 of the lemma we have that $(g_1^{-1}g_2\mathrm{F}Z(\mathrm{GL}(2,\mathrm{C})))$ is in the centralizer of $\mathrm{T}/\mathrm{F}Z(\mathrm{GL}(2,\mathrm{C}))$. One can see this by observing the following computation. Let $h = \begin{bmatrix} \mathrm{A} & 0 \\ 0 & \xi_{\mathrm{A}}\mathrm{B} \end{bmatrix} \in \mathrm{H}$ Then $\gamma h\gamma^{-1} = \begin{bmatrix} \xi_{\mathrm{A}}g_1\mathrm{B}g_1^{-1} & 0 \\ 0 & g_2\mathrm{A}g_2^{-1} \end{bmatrix} \in \mathrm{H}$ implies that there exists a scalar ξ such that $\xi^{-1}g_2\mathrm{A}g_2^{-1} \in \xi_{\mathrm{A}}g_1\mathrm{B}g_1^{-1}\mathrm{F}$ which implies that $g_1^{-1}g_2\mathrm{A}(g_1^{-1}g_2)^{-1} \in \xi\xi_{\mathrm{A}}\mathrm{BF}$ which is equal to $\xi\xi_{\mathrm{A}}\mathrm{F}$. Since $\pi_1(\mathrm{H}) = \mathrm{T}$, this implies that $g_1^{-1}g_2$ centralizes $\mathrm{T}/(\mathrm{F}Z(\mathrm{GL}(2,\mathrm{C})))$.

Case 2.(a) $\mathbf{T} \in \mathcal{A}_5^{GL_2}$. In this case $\pi(\mathrm{T})$ has only the trivial subgroups, hence we have only two subcases to consider here.

- Suppose $\tilde{\mathrm{F}} = Z(\mathrm{T})$. Then $\mathrm{T}/\tilde{\mathrm{F}} \cong \mathrm{A}_5$ which is a complete group (see [ROT] pp. 131-132) and evidently $Aut(\mathrm{T}/\tilde{\mathrm{F}}) = Inn(\mathrm{T}/\tilde{\mathrm{F}})$. Therefore H is conjugate to a group of the desired form. Now consider γ. By the remark following case 2. each g_i normalizes T. Lemma 4.11 and Lemma 4.8 then imply that each g_i is in $\mathrm{T}Z(\mathrm{GL}(2,\mathrm{C}))$. Since $\mathrm{T}/Z(\mathrm{T})$ is complete it has trivial center. Therefore again by the remark following case 2., $(g_1^{-1}g_2)$ is a scalar matrix. Hence $\gamma = \begin{bmatrix} 0 & g \\ \xi g & 0 \end{bmatrix}$ where $g = \xi_g t_g$ with $t_g \in$ T. Now if we let $y = \begin{bmatrix} \mathrm{C} & 0 \\ 0 & \mathrm{D} \end{bmatrix}$ with C = I and D = ξ_g a computation shows that $y\mathrm{H}y^{-1} = \mathrm{H}$ and $y\gamma y^{-1} \in \begin{bmatrix} 0 & \mathrm{I} \\ \xi & 0 \end{bmatrix}$H.

- $\tilde{\mathrm{F}} = \mathrm{T}$. Then H is of the desired form. Now consider γ. As in the previous case we have that each g_i is in $\mathrm{T}Z(\mathrm{GL}(2,\mathrm{C}))$. Let $g_i = \xi_i t_i$. Now if we let $y = \begin{bmatrix} \mathrm{C} & 0 \\ 0 & \mathrm{D} \end{bmatrix}$ with C = I and D = ξ_1 a computation shows that $y\gamma y^{-1} \in \begin{bmatrix} 0 & \mathrm{I} \\ \xi & 0 \end{bmatrix}$H, where $\xi = \xi_1\xi_2$.

Case 2.(b) $\mathbf{T} \in \mathcal{S}_4^{GL_2}$. In addition to the trivial subgroups, $\pi(\mathrm{T})$ has normal subgroups which are isomorphic to the Klein four group, K, and A_4. Since $\mathrm{T}/Z(\mathrm{T}) \cong \mathrm{S}_4$ is complete and since the normalizer of T is $\mathrm{T}Z(\mathrm{GL}(2,\mathrm{C}))$, the cases where $\tilde{\mathrm{F}} = Z(\mathrm{T})$ and where $\tilde{\mathrm{F}} = \mathrm{T}$ are proven exactly as in Case 2.(a). Hence we have only two new cases to consider here.

- Suppose $\tilde{\mathrm{F}} = \pi^{-1}(\mathrm{K})$. Then $\mathrm{T}/\tilde{\mathrm{F}} \cong \mathrm{S}_3$ which is also a complete group and evidently $Aut(\mathrm{T}/\tilde{\mathrm{F}}) = Inn(\mathrm{T}/\tilde{\mathrm{F}})$. Therefore H is conjugate to a group of the desired form. Now consider γ. Note that as in the previous case $g_i \in \mathrm{T}Z(\mathrm{GL}(2,\mathrm{C}))$ and the element $g_1^{-1}g_2\mathrm{F}Z(\mathrm{GL}(2,\mathrm{C}))$, centralizes $\mathrm{T}/\mathrm{F}Z(\mathrm{GL}(2,\mathrm{C}))$. Since $\mathrm{T}/\tilde{\mathrm{F}}$ has trivial center, $g_1^{-1}g_2\mathrm{F}Z(\mathrm{GL}(2,\mathrm{C}))$ is contained in $\mathrm{F}Z(\mathrm{GL}(2,\mathrm{C}))$. Therefore we have that $g_2 \in g_1\mathrm{F}Z(\mathrm{GL}(2,\mathrm{C}))$ and hence $\gamma \in \begin{bmatrix} 0 & \mathrm{I} \\ \xi & 0 \end{bmatrix}$H.
- Suppose $\tilde{\mathrm{F}} = \pi^{-1}(\mathrm{A}_4)$. Then $\mathrm{T}/\tilde{\mathrm{F}}$ is a group of order two and hence has trivial automorphism group. Therefore H is of the stated form. Consider γ. There are two cases to consider. Either g_1 and g_2 are in the same or different $\tilde{\mathrm{F}}\cdot Z(\mathrm{GL}(2,\mathrm{C}))$ cosets in $\mathrm{T}\cdot Z(\mathrm{GL}(2,\mathrm{C}))$. Suppose they are in the same coset. Then we may rewrite γ as $\begin{bmatrix} 0 & g \\ g & 0 \end{bmatrix}$ where g is a representative of this coset. Since the group $\mathrm{S}_4/\mathrm{A}_4$ has order 2, $g^2 \in \mathrm{T}Z(\mathrm{GL}(2,\mathrm{C}))$. Let $y = \begin{bmatrix} \mathrm{C} & 0 \\ 0 & \mathrm{D} \end{bmatrix}$

where C = I and D = g. Then $y\gamma y^{-1} = \begin{bmatrix} 0 & \mathrm{I} \\ g^2 & 0 \end{bmatrix} \in \begin{bmatrix} 0 & \mathrm{I} \\ \xi & 0 \end{bmatrix}$H. On the other hand, if g_1 and g_2 are in different cosets, then one can show that H$\cup\gamma$H is not a group and hence this case is not a part of our results.

Case 2.(c) $\mathbf{T} \in \mathcal{A}_4^{GL_2}$. In addition to the trivial subgroups, $\pi(\mathrm{T})$ has a normal subgroup which is isomorphic to K. In each of the following subcases, Lemma 4.11 and Lemma 4.8 imply that γ is such that $g_i \in \overline{\mathrm{T}}Z(\mathrm{GL}(2,\mathrm{C}))$ where $\mathrm{T} \subset \overline{\mathrm{T}}$, $\overline{\mathrm{T}} \in \mathcal{S}_4^{GL_2}$.

- Suppose $\tilde{\mathrm{F}} = Z(\mathrm{T})$. Unfortunately $\mathrm{T}/Z(\mathrm{T}) \cong \mathrm{A}_4$ is not a complete group. Nonetheless, one can find in [SC] within the proof of Theorem 11.4.6 p.313 that any automorphism of A_4 can be realized as the restriction of some inner automorphism of S_4. Therefore H is of the desired form. Consider $\gamma = \begin{bmatrix} 0 & g_1 \\ g_2 & 0 \end{bmatrix}$. In this case $(g_1^{-1}g_2\mathrm{F})$ which is an element of $\overline{\mathrm{T}}Z(\mathrm{GL}(2,\mathrm{C}))/Z(\mathrm{T})$ centralizes $\mathrm{T}/Z(\mathrm{T}) \cong \mathrm{A}_4$. This implies that $g_1^{-1}g_2 \in Z(\mathrm{GL}(2,\mathrm{C}))$. Let $\gamma = \begin{bmatrix} 0 & g \\ \xi g & 0 \end{bmatrix}$. Let n be the order of g in $\overline{\mathrm{T}}/Z(\mathrm{GL}(2,\mathrm{C}))$. Let $y = \begin{bmatrix} \mathrm{C} & 0 \\ 0 & \mathrm{D} \end{bmatrix}$ where C = I and D = g^{n-1}. Then $y\gamma y^{-1} = \begin{bmatrix} 0 & \mathrm{I} \\ \xi & 0 \end{bmatrix}$ which is clearly in $\begin{bmatrix} 0 & \mathrm{I} \\ \xi & 0 \end{bmatrix}$H.
- Suppose $\tilde{\mathrm{F}} = \pi^{-1}(\mathrm{K})$. Consider the automorphisms of $\mathrm{T}/\tilde{\mathrm{F}}$. This group is isomorphic to A_4/K, a group of order three and hence isomorphic to A_3. The only automorphisms of such a group are the identity and inverse maps. Let x be an element in A_4 whose coset $\overline{x}$ generates $\mathrm{A}_4/\mathrm{K} = \mathrm{A}_3$. There exists an element $y \in \mathrm{S}_3 \subseteq \mathrm{S}_4 \subset \mathrm{PGL}(2,\mathrm{C})$ such that $yxy^{-1} = x^{-1}$. Therefore the automorphism of A_4/K given by $x \to x^{-1}$ is induced by the automorphism $x \to yxy^{-1}$ of A_4. Therefore both the identity map and the inverse map on A_4/K are induced by conjugation of elements in $\mathrm{PGL}(2,\mathrm{C})$. Therefore, in $\mathrm{GL}(4,\mathrm{C})$, H is conjugate to a group of the given form. Consider $\gamma = \begin{bmatrix} 0 & g_1 \\ g_2 & 0 \end{bmatrix}$. $(g_1^{-1}g_2\tilde{\mathrm{F}})$ which is in $\overline{\mathrm{T}}/\tilde{\mathrm{F}}$ must centralize $\mathrm{T}/\tilde{\mathrm{F}}$. One can show that $\overline{\mathrm{T}}/(\tilde{\mathrm{F}}Z(\mathrm{GL}(2,\mathrm{C}))) \cong \mathrm{S}_3$ and $\mathrm{T}/\tilde{\mathrm{F}} \cong \mathrm{A}_3$. Note that the identity element

is the only element of S_3 which centralizes A_3. Therefore we must have that $(g_1^{-1}g_2\tilde{F}) \in \tilde{F}Z(GL(2,C))$. One can also show that if $\tilde{g}$ is any element in $g\tilde{F}$ and if $\gamma_1 = \begin{bmatrix} 0 & g \\ g & 0 \end{bmatrix}$ and $\gamma_2 = \begin{bmatrix} 0 & g \\ \tilde{g} & 0 \end{bmatrix}$, then the cosets $\gamma_1 H$ and $\gamma_2 H$ are equal. Hence we may assume that $\gamma = \begin{bmatrix} 0 & g \\ g & 0 \end{bmatrix}$ where $g \in S_4^{SL_2}$. Now let $y = \begin{bmatrix} g^{-1} & 0 \\ 0 & g^2 \end{bmatrix}$. Then $y\gamma y^{-1} = \begin{bmatrix} 0 & g^{-2} \\ g^4 & 0 \end{bmatrix}$. Since $\overline{T}/(TZ(GL(2,C)))$ is a group of order two, each nonzero entry of this matrix is in $TZ(GL(2,C))$. Since $\overline{T}/(\tilde{F}Z(GL(2,C)))$ is a group of order six, $g^4 \in g^{-2}\tilde{F}Z(GL(2,C))$ Therefore $y\gamma y^{-1} \in \begin{bmatrix} 0 & I \\ \xi & 0 \end{bmatrix} H$, for some scalar ξ.

- Suppose $\tilde{F} = T$. Then again H has the form stated in part 1 of the lemma. Now consider γ. As in the previous case we have that each $g_i \in \overline{T}Z(GL(2,C))$ where $\pi(\overline{T})$ is isomorphic to S_4. In [BL3] we find that $O_{S_4} = \begin{bmatrix} 0 & 1 \\ i & 0 \end{bmatrix}$ is a generator of the degree two projective representation of S_4 that is not a generator of a degree two projective representation of A_4. Since A_4 is a normal subgroup of S_4 of index 2, any representative of an element in this projective representation of S_4 which is not in $\pi(T) \cong A_4$ can be written as $\xi A O_{S_4}$ for some $\xi \in Z(GL(2,C))$, $A \in \pi(T)$. Therefore we may assume that each g_i is in $O_{S_4}TZ(GL(2,C))$. $\gamma^2 \in H$ implies that γ can be written as either $\begin{bmatrix} 0 & \xi_1 \\ \xi_2 & 0 \end{bmatrix}$ or $\begin{bmatrix} 0 & \xi_1 O_{S_4} \\ \xi_2 O_{S_4} & 0 \end{bmatrix}$ where $\xi_i \in Z(GL(2,C))$ and $\xi_1\xi_2 \in Z(T)$. Let $y = \begin{bmatrix} C & 0 \\ 0 & D \end{bmatrix}$ where $C \in Z(T)$ and $D = \xi_1$. This then implies the result. ∎

Theorem 4.2 *Let $G \subseteq SL(4,C)$ be an imprimitive nonmonomial linear algebraic group. Then one of the following holds:*

1. *G contains a normal subgroup H of index two such that $H \subseteq T \times T$ where T is projectively finite.*

2. *$\exists\ n = 0, 1, \ldots$ such that G is conjugate to one of the following algebraic groups.*

$$\left\{ \begin{bmatrix} A & 0 \\ 0 & B \end{bmatrix} : A, B \in G_n, det(AB) = 1 \right\} \cup \left\{ \begin{bmatrix} 0 & A \\ B & 0 \end{bmatrix} : A, B \in G_n, det(AB) = 1 \right\}$$

$$\left\{ \begin{bmatrix} B & 0 \\ 0 & \pm d_B B \end{bmatrix} : B \in G_n, d_B = \frac{1}{det(B)} \right\} \cup \left\{ \begin{bmatrix} 0 & \pm d_B B \\ B & 0 \end{bmatrix} : B \in G_n, d_B = \frac{1}{det(B)} \right\}$$

$$\left\{ \begin{bmatrix} B & 0 \\ 0 & d_B B \end{bmatrix} : B \in G_n, d_B = \frac{1}{det(B)} \right\} \cup \left\{ \begin{bmatrix} 0 & d_B B \\ B & 0 \end{bmatrix} : B \in G_n, d_B = \frac{1}{det(B)} \right\}$$

Proof: Let $\mathrm{G} \subseteq \mathrm{SL}(4,\mathrm{C})$ be an infinite, imprimitive, nonmonomial linear algebraic group. Let π_1 and π_2 be the coordinate projection maps as defined in Lemma 4.7. Then Lemma 4.7 implies that G has a normal subgroup, H, of index two such that $\mathrm{H} \subseteq \left\{ \begin{bmatrix} \mathrm{A} & 0 \\ 0 & \mathrm{B} \end{bmatrix} : \mathrm{A}, \mathrm{B} \in \mathrm{GL}(2,\mathrm{C}) \right\}$, where $\pi_1(\mathrm{H}) = \pi_2(\mathrm{H}) = \mathrm{T}$ with T a primitive subgroup of $\mathrm{GL}(2,\mathrm{C})$. Let $\pi : \mathrm{GL}(2,\mathrm{C}) \to \mathrm{PGL}(2,\mathrm{C})$ be the canonical projection. Assume $\pi(\mathrm{T})$ is infinite. By Lemma 4.8, $\mathrm{T} = \mathrm{G}_n$ for some nonnegative integer n. By Lemma 4.7, there exists an element $\gamma \in \mathrm{G} \backslash \mathrm{H}$ of the form $\begin{bmatrix} 0 & g_1 \\ g_2 & 0 \end{bmatrix}$ where $g_i \in \mathrm{GL}(2,\mathrm{C})$ and $\det(g_1 g_2) = 1$. Since H is of index two in G, once we find such an element γ we will have all of G. Indeed $\mathrm{G} = \mathrm{H} \cup \gamma\mathrm{H}$.

Let $ker(\pi_1) = \{\mathrm{I}\} \times \mathrm{F}$. By Lemma 4.9 there is a mapping $f \in Aut(\mathrm{T}/\tilde{\mathrm{F}})$ where $\tilde{\mathrm{F}} = \pi^{-1}(\pi(\mathrm{F}))$ such that $f(\mathrm{A}\tilde{\mathrm{F}}) = \mathrm{B}\tilde{\mathrm{F}}$ for each $(\mathrm{A}, \mathrm{B}) \in \mathrm{H}$. Given this map and the element γ described above, Lemma 4.13 implies that H is conjugate to a group of the form $\left\{ \begin{bmatrix} \mathrm{A} & 0 \\ 0 & \xi_{\mathrm{A}}\mathrm{B} \end{bmatrix} : \mathrm{A}, \mathrm{B} \in \mathrm{T}, \mathrm{B} \in \mathrm{AF} \right\}$ where each $\xi_{\mathrm{A}} \in Z(\mathrm{GL}(2,\mathrm{C}))$. Also, $\exists$ an element $y = \begin{bmatrix} \mathrm{C} & 0 \\ 0 & \mathrm{D} \end{bmatrix}$, $\mathrm{C}, \mathrm{D} \in \mathrm{GL}(2,\mathrm{C})$, and a scalar ξ such that $y\mathrm{G}y^{-1}$ is equal to $\mathrm{H} \cup \begin{bmatrix} 0 & \mathrm{I} \\ \xi & 0 \end{bmatrix} \mathrm{H}$. Now since $\mathrm{G} \subseteq \mathrm{SL}(4,\mathrm{C})$, $\mathrm{H} \subseteq \mathrm{SL}(4,\mathrm{C})$. This implies that for each $x \in \mathrm{F}$ and for every $\mathrm{A} \in \mathrm{G}_n$ $\det(\xi_{\mathrm{A}}\mathrm{A}^2 x) = 1$. This implies that $\mathrm{F} \subseteq \mathrm{SL}(2,\mathrm{C})$ and $\xi_{\mathrm{A}} = \pm\det(\mathrm{A}^{-1})$. The subgroups of G_n contained in $\mathrm{SL}(2,\mathrm{C})$ are $\{\mathrm{I}\}$, $\{\pm\mathrm{I}\}$ and $\mathrm{SL}(2,\mathrm{C})$.

Allowing F to be any one of these subgroups we obtain the following possibilities for the group, H.

1. $\left\{\begin{bmatrix} \mathrm{B} & 0 \\ 0 & d_{\mathrm{B}}\mathrm{B} \end{bmatrix} : \mathrm{B} \in \mathrm{T},\ d_{\mathrm{B}} = \frac{1}{\det\mathrm{B}}\right\}$

2. $\left\{\begin{bmatrix} \mathrm{B} & 0 \\ 0 & \pm d_{\mathrm{B}}\mathrm{B} \end{bmatrix} : \mathrm{B} \in \mathrm{T},\ d_{\mathrm{B}} = \frac{1}{\det\mathrm{B}}\}\right\}$

3. $\left\{\begin{bmatrix} \mathrm{A} & 0 \\ 0 & \mathrm{B} \end{bmatrix} : \mathrm{A}, \mathrm{B} \in \mathrm{T}, \det(\mathrm{AB}) = 1\right\}$

Now since G and H are in SL(4, C), $\gamma = \begin{bmatrix} 0 & \mathrm{I} \\ \xi & 0 \end{bmatrix} \in \mathrm{SL}(4,\mathrm{C})$. Therefore $\xi = \pm\mathrm{I}$. When H is either of the second two groups $\begin{bmatrix} 0 & \mathrm{I} \\ \mathrm{I} & 0 \end{bmatrix}\mathrm{H} = \begin{bmatrix} 0 & \mathrm{I} \\ -\mathrm{I} & 0 \end{bmatrix}\mathrm{H}$. When $\mathrm{H} = \left\{\begin{bmatrix} \mathrm{B} & 0 \\ 0 & d_{\mathrm{B}}\mathrm{B} \end{bmatrix} : \mathrm{B} \in \mathrm{T},\ d_{\mathrm{B}} = \frac{1}{\det\mathrm{B}}\right\}$, $\mathrm{H} \cup \begin{bmatrix} 0 & \mathrm{I} \\ -\mathrm{I} & 0 \end{bmatrix}\mathrm{H}$ is not a group. Hence there are only three distinct possibilities for $\mathrm{H} \cup \gamma\mathrm{H}$. Lemma 4.12 ensures that each of these three possibilities is a group. The result follows. ∎

In [BL1] the author gives a list of the nonconjugate imprimitive projective groups via generating substitutions. The classification given in the following result was constructed by the author prior to her knowledge of this paper by Blichfeldt.

Theorem 4.3 *Let G be an imprimitive, nonmonomial subgroup of SL(4, C) containing a normal subgroup H of index two where $H \subset T \times T$, with T projectively finite. Then there exists a nonnegative integer, n, and a subgroup $F \subset SL(2, C)$ of T such that G is conjugate to one of the following groups.*

- $\left\{\begin{bmatrix} A & 0 \\ 0 & \xi_{\mathrm{A}}B \end{bmatrix} : A, B \in P_n, B \in AF\right\} \bigcup \left\{\begin{bmatrix} 0 & A \\ \xi_{\mathrm{A}}B & 0 \end{bmatrix} : A, B \in P_n, B \in AF\right\}$

  where $\xi_{\mathrm{A}} \in \{\frac{\pm 1}{det(A)}\}$ and one of the following holds.

  1. *$P = A_5$ and $\pi(F) \cong \{I\}$ or A_5.*

  2. *$P = S_4$ and $\pi(F) \cong \{I\}, K, A_4$ or S_4.*

  3. *$P = A_4$ and $\pi(F) \cong \{I\}, K$ or A_4.*

- $\left\{\begin{bmatrix} A & 0 \\ 0 & \xi_{\mathrm{A}} B \end{bmatrix} : A, B \in (A_4)_n, B \in AF\right\} \bigcup \left\{\begin{bmatrix} 0 & A \\ \xi_{\mathrm{A}} B & 0 \end{bmatrix} : A, B \in O_{S_4}(A_4)_n, B \in AF\right\}$ *where* $\pi(F) \cong \{I\}, K$ *or* A_4 *and* $O_{S_4} = \begin{bmatrix} 0 & 1 \\ i & 0 \end{bmatrix}$.

Proof: Let G be such a group. Let π_1 and π_2 be the coordinate projections given in Lemma 4.7. Let $\pi : \mathrm{GL}(2, \mathrm{C}) \to \mathrm{PGL}(2, \mathrm{C})$ be the canonical projection. By Lemma 4.7, G has a normal subgroup H of index two where $\mathrm{H} \subseteq \left\{\begin{bmatrix} \mathrm{A} & 0 \\ 0 & \mathrm{B} \end{bmatrix} : \mathrm{A}, \mathrm{B} \in \mathrm{GL}(2, \mathrm{C})\right\}$ and $\pi_1(\mathrm{H}) = \pi_2(\mathrm{H}) = \mathrm{T}$. Since we have assumed that T is projectively finite, $\pi(\mathrm{T})$ is a finite primitive subgroup of $\mathrm{GL}(2, \mathrm{C})$. Blichfeldt shows in [BL3] that $\pi(\mathrm{T})$ is isomorphic to a permutation group, $\mathrm{P} \in \{\mathrm{A}_4, \mathrm{S}_4, \mathrm{A}_5\}$. Let $ker(\pi_1) = \{\mathrm{I}\} \times \mathrm{F}$. F must be a normal subgroup of T. Therefore $\mathrm{P} = \mathrm{A}_5$ implies that $\pi(\mathrm{F}) \cong \{\mathrm{I}\}$ or A_5. $\mathrm{P} = \mathrm{S}_4$ implies that $\pi(\mathrm{F}) \cong \{\mathrm{I}\}, \mathrm{K}, \mathrm{A}_4$ or S_4 and $\mathrm{P} = \mathrm{A}_4$ implies that $\pi(\mathrm{F}) \cong \{\mathrm{I}\}, \mathrm{K}$ or A_4. By Lemma 4.9, there is a mapping $f \in Aut(\mathrm{T}/\tilde{\mathrm{F}})$ where $\tilde{\mathrm{F}} = \pi^{-1}(\pi(\mathrm{F}))$ such that $f(\mathrm{A}\tilde{\mathrm{F}}) = \mathrm{B}\tilde{\mathrm{F}}$ for each $(\mathrm{A}, \mathrm{B}) \in \mathrm{H}$. Given this map and the element γ of Lemma 4.7, Lemma 4.13, implies that H is conjugate to $\left\{\begin{bmatrix} \mathrm{A} & 0 \\ 0 & \xi_{\mathrm{A}}\mathrm{B} \end{bmatrix} : \mathrm{A}, \mathrm{B} \in \mathrm{T}, \mathrm{B} \in \mathrm{AF}\right\}$, where each $\xi_{\mathrm{A}} \in Z(\mathrm{GL}(2, \mathrm{C}))$, and $\exists$ another scalar, ξ, and an element $y = \begin{bmatrix} \mathrm{C} & 0 \\ 0 & \mathrm{D} \end{bmatrix}$, $\mathrm{C}, \mathrm{D} \in \mathrm{GL}(2, \mathrm{C})$, such that one of the following holds.

- $\pi_i(\mathrm{H}) \cong \mathrm{T}$ where $\pi(\mathrm{T}) \cong \mathrm{A}_4$ in which case $y\mathrm{G}y^{-1}$ is equal to either $\mathrm{H} \cup \begin{bmatrix} 0 & \mathrm{I} \\ \xi & 0 \end{bmatrix} \mathrm{H}$ or $\mathrm{H} \cup \begin{bmatrix} 0 & \mathrm{O}_{\mathrm{S}_4} \\ \xi \mathrm{O}_{\mathrm{S}_4} & 0 \end{bmatrix} \mathrm{H}$ where $\mathrm{O}_{\mathrm{S}_4} = \begin{bmatrix} 0 & 1 \\ i & 0 \end{bmatrix}$.
- $y\mathrm{G}y^{-1}$ is equal to $\mathrm{H} \cup \begin{bmatrix} 0 & \mathrm{I} \\ \xi & 0 \end{bmatrix} \mathrm{H}$.

Since $\mathrm{G} \subseteq \mathrm{SL}(4, \mathrm{C})$, both H and γ are in $\mathrm{SL}(4, \mathrm{C})$. $\mathrm{H} \subseteq \mathrm{SL}(4, \mathrm{C})$ implies that $\mathrm{F} \subseteq \mathrm{SL}(2, \mathrm{C})$ and $\xi_{\mathrm{A}} = \pm\det(\mathrm{A}^{-1})$. $\gamma \in \mathrm{SL}(4, \mathrm{C})$ implies that $\xi \in \{\pm\mathrm{I}\}$. The result follows.∎

4.2 The Primitive Groups of Degree Four

4.2.1 The Infinite Primitive Subroups of $\mathbf{SL}(4, \mathbf{C})$

Given an infinite primitive group, G, Proposition 2.3 of [SU1] implies that the connected component of the identity, G^0, is a semisimple group. We will consider two cases separately, that when G^0 is irreducible, in which case we will employ a bit of Lie algebra, and that when G^0 is reducible.

Lemma 4.14 *(Singer) Let G be an infinite primitive subgroup of $SL(4, C)$. Then either G^0 is an irreducible group or G^0 is conjugate to*

$$\left\{ \begin{bmatrix} A & 0 \\ 0 & A \end{bmatrix} : A \in SL(2, C) \right\}.$$

Proof: Since G is primitive and hence an irreducible group G^0 is reductive and hence completely reducible. If G^0 is not irreducible then we can write $C^4 = \bigoplus_{i=1}^m V_i$ where each V_i is an irreducible G^0- module. Since G is primitive, the V_i must be isomorphic G^0- modules. If $\dim V_i = 1$ then G^0 would be conjugate to a subgroup of constant matrices in $SL(4, C)$ and hence would be trivial. Therefore G^0 is conjugate to a subgroup of $\left\{ \begin{bmatrix} A & 0 \\ 0 & A \end{bmatrix} : A \in GL(2, C) \right\}$. Since $G \subseteq SL(4, C)$ we have that $(\det A)^2 = 1$ and since G^0 is connected, $G^0 \subseteq \left\{ \begin{bmatrix} A & 0 \\ 0 & A \end{bmatrix} : A \in SL(2, C) \right\}$. Now, let H be the image of G^0 under the map which takes $\begin{bmatrix} A & 0 \\ 0 & A \end{bmatrix}$ to A. If H is a proper subgroup of $SL(2, C)$ then V_i is not an irreducible G_0- module therefore we must have that $G^0 = \left\{ \begin{bmatrix} A & 0 \\ 0 & A \end{bmatrix} : A \in SL(2, C) \right\}$. ∎

Lemma 4.15 *(Singer) Let* $G = \left\{ \begin{bmatrix} A & 0 \\ 0 & A \end{bmatrix} : A \in SL(2, C) \right\}$.

1. *The centralizer of G in $GL(4, C)$ is* $\tilde{C} = \left\{ \begin{bmatrix} aI & bI \\ cI & dI \end{bmatrix} : (ad - bc) \neq 0 \right\}$.
2. *The normalizer of G in $GL(4, C)$ is $\tilde{C} \cdot G$.*

Proof: Let $\mathrm{G} = \left\{ \begin{bmatrix} \mathrm{A} & 0 \\ 0 & \mathrm{A} \end{bmatrix} : \mathrm{A} \in \mathrm{SL}(2,\mathrm{C}) \right\}$.

(Proof of 1) Let $\mathrm{U} = \begin{bmatrix} u_1 & u_2 \\ u_3 & u_4 \end{bmatrix}$, where each $u_i \in \mathrm{GL}(2,\mathrm{C})$, centralize G. Then $u_i\mathrm{A} = \mathrm{A}u_i$ for each i and all $\mathrm{A} \in \mathrm{SL}(2,\mathrm{C})$. Schur's Lemma then implies that each u_i is a constant matrix and so we may write $\mathrm{U} = \begin{bmatrix} a\mathrm{I} & b\mathrm{I} \\ c\mathrm{I} & d\mathrm{I} \end{bmatrix}$. The determinant of this matrix is $(ad-bc)^2$ and hence this quantity must be equal to 1.

(Proof of 2) Let $\mathrm{U} \in \mathrm{GL}(4,\mathrm{C})$ normalize G. Since G is isomorphic to $\mathrm{SL}(2,\mathrm{C})$ whose automorphism group contains only inner automorphisms, we have that UV^{-1} centralizes G for some $\mathrm{V} \in \mathrm{G}$. The result now follows from the previous lemma. ∎

G^0 reducible

Theorem 4.4 *(Singer) Let $G_2 = \{A \in GL(2,C) : detA^2 = 1\}$.*

1. *Let G be a primitive subgroup of $SL(4,C)$ with $G^0 = \left\{ \begin{bmatrix} A & 0 \\ 0 & A \end{bmatrix} : A \in SL(2,C) \right\}$. Then there exists a finite primitive subgroup, $\tilde{H}$, of G_2 such that $G = HG^0$ where $H = \left\{ \begin{bmatrix} aI & bI \\ cI & dI \end{bmatrix} : \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in \tilde{H}, I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \right\}$.*
2. *Furthermore, for any primitive subgroup $\tilde{H}$ of G_2, HG^0 is a primitive subgroup of $SL(4,C)$ where H is defined as above.*

Proof:

(Proof of 1) Let $\tilde{C}$ be as in Lemma 4.15 and let $\mathrm{G} \cap \tilde{C} = \mathrm{H}$. Since $\mathrm{G} \subseteq \tilde{C}\mathrm{G}^0$, the group H maps surjectively onto G/G^0. Therefore $\mathrm{HG}^0 = \mathrm{G}$. Since $\tilde{C} \cap \mathrm{G}^0 = \left\{ \begin{bmatrix} \mathrm{I} & 0 \\ 0 & \mathrm{I} \end{bmatrix}, \begin{bmatrix} 0 & -\mathrm{I} \\ -\mathrm{I} & 0 \end{bmatrix} \right\}$, H is finite. Since $\mathrm{H} \subseteq \mathrm{SL}(4,\mathrm{C})$, we have that

$\tilde{\mathrm{H}} = \left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} : \begin{bmatrix} a\mathrm{I} & b\mathrm{I} \\ c\mathrm{I} & d\mathrm{I} \end{bmatrix} \in \mathrm{H} \right\}$ is a finite subgroup of G_2. We will show now that $\tilde{\mathrm{H}}$ is primitive. Assume not and let $g = \begin{bmatrix} u_1 & u_2 \\ u_3 & u_4 \end{bmatrix} \in \mathrm{SL}(2,\mathrm{C})$ satisfy $g\mathrm{H}g^{-1} \subseteq \left\{ \begin{bmatrix} a & 0 \\ 0 & a^{-1} \end{bmatrix} : a \in \mathrm{C}^* \right\} \cup \left\{ \begin{bmatrix} 0 & -a \\ a^{-1} & 0 \end{bmatrix} : a \in \mathrm{C}^* \right\}$. Then conjugating by $h = \begin{bmatrix} u_1\mathrm{I} & u_2\mathrm{I} \\ u_3\mathrm{I} & u_4\mathrm{I} \end{bmatrix}$ allows us to assume that $\mathrm{G} = \mathrm{HG}^0$ where $\mathrm{H} \subseteq \left\{ \begin{bmatrix} a\mathrm{I} & 0 \\ 0 & a^{-1}\mathrm{I} \end{bmatrix} : a \in \mathrm{C}^* \right\} \cup \left\{ \begin{bmatrix} 0 & -a\mathrm{I} \\ a^{-1}\mathrm{I} & 0 \end{bmatrix} : a \in \mathrm{C}^* \right\}$

and $\mathrm{G}^0 = \left\{ \begin{bmatrix} \mathrm{A} & 0 \\ 0 & \mathrm{A} \end{bmatrix} : \mathrm{A} \in \mathrm{SL}(2,\mathrm{C}) \right\}$.
This would imply that G would be either irreducible or imprimitive, a contradiction.

(Proof of 2) Let $\{e_1, e_2, e_3, e_4\}$ be the standard basis of C^4, that is the basis used to define $\mathrm{G}^0 = \left\{ \begin{bmatrix} \mathrm{A} & 0 \\ 0 & \mathrm{A} \end{bmatrix} : \mathrm{A} \in \mathrm{SL}(2,\mathrm{C}) \right\}$. Assume G is not primitive and let $\mathrm{C}^4 = \bigoplus_{i=1}^m \mathrm{V}_i$ where G permutes the V_i. Since G^0 is connected, we have that G^0 leaves each V_i invariant. Therefore it must be that $m = 2$, each V_i has dimension 2 and V_1 and V_2 are isomorphic G^0- modules. Furthermore, the V_1 and V_2 are G^0- isomorphic to the 2-dimensional irreducible $\mathrm{SL}(2,\mathrm{C})-$ modules. We therefore can select a basis $\{f_1, f_2\}$ of V_1 and $\{f_3, f_4\}$ of V_2 such that for any $g \in \mathrm{G}^0$, the matrices of g with respect to $\{e_1, e_2, e_3, e_4\}$ and $\{f_1, f_2, f_3.f_4\}$ are the same. That is, for some $\mathrm{A} \in \mathrm{SL}(2,\mathrm{C})$, $g = \begin{bmatrix} \mathrm{A} & 0 \\ 0 & \mathrm{A} \end{bmatrix}$ with respect to both bases. This implies that if U is the change of basis matrix then U commutes with G^0 and hence by Lemma 4.15, U is of the form $\mathrm{U} = \begin{bmatrix} u_1\mathrm{I} & u_2\mathrm{I} \\ u_3\mathrm{I} & u_4\mathrm{I} \end{bmatrix}$. Furthermore, all matrices of UHU^{-1} are either block diagonal or block skew diagonal. Therefore $\begin{bmatrix} u_1 & u_2 \\ u_3 & u_4 \end{bmatrix} \tilde{\mathrm{H}} \begin{bmatrix} u_1 & u_2 \\ u_3 & u_4 \end{bmatrix}^{-1}$ is contained in $\left\{ \begin{bmatrix} a & 0 \\ 0 & a^{-1} \end{bmatrix} : a \in \mathrm{C}^* \right\} \cup \left\{ \begin{bmatrix} 0 & -a \\ a^{-1} & 0 \end{bmatrix} : a \in \mathrm{C}^* \right\}$ contradicting the fact that $\tilde{\mathrm{H}}$ is primitive. ∎

Remarks: 1.) Note that the finite primitive subgroups of $\mathrm{GL}(2,\mathrm{C})$ are the groups $A_4^{SL_2}, S_4^{SL_2}$, and $A_5^{SL_2}$ or an extension of one of these groups by $\sqrt{-1}\mathrm{I}$, where I is the identity in $\mathrm{GL}(2,\mathrm{C})$. These extensions will be denoted by $(A_4^{SL_2})_2, (S_4^{SL_2})_2$, and $(A_5^{SL_2})_2$.

2.) $\mathrm{GL}(2,\mathrm{C}) \times \mathrm{GL}(2,\mathrm{C})$ acts in a natural way on $\mathrm{C}^2 \otimes \mathrm{C}^2$ via the maps $\begin{bmatrix} a & b \\ c & d \end{bmatrix} \times 1 \to \begin{bmatrix} a\mathrm{I} & b\mathrm{I} \\ c\mathrm{I} & d\mathrm{I} \end{bmatrix}$ and $1 \times \begin{bmatrix} a & b \\ c & d \end{bmatrix} \to \begin{bmatrix} a & b & 0 & 0 \\ c & d & 0 & 0 \\ 0 & 0 & a & b \\ 0 & 0 & c & d \end{bmatrix}$. If $\tilde{\mathrm{H}}$ is any of the above six finite groups, we have a homomorphism of $\tilde{H} \times \mathrm{SL}(2,\mathrm{C})$ into $\mathrm{GL}(4,\mathrm{C})$. (Note: there is a nontrivial kernel.) The groups described in Theorem 4.4 are just the images of the groups $\tilde{\mathrm{H}} \times \mathrm{SL}(2,\mathrm{C})$.

3.) Note that the image of $SL(2,C) \times SL(2,C)$ in $GL(4,C)$ given by the action on $C^2 \otimes C^2$ is $SO(4,C)$. Therefore, if $\tilde{H} \subseteq SL(2,C)$, then the image of $\tilde{H} \times SL(2,C)$ is in $SO(4,C)$.

G^0 irreducible

Lie algebra considerations will yield the following:

Theorem 4.5 *Let G be an infinite primitive subgroup of $SL(4,C)$. Then one of the following holds:*

1. *$G \subseteq \bigcup_{\omega^4=1} \omega G^0$, where G^0 is one of*
   - *(a) $SL(2,C)$ acting on $Sym^3 C^2$*
   - *(b) $SP(4,C)$*
2. *$G \subseteq (\cup_{\omega^4=1} \omega G^0) \bigcup J(\cup_{\omega^4=1} \omega G^0)$, where $J = \begin{bmatrix} 0 & I \\ I & 0 \end{bmatrix}$, $I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$, and G^0 is the usual representation of $SO(4,C)$.*
3. *$G = SL(4,C)$ acting on W, where W is one of*
   - *(a) the usual representation space, $V \approx C^4$*
   - *(b) the dual space, V^*, of V.*

Proof: Let G be an infinite primitive subgroup of $SL(4,C)$. Then the connected component of the identity, G^0, is also an infinite primitive subgroup of $SL(4,C)$ which, by Proposition 2.3 of [SU1], is a semisimple group. Let $\mathcal{G}$ be the Lie algebra associated with G^0. Then $\mathcal{G}$ is a semisimple Lie subalgebra of $sl(4,C)$ which has an irreducible representation of degree four. Up to isomorphism the simple Lie subalgebras of $sl(4,C)$ which can be found in [HU1] sections 1 and 19 are:

- $sl(4,C)$ of dimension 15.
- $sl(2,C)$ of dimension 3.
- $sp(4,C)$ of dimension 10.

Any semisimple Lie algebra is a direct sum of simple ones. Therefore using a dimension argument and the above list we can construct a list of irreducible semisimple Lie subalgebras of $\mathrm{sl}(4, \mathrm{C})$. Up to isomorphism these are:

1. $\mathrm{sl}(4, \mathrm{C})$ of dimension 15
2. $\mathrm{sl}(3, \mathrm{C})$ of dimension 8
3. $\mathrm{sl}(2, \mathrm{C})$ of dimension 3
4. $\mathrm{sp}(4, \mathrm{C})$ of dimension 10
5. $\mathrm{so}(4, \mathrm{C}) \approx \mathrm{sl}(2, \mathrm{C}) \times \mathrm{sl}(2, \mathrm{C})$ of dimension 9

Let V denote the standard representation space for each of the listed simple Lie algebras. Then the only four dimensional vector space on which $\mathrm{sl}(2, \mathrm{C})$ acts irreducibly is $\mathrm{Sym}^3 V$. $\mathrm{sl}(3, \mathrm{C})$ has no such representation. $\mathrm{sl}(4, \mathrm{C})$ acts irreducibly and distinctly on both V and V^*. Finally, the only four dimensional vector space on which $\mathrm{sp}(4, \mathrm{C})$ acts irreducibly is V itself. Now any vector space on which a direct sum of infinite Lie algebras acts irreducibly is a tensor product of spaces on which the factors act irreducibly. Therefore the only four dimensional vector space on which $\mathrm{so}(4, \mathrm{C}) \cong \mathrm{sl}(2, \mathrm{C}) \times \mathrm{sl}(2, \mathrm{C})$ acts irreducibly is $\mathrm{V} \otimes \mathrm{V}$ where $\mathrm{V} \cong \mathrm{C}^2$.

There is a one-to-one correspondence between the representations of a Lie algebra and its associated simply connected algebraic group. Therefore we have constructed a complete list of connected irreducible linear algebraic subgroups of $\mathrm{SL}(4, \mathrm{C})$. They are:

- $\mathrm{SL}(4, \mathrm{C})$ acting on its standard representation, V.
- $\mathrm{SL}(4, \mathrm{C})$ acting on the dual, V^*.
- $\mathrm{SL}(2, \mathrm{C})$ acting on $\mathrm{Sym}^3 V$ where V is its standard representation.
- $\mathrm{SP}(4, \mathrm{C})$ acting on its standard representation, V.
- $\mathrm{SO}(4, \mathrm{C})$ is the image of $\mathrm{SL}(2, \mathrm{C}) \times \mathrm{SL}(2, \mathrm{C})$ acting on $V \otimes V$ where V is the standard representation for $\mathrm{SL}(2, \mathrm{C})$.

The above are all of the possibilities for the connected component of the identity, G^0, of G when G is an infinite primitive subgroup of $\mathrm{SL}(4,\mathrm{C})$. Finally, we will determine all of the possibilities for the group G itself. Recall that any element of G not in G^0 induces an automorphism on G^0 via conjugation. Note that automorphism groups are independent of the representation.

In [HU2] we find that for any semisimple algebraic group G we have that $Aut(\mathrm{G})$ is the semidirect product of the inner automorphisms of G and the automorphism group of the Dynkin diagram associated to the Lie algebra that corresponds to G. In [HU1] section 11 one finds that the only two groups out of the list above leading to nontrivial automorphism groups of a Dynkin diagram are $\mathrm{SL}(4,\mathrm{C})$ and $\mathrm{SO}(4,\mathrm{C})$. The diagram automorphisms associated to these groups are elements of $\mathbb{Z}_2 = \mathbb{Z}/2\mathbb{Z}$.

Case 1. Suppose G^0 is either $\mathrm{SL}(2,\mathrm{C})$ or $\mathrm{SP}(4,\mathrm{C})$ Then $Aut(\mathrm{G}^0) = Inn(\mathrm{G}^0)$. Let G be an algebraic subgroup of $\mathrm{SL}(4,\mathrm{C})$ with component of the identity G^0. Note that G^0 is a normal subgroup of finite index in G. By the preceeding remarks, conjugation by any element g of G induces an inner automorphism on G^0. That is for each $g \in \mathrm{G}$ there is an $h \in \mathrm{G}^0$ such that $gxg^{-1} = hxh^{-1}$ for every $x \in \mathrm{G}^0$. That is $h^{-1}g$ centralizes the irreducible group G^0. Schur's Lemma implies that $h^{-1}g$ must be scalar. Since $h^{-1}g \in \mathrm{SL}(4,\mathrm{C})$, $h^{-1}g = \xi\mathrm{I}$ where ξ is a fourth root of unity.

Therefore we have that $\mathrm{G} \subseteq \{\mathrm{I}, \omega\mathrm{I}, \omega^2\mathrm{I}, \omega^3\mathrm{I}\} \cdot \mathrm{G}^0$, where ω is a primitive forth root of unity.

Case 2. Suppose $\mathrm{G}^0 = \mathrm{SO}(4,\mathrm{C})$. The automorphism group of G^0 is given by the semidirect product of $Inn(\mathrm{G}^0)$ and $\mathbb{Z}_2 = \mathbb{Z}/2\mathbb{Z}$. Recall that $\mathrm{SO}(4,\mathrm{C}) \approx \mathrm{SL}(2,\mathrm{C}) \times \mathrm{SL}(2,\mathrm{C})$. The action of $\mathbb{Z}_2$ interchanges the copies of $\mathrm{SL}(2,\mathrm{C})$, so in matrix form this automorphism is equivalent to conjugation by the element $\begin{bmatrix} 0 & \mathrm{I} \\ \mathrm{I} & 0 \end{bmatrix}$, where I is the identity in $\mathrm{GL}(2,\mathrm{C})$. Call this matrix J. Then, as above, since conjugation by any element of G induces an automorphism on G^0, we have that $\mathrm{G} \subseteq (\cup_\xi \xi \cdot \mathrm{I} \cdot \mathrm{G}^0)$, where ξ is either a fourth root of unity or a fourth root of unity times J.

Case 3. Suppose $\mathrm{G}^0 = \mathrm{SL}(4,\mathrm{C})$. Since we have assumed that $\mathrm{G} \subseteq \mathrm{SL}(4,\mathrm{C})$ we take $\mathrm{G} = \mathrm{SL}(4,\mathrm{C})$. ∎

4.2.2 The Finite Primitive Subgroups of $\mathbf{SL}(4, \mathbf{C})$.

In [BL2] Blichfeldt gives a list of the 30 finite primitive projective groups of degree four. Let $\pi : \mathrm{GL}(4, \mathrm{C}) \to \mathrm{PGL}(4, \mathrm{C})$ be the canonical projection. Any finite primitive subgroup, g, of $\mathrm{SL}(4, \mathrm{C})$ must project onto one of the groups in Blichfeldt's list. Let $\overline{\mathrm{G}}$ denote a primitive projective group. Given determinant one generating representatives for this group, one can obtain a subgroup $\tilde{\mathrm{G}}$ of $\mathrm{SL}(4, \mathrm{C})$ which projects onto $\overline{\mathrm{G}}$ and which is largest in the sense that it contains any other subgroup of $\mathrm{SL}(4, \mathrm{C})$ projecting onto $\overline{\mathrm{G}}$. $\tilde{\mathrm{G}}$ is the subgroup of $\mathrm{SL}(4, \mathrm{C})$ generated by the given set of generating representatives of $\overline{\mathrm{G}}$ along with a generator for the degree four representation of the group of units, U_4. Using the computer algebra package *MAGMA* we have constructed $\tilde{\mathrm{G}}$ for each of the finite primitive groups given in [BL3]. These will constitute the list of finite primitive subgroups of $\mathrm{SL}(4, \mathrm{C})$ and will be donoted by FP1,. . ., FP30.

Matrix Generators for the Finite Primitive Subgroups of $\mathbf{SL}(4, \mathbf{C})$

In the following ζ_i denotes a primitive i^{th} root of unity. The first 12 groups are generated by tensor products of the generators of the finite primitive subgroups of $\mathrm{SL}(2, \mathrm{C})$.

FP1 (Order = 576)

$$\begin{bmatrix} \zeta_4 & 0 & 0 & 0 \\ 0 & -\zeta_4 & 0 & 0 \\ 0 & 0 & \zeta_4 & 0 \\ 0 & 0 & 0 & -\zeta_4 \end{bmatrix}, \begin{bmatrix} \frac{1}{2}\zeta_4 - \frac{1}{2} & \frac{1}{2}\zeta_4 - \frac{1}{2} & 0 & 0 \\ \frac{1}{2}\zeta_4 + \frac{1}{2} & -\frac{1}{2}\zeta_4 - \frac{1}{2} & 0 & 0 \\ 0 & 0 & \frac{1}{2}\zeta_4 - \frac{1}{2} & \frac{1}{2}\zeta_4 - \frac{1}{2} \\ 0 & 0 & \frac{1}{2}\zeta_4 + \frac{1}{2} & -\frac{1}{2}\zeta_4 - \frac{1}{2} \end{bmatrix}, \begin{bmatrix} \zeta_4 & 0 & 0 & 0 \\ 0 & \zeta_4 & 0 & 0 \\ 0 & 0 & -\zeta_4 & 0 \\ 0 & 0 & 0 & -\zeta_4 \end{bmatrix},$$

$$\begin{bmatrix} \frac{1}{2}\zeta_4 - \frac{1}{2} & 0 & \frac{1}{2}\zeta_4 - \frac{1}{2} & 0 \\ 0 & \frac{1}{2}\zeta_4 - \frac{1}{2} & 0 & \frac{1}{2}\zeta_4 - \frac{1}{2} \\ \frac{1}{2}\zeta_4 + \frac{1}{2} & 0 & -\frac{1}{2}\zeta_4 - \frac{1}{2} & 0 \\ 0 & \frac{1}{2}\zeta_4 + \frac{1}{2} & 0 & -\frac{1}{2}\zeta_4 - \frac{1}{2} \end{bmatrix}$$

FP2 (Order = 1152)

$$\begin{bmatrix} \frac{1}{2}\zeta_{40}^{10} - \frac{1}{2} & 0 & \frac{1}{2}\zeta_{40}^{10} - \frac{1}{2} & 0 \\ 0 & \frac{1}{2}\zeta_{40}^{10} - \frac{1}{2} & 0 & \frac{1}{2}\zeta_{40}^{10} - \frac{1}{2} \\ \frac{1}{2}\zeta_{40}^{10} + \frac{1}{2} & 0 & -\frac{1}{2}\zeta_{40}^{10} - \frac{1}{2} & 0 \\ 0 & \frac{1}{2}\zeta_{40}^{10} + \frac{1}{2} & 0 & -\frac{1}{2}\zeta_{40}^{10} - \frac{1}{2} \end{bmatrix}, \begin{bmatrix} \zeta_{40}^{5} & 0 & 0 & 0 \\ 0 & \zeta_{40}^{5} & 0 & 0 \\ 0 & 0 & -\zeta_{40}^{15} & 0 \\ 0 & 0 & 0 & -\zeta_{40}^{15} \end{bmatrix},$$

$$\begin{bmatrix} \frac{1}{2}\zeta_{40}^{10} - \frac{1}{2} & \frac{1}{2}\zeta_{40}^{10} - \frac{1}{2} & 0 & 0 \\ \frac{1}{2}\zeta_{40}^{10} + \frac{1}{2} & -\frac{1}{2}\zeta_{40}^{10} - \frac{1}{2} & 0 & 0 \\ 0 & 0 & \frac{1}{2}\zeta_{40}^{10} - \frac{1}{2} & \frac{1}{2}\zeta_{40}^{10} - \frac{1}{2} \\ 0 & 0 & \frac{1}{2}\zeta_{40}^{10} + \frac{1}{2} & -\frac{1}{2}\zeta_{40}^{10} - \frac{1}{2} \end{bmatrix}, \begin{bmatrix} \zeta_{40}^{10} & 0 & 0 & 0 \\ 0 & -\zeta_{40}^{10} & 0 & 0 \\ 0 & 0 & \zeta_{40}^{10} & 0 \\ 0 & 0 & 0 & -\zeta_{40}^{10} \end{bmatrix}, \begin{bmatrix} \zeta_{40}^{10} & 0 & 0 & 0 \\ 0 & \zeta_{40}^{10} & 0 & 0 \\ 0 & 0 & \zeta_{40}^{10} & 0 \\ 0 & 0 & 0 & \zeta_{40}^{10} \end{bmatrix}$$

FP3 (Order = 2880)

$$\begin{bmatrix} \frac{1}{2}\zeta_{40}^{10}-\frac{1}{2} & 0 & \frac{1}{2}\zeta_{40}^{10}-\frac{1}{2} & 0 \\ 0 & \frac{1}{2}\zeta_{40}^{10}-\frac{1}{2} & 0 & \frac{1}{2}\zeta_{40}^{10}-\frac{1}{2} \\ \frac{1}{2}\zeta_{40}^{10}+\frac{1}{2} & 0 & -\frac{1}{2}\zeta_{40}^{10}-\frac{1}{2} & 0 \\ 0 & \frac{1}{2}\zeta_{40}^{10}+\frac{1}{2} & 0 & -\frac{1}{2}\zeta_{40}^{10}-\frac{1}{2} \end{bmatrix}, \begin{bmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{bmatrix},$$

$$\begin{bmatrix} \zeta_{40}^{10} & 0 & 0 & 0 \\ 0 & \zeta_{40}^{10} & 0 & 0 \\ 0 & 0 & -\zeta_{40}^{10} & 0 \\ 0 & 0 & 0 & -\zeta_{40}^{10} \end{bmatrix}, \begin{bmatrix} -\zeta_{40}^{4} & 0 & 0 & 0 \\ 0 & \zeta_{40}^{12}-\zeta_{40}^{8}+\zeta_{40}^{4}-1 & 0 & 0 \\ 0 & 0 & -\zeta_{40}^{4} & 0 \\ 0 & 0 & 0 & \zeta_{40}^{12}-\zeta_{40}^{8}+\zeta_{40}^{4}-1 \end{bmatrix},$$

$$\begin{bmatrix} -\frac{3}{5}\zeta_{40}^{12}+\frac{1}{5}\zeta_{40}^{8}-\frac{4}{5}\zeta_{40}^{4}+\frac{2}{5} & \frac{1}{5}\zeta_{40}^{12}+\frac{3}{5}\zeta_{40}^{8}-\frac{2}{5}\zeta_{40}^{4}+\frac{1}{5} & 0 & 0 \\ \frac{1}{5}\zeta_{40}^{12}+\frac{3}{5}\zeta_{40}^{8}-\frac{2}{5}\zeta_{40}^{4}+\frac{1}{5} & \frac{3}{5}\zeta_{40}^{12}-\frac{1}{5}\zeta_{40}^{8}+\frac{4}{5}\zeta_{40}^{4}-\frac{2}{5} & 0 & 0 \\ 0 & 0 & -\frac{3}{5}\zeta_{40}^{12}+\frac{1}{5}\zeta_{40}^{8}-\frac{4}{5}\zeta_{40}^{4}+\frac{2}{5} & \frac{1}{5}\zeta_{40}^{12}+\frac{3}{5}\zeta_{40}^{8}-\frac{2}{5}\zeta_{40}^{4}+\frac{1}{5} \\ 0 & 0 & \frac{1}{5}\zeta_{40}^{12}+\frac{3}{5}\zeta_{40}^{8}-\frac{2}{5}\zeta_{40}^{4}+\frac{1}{5} & \frac{3}{5}\zeta_{40}^{12}-\frac{1}{5}\zeta_{40}^{8}+\frac{4}{5}\zeta_{40}^{4}-\frac{2}{5} \end{bmatrix}$$

FP4 (Order = 5760)

$$\begin{bmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{bmatrix}, \begin{bmatrix} \frac{1}{2}\zeta_{40}^{10}-\frac{1}{2} & 0 & \frac{1}{2}\zeta_{40}^{10}-\frac{1}{2} & 0 \\ 0 & \frac{1}{2}\zeta_{40}^{10}-\frac{1}{2} & 0 & \frac{1}{2}\zeta_{40}^{10}-\frac{1}{2} \\ \frac{1}{2}\zeta_{40}^{10}+\frac{1}{2} & 0 & -\frac{1}{2}\zeta_{40}^{10}-\frac{1}{2} & 0 \\ 0 & \frac{1}{2}\zeta_{40}^{10}+\frac{1}{2} & 0 & -\frac{1}{2}\zeta_{40}^{10}-\frac{1}{2} \end{bmatrix},$$

$$\begin{bmatrix} \zeta_{40}^{5} & 0 & 0 & 0 \\ 0 & \zeta_{40}^{5} & 0 & 0 \\ 0 & 0 & -\zeta_{40}^{15} & 0 \\ 0 & 0 & 0 & -\zeta_{40}^{15} \end{bmatrix}, \begin{bmatrix} -\zeta_{40}^{4} & 0 & 0 & 0 \\ 0 & \zeta_{40}^{12}-\zeta_{40}^{8}+\zeta_{40}^{4}-1 & 0 & 0 \\ 0 & 0 & -\zeta_{40}^{4} & 0 \\ 0 & 0 & 0 & \zeta_{40}^{12}-\zeta_{40}^{8}+\zeta_{40}^{4}-1 \end{bmatrix},$$

$$\begin{bmatrix} -\frac{3}{5}\zeta_{40}^{12}+\frac{1}{5}\zeta_{40}^{8}-\frac{4}{5}\zeta_{40}^{4}+\frac{2}{5} & \frac{1}{5}\zeta_{40}^{12}+\frac{3}{5}\zeta_{40}^{8}-\frac{2}{5}\zeta_{40}^{4}+\frac{1}{5} & 0 & 0 \\ \frac{1}{5}\zeta_{40}^{12}+\frac{3}{5}\zeta_{40}^{8}-\frac{2}{5}\zeta_{40}^{4}+\frac{1}{5} & \frac{3}{5}\zeta_{40}^{12}-\frac{1}{5}\zeta_{40}^{8}+\frac{4}{5}\zeta_{40}^{4}-\frac{2}{5} & 0 & 0 \\ 0 & 0 & -\frac{3}{5}\zeta_{40}^{12}+\frac{1}{5}\zeta_{40}^{8}-\frac{4}{5}\zeta_{40}^{4}+\frac{2}{5} & \frac{1}{5}\zeta_{40}^{12}+\frac{3}{5}\zeta_{40}^{8}-\frac{2}{5}\zeta_{40}^{4}+\frac{1}{5} \\ 0 & 0 & \frac{1}{5}\zeta_{40}^{12}+\frac{3}{5}\zeta_{40}^{8}-\frac{2}{5}\zeta_{40}^{4}+\frac{1}{5} & \frac{3}{5}\zeta_{40}^{12}-\frac{1}{5}\zeta_{40}^{8}+\frac{4}{5}\zeta_{40}^{4}-\frac{2}{5} \end{bmatrix}$$

FP5 (Order = 2304)

$$\begin{bmatrix} \frac{1}{2}\zeta_{40}^{10}-\frac{1}{2} & 0 & \frac{1}{2}\zeta_{40}^{10}-\frac{1}{2} & 0 \\ 0 & \frac{1}{2}\zeta_{40}^{10}-\frac{1}{2} & 0 & \frac{1}{2}\zeta_{40}^{10}-\frac{1}{2} \\ \frac{1}{2}\zeta_{40}^{10}+\frac{1}{2} & 0 & -\frac{1}{2}\zeta_{40}^{10}-\frac{1}{2} & 0 \\ 0 & \frac{1}{2}\zeta_{40}^{10}+\frac{1}{2} & 0 & -\frac{1}{2}\zeta_{40}^{10}-\frac{1}{2} \end{bmatrix}, \begin{bmatrix} \zeta_{40}^{5} & 0 & 0 & 0 \\ 0 & \zeta_{40}^{5} & 0 & 0 \\ 0 & 0 & -\zeta_{40}^{15} & 0 \\ 0 & 0 & 0 & -\zeta_{40}^{15} \end{bmatrix},$$

$$\begin{bmatrix} \frac{1}{2}\zeta_{40}^{10}-\frac{1}{2} & \frac{1}{2}\zeta_{40}^{10}-\frac{1}{2} & 0 & 0 \\ \frac{1}{2}\zeta_{40}^{10}+\frac{1}{2} & -\frac{1}{2}\zeta_{40}^{10}-\frac{1}{2} & 0 & 0 \\ 0 & 0 & \frac{1}{2}\zeta_{40}^{10}-\frac{1}{2} & \frac{1}{2}\zeta_{40}^{10}-\frac{1}{2} \\ 0 & 0 & \frac{1}{2}\zeta_{40}^{10}+\frac{1}{2} & -\frac{1}{2}\zeta_{40}^{10}-\frac{1}{2} \end{bmatrix}, \begin{bmatrix} \zeta_{40}^{5} & 0 & 0 & 0 \\ 0 & -\zeta_{40}^{15} & 0 & 0 \\ 0 & 0 & \zeta_{40}^{5} & 0 \\ 0 & 0 & 0 & -\zeta_{40}^{15} \end{bmatrix}$$

FP6 (Order = 14400)

$$\begin{bmatrix} -\zeta_{40}^{4} & 0 & 0 & 0 \\ 0 & -\zeta_{40}^{4} & 0 & 0 \\ 0 & 0 & \zeta_{40}^{12}-\zeta_{40}^{8}+\zeta_{40}^{4}-1 & 0 \\ 0 & 0 & 0 & \zeta_{40}^{12}-\zeta_{40}^{8}+\zeta_{40}^{4}-1 \end{bmatrix}, \begin{bmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{bmatrix},$$

$$\begin{bmatrix} -\frac{3}{5}\zeta_{40}^{12}+\frac{1}{5}\zeta_{40}^{8}-\frac{4}{5}\zeta_{40}^{4}+\frac{2}{5} & 0 & \frac{1}{5}\zeta_{40}^{12}+\frac{3}{5}\zeta_{40}^{8}-\frac{2}{5}\zeta_{40}^{4}+\frac{1}{5} & 0 \\ 0 & -\frac{3}{5}\zeta_{40}^{12}+\frac{1}{5}\zeta_{40}^{8}-\frac{4}{5}\zeta_{40}^{4}+\frac{2}{5} & 0 & \frac{1}{5}\zeta_{40}^{12}+\frac{3}{5}\zeta_{40}^{8}-\frac{2}{5}\zeta_{40}^{4}+\frac{1}{5} \\ \frac{1}{5}\zeta_{40}^{12}+\frac{3}{5}\zeta_{40}^{8}-\frac{2}{5}\zeta_{40}^{4}+\frac{1}{5} & 0 & \frac{3}{5}\zeta_{40}^{12}-\frac{1}{5}\zeta_{40}^{8}+\frac{4}{5}\zeta_{40}^{4}-\frac{2}{5} & 0 \\ 0 & \frac{1}{5}\zeta_{40}^{12}+\frac{3}{5}\zeta_{40}^{8}-\frac{2}{5}\zeta_{40}^{4}+\frac{1}{5} & 0 & \frac{3}{5}\zeta_{40}^{12}-\frac{1}{5}\zeta_{40}^{8}+\frac{4}{5}\zeta_{40}^{4}-\frac{2}{5} \end{bmatrix},$$

$$\begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{bmatrix}, \begin{bmatrix} -\zeta_{40}^{4} & 0 & 0 & 0 \\ 0 & \zeta_{40}^{12}-\zeta_{40}^{8}+\zeta_{40}^{4}-1 & 0 & 0 \\ 0 & 0 & -\zeta_{40}^{4} & 0 \\ 0 & 0 & 0 & \zeta_{40}^{12}-\zeta_{40}^{8}+\zeta_{40}^{4}-1 \end{bmatrix},$$

$$\begin{bmatrix} -\frac{3}{5}\zeta_{40}^{12}+\frac{1}{5}\zeta_{40}^{8}-\frac{4}{5}\zeta_{40}^{4}+\frac{2}{5} & \frac{1}{5}\zeta_{40}^{12}+\frac{3}{5}\zeta_{40}^{8}-\frac{2}{5}\zeta_{40}^{4}+\frac{1}{5} & 0 & 0 \\ \frac{1}{5}\zeta_{40}^{12}+\frac{3}{5}\zeta_{40}^{8}-\frac{2}{5}\zeta_{40}^{4}+\frac{1}{5} & \frac{3}{5}\zeta_{40}^{12}-\frac{1}{5}\zeta_{40}^{8}+\frac{4}{5}\zeta_{40}^{4}-\frac{2}{5} & 0 & 0 \\ 0 & 0 & -\frac{3}{5}\zeta_{40}^{12}+\frac{1}{5}\zeta_{40}^{8}-\frac{4}{5}\zeta_{40}^{4}+\frac{2}{5} & \frac{1}{5}\zeta_{40}^{12}+\frac{3}{5}\zeta_{40}^{8}-\frac{2}{5}\zeta_{40}^{4}+\frac{1}{5} \\ 0 & 0 & \frac{1}{5}\zeta_{40}^{12}+\frac{3}{5}\zeta_{40}^{8}-\frac{2}{5}\zeta_{40}^{4}+\frac{1}{5} & \frac{3}{5}\zeta_{40}^{12}-\frac{1}{5}\zeta_{40}^{8}+\frac{4}{5}\zeta_{40}^{4}-\frac{2}{5} \end{bmatrix}$$

FP7 (Order = 1152)

$$\begin{bmatrix} \zeta_{40}^{10} & 0 & 0 & 0 \\ 0 & -\zeta_{40}^{10} & 0 & 0 \\ 0 & 0 & \zeta_{40}^{10} & 0 \\ 0 & 0 & 0 & -\zeta_{40}^{10} \end{bmatrix}, \begin{bmatrix} \frac{1}{2}\zeta_{40}^{10}-\frac{1}{2} & \frac{1}{2}\zeta_{40}^{10}-\frac{1}{2} & 0 & 0 \\ \frac{1}{2}\zeta_{40}^{10}+\frac{1}{2} & -\frac{1}{2}\zeta_{40}^{10}-\frac{1}{2} & 0 & 0 \\ 0 & 0 & \frac{1}{2}\zeta_{40}^{10}-\frac{1}{2} & \frac{1}{2}\zeta_{40}^{10}-\frac{1}{2} \\ 0 & 0 & \frac{1}{2}\zeta_{40}^{10}+\frac{1}{2} & -\frac{1}{2}\zeta_{40}^{10}-\frac{1}{2} \end{bmatrix}, \begin{bmatrix} \zeta_{40}^{10} & 0 & 0 & 0 \\ 0 & \zeta_{40}^{10} & 0 & 0 \\ 0 & 0 & -\zeta_{40}^{10} & 0 \\ 0 & 0 & 0 & -\zeta_{40}^{10} \end{bmatrix},$$

$$\begin{bmatrix} \frac{1}{2}\zeta_{40}^{10}-\frac{1}{2} & 0 & \frac{1}{2}\zeta_{40}^{10}-\frac{1}{2} & 0 \\ 0 & \frac{1}{2}\zeta_{40}^{10}-\frac{1}{2} & 0 & \frac{1}{2}\zeta_{40}^{10}-\frac{1}{2} \\ \frac{1}{2}\zeta_{40}^{10}+\frac{1}{2} & 0 & -\frac{1}{2}\zeta_{40}^{10}-\frac{1}{2} & 0 \\ 0 & \frac{1}{2}\zeta_{40}^{10}+\frac{1}{2} & 0 & -\frac{1}{2}\zeta_{40}^{10}-\frac{1}{2} \end{bmatrix}, \begin{bmatrix} -1 & 0 & 0 & 0 \\ 0 & \zeta_{40}^{10} & 0 & 0 \\ 0 & 0 & \zeta_{40}^{10} & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

FP8 (Order = 1152)

$$\begin{bmatrix} \zeta_{40}^{10} & 0 & 0 & 0 \\ 0 & -\zeta_{40}^{10} & 0 & 0 \\ 0 & 0 & \zeta_{40}^{10} & 0 \\ 0 & 0 & 0 & -\zeta_{40}^{10} \end{bmatrix}, \begin{bmatrix} \frac{1}{2}\zeta_{40}^{10}-\frac{1}{2} & \frac{1}{2}\zeta_{40}^{10}-\frac{1}{2} & 0 & 0 \\ \frac{1}{2}\zeta_{40}^{10}+\frac{1}{2} & -\frac{1}{2}\zeta_{40}^{10}-\frac{1}{2} & 0 & 0 \\ 0 & 0 & \frac{1}{2}\zeta_{40}^{10}-\frac{1}{2} & \frac{1}{2}\zeta_{40}^{10}-\frac{1}{2} \\ 0 & 0 & \frac{1}{2}\zeta_{40}^{10}+\frac{1}{2} & -\frac{1}{2}\zeta_{40}^{10}-\frac{1}{2} \end{bmatrix}, \begin{bmatrix} \zeta_{40}^{10} & 0 & 0 & 0 \\ 0 & \zeta_{40}^{10} & 0 & 0 \\ 0 & 0 & -\zeta_{40}^{10} & 0 \\ 0 & 0 & 0 & -\zeta_{40}^{10} \end{bmatrix},$$

$$\begin{bmatrix} \frac{1}{2}\zeta_{40}^{10}-\frac{1}{2} & 0 & \frac{1}{2}\zeta_{40}^{10}-\frac{1}{2} & 0 \\ 0 & \frac{1}{2}\zeta_{40}^{10}-\frac{1}{2} & 0 & \frac{1}{2}\zeta_{40}^{10}-\frac{1}{2} \\ \frac{1}{2}\zeta_{40}^{10}+\frac{1}{2} & 0 & -\frac{1}{2}\zeta_{40}^{10}-\frac{1}{2} & 0 \\ 0 & \frac{1}{2}\zeta_{40}^{10}+\frac{1}{2} & 0 & -\frac{1}{2}\zeta_{40}^{10}-\frac{1}{2} \end{bmatrix}, \begin{bmatrix} \zeta_{40}^{5} & 0 & 0 & 0 \\ 0 & 0 & \zeta_{40}^{5} & 0 \\ 0 & \zeta_{40}^{5} & 0 & 0 \\ 0 & 0 & 0 & \zeta_{40}^{5} \end{bmatrix}$$

FP9 (Order = 2304)

$$\begin{bmatrix} \zeta_{40}^{10} & 0 & 0 & 0 \\ 0 & -\zeta_{40}^{10} & 0 & 0 \\ 0 & 0 & \zeta_{40}^{10} & 0 \\ 0 & 0 & 0 & -\zeta_{40}^{10} \end{bmatrix}, \begin{bmatrix} \frac{1}{2}\zeta_{40}^{10}-\frac{1}{2} & \frac{1}{2}\zeta_{40}^{10}-\frac{1}{2} & 0 & 0 \\ \frac{1}{2}\zeta_{40}^{10}+\frac{1}{2} & -\frac{1}{2}\zeta_{40}^{10}-\frac{1}{2} & 0 & 0 \\ 0 & 0 & \frac{1}{2}\zeta_{40}^{10}-\frac{1}{2} & \frac{1}{2}\zeta_{40}^{10}-\frac{1}{2} \\ 0 & 0 & \frac{1}{2}\zeta_{40}^{10}+\frac{1}{2} & -\frac{1}{2}\zeta_{40}^{10}-\frac{1}{2} \end{bmatrix}, \begin{bmatrix} \zeta_{40}^{10} & 0 & 0 & 0 \\ 0 & \zeta_{40}^{10} & 0 & 0 \\ 0 & 0 & -\zeta_{40}^{10} & 0 \\ 0 & 0 & 0 & -\zeta_{40}^{10} \end{bmatrix},$$

$$\begin{bmatrix} \frac{1}{2}\zeta_{40}^{10}-\frac{1}{2} & 0 & \frac{1}{2}\zeta_{40}^{10}-\frac{1}{2} & 0 \\ 0 & \frac{1}{2}\zeta_{40}^{10}-\frac{1}{2} & 0 & \frac{1}{2}\zeta_{40}^{10}-\frac{1}{2} \\ \frac{1}{2}\zeta_{40}^{10}+\frac{1}{2} & 0 & -\frac{1}{2}\zeta_{40}^{10}-\frac{1}{2} & 0 \\ 0 & \frac{1}{2}\zeta_{40}^{10}+\frac{1}{2} & 0 & -\frac{1}{2}\zeta_{40}^{10}-\frac{1}{2} \end{bmatrix}, \begin{bmatrix} -1 & 0 & 0 & 0 \\ 0 & \zeta_{40}^{10} & 0 & 0 \\ 0 & 0 & \zeta_{40}^{10} & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \begin{bmatrix} \zeta_{40}^{5} & 0 & 0 & 0 \\ 0 & 0 & \zeta_{40}^{5} & 0 \\ 0 & \zeta_{40}^{5} & 0 & 0 \\ 0 & 0 & 0 & \zeta_{40}^{5} \end{bmatrix}$$

FP10 (Order = 4608)

$$\begin{bmatrix} \frac{1}{2}\zeta_{40}^{10}-\frac{1}{2} & 0 & \frac{1}{2}\zeta_{40}^{10}-\frac{1}{2} & 0 \\ 0 & \frac{1}{2}\zeta_{40}^{10}-\frac{1}{2} & 0 & \frac{1}{2}\zeta_{40}^{10}-\frac{1}{2} \\ \frac{1}{2}\zeta_{40}^{10}+\frac{1}{2} & 0 & -\frac{1}{2}\zeta_{40}^{10}-\frac{1}{2} & 0 \\ 0 & \frac{1}{2}\zeta_{40}^{10}+\frac{1}{2} & 0 & -\frac{1}{2}\zeta_{40}^{10}-\frac{1}{2} \end{bmatrix}, \begin{bmatrix} \zeta_{40}^{5} & 0 & 0 & 0 \\ 0 & \zeta_{40}^{5} & 0 & 0 \\ 0 & 0 & -\zeta_{40}^{15} & 0 \\ 0 & 0 & 0 & -\zeta_{40}^{15} \end{bmatrix},$$

$$\begin{bmatrix} \frac{1}{2}\zeta_{40}^{10}-\frac{1}{2} & \frac{1}{2}\zeta_{40}^{10}-\frac{1}{2} & 0 & 0 \\ \frac{1}{2}\zeta_{40}^{10}+\frac{1}{2} & -\frac{1}{2}\zeta_{40}^{10}-\frac{1}{2} & 0 & 0 \\ 0 & 0 & \frac{1}{2}\zeta_{40}^{10}-\frac{1}{2} & \frac{1}{2}\zeta_{40}^{10}-\frac{1}{2} \\ 0 & 0 & \frac{1}{2}\zeta_{40}^{10}+\frac{1}{2} & -\frac{1}{2}\zeta_{40}^{10}-\frac{1}{2} \end{bmatrix}, \begin{bmatrix} \zeta_{40}^{5} & 0 & 0 & 0 \\ 0 & -\zeta_{40}^{15} & 0 & 0 \\ 0 & 0 & \zeta_{40}^{5} & 0 \\ 0 & 0 & 0 & -\zeta_{40}^{15} \end{bmatrix}, \begin{bmatrix} \zeta_{40}^{5} & 0 & 0 & 0 \\ 0 & 0 & \zeta_{40}^{5} & 0 \\ 0 & \zeta_{40}^{5} & 0 & 0 \\ 0 & 0 & 0 & \zeta_{40}^{5} \end{bmatrix}$$

FP11 (Order = 28800)

$$\begin{bmatrix} -\zeta_{40}^{4} & 0 & 0 & 0 \\ 0 & -\zeta_{40}^{4} & 0 & 0 \\ 0 & 0 & \zeta_{40}^{12}-\zeta_{40}^{8}+\zeta_{40}^{4}-1 & 0 \\ 0 & 0 & 0 & \zeta_{40}^{12}-\zeta_{40}^{8}+\zeta_{40}^{4}-1 \end{bmatrix},$$

$$\begin{bmatrix} -\frac{3}{5}\zeta_{40}^{12}+\frac{1}{5}\zeta_{40}^{8}-\frac{4}{5}\zeta_{40}^{4}+\frac{2}{5} & 0 & \frac{1}{5}\zeta_{40}^{12}+\frac{3}{5}\zeta_{40}^{8}-\frac{2}{5}\zeta_{40}^{4}+\frac{1}{5} & 0 \\ 0 & -\frac{3}{5}\zeta_{40}^{12}+\frac{1}{5}\zeta_{40}^{8}-\frac{4}{5}\zeta_{40}^{4}+\frac{2}{5} & 0 & \frac{1}{5}\zeta_{40}^{12}+\frac{3}{5}\zeta_{40}^{8}-\frac{2}{5}\zeta_{40}^{4}+\frac{1}{5} \\ \frac{1}{5}\zeta_{40}^{12}+\frac{3}{5}\zeta_{40}^{8}-\frac{2}{5}\zeta_{40}^{4}+\frac{1}{5} & 0 & \frac{3}{5}\zeta_{40}^{12}-\frac{1}{5}\zeta_{40}^{8}+\frac{4}{5}\zeta_{40}^{4}-\frac{2}{5} & 0 \\ 0 & \frac{1}{5}\zeta_{40}^{12}+\frac{3}{5}\zeta_{40}^{8}-\frac{2}{5}\zeta_{40}^{4}+\frac{1}{5} & 0 & \frac{3}{5}\zeta_{40}^{12}-\frac{1}{5}\zeta_{40}^{8}+\frac{4}{5}\zeta_{40}^{4}-\frac{2}{5} \end{bmatrix},$$

$$\begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{bmatrix}, \begin{bmatrix} -\zeta_{40}^{4} & 0 & 0 & 0 \\ 0 & \zeta_{40}^{12}-\zeta_{40}^{8}+\zeta_{40}^{4}-1 & 0 & 0 \\ 0 & 0 & -\zeta_{40}^{4} & 0 \\ 0 & 0 & 0 & \zeta_{40}^{12}-\zeta_{40}^{8}+\zeta_{40}^{4}-1 \end{bmatrix},$$

$$\begin{bmatrix} -\frac{3}{5}\zeta_{40}^{12} + \frac{1}{5}\zeta_{40}^{8} - \frac{4}{5}\zeta_{40}^{4} + \frac{2}{5} & \frac{1}{5}\zeta_{40}^{12} + \frac{3}{5}\zeta_{40}^{8} - \frac{2}{5}\zeta_{40}^{4} + \frac{1}{5} & 0 & 0 \\ \frac{1}{5}\zeta_{40}^{12} + \frac{3}{5}\zeta_{40}^{8} - \frac{2}{5}\zeta_{40}^{4} + \frac{1}{5} & \frac{3}{5}\zeta_{40}^{12} - \frac{1}{5}\zeta_{40}^{8} + \frac{4}{5}\zeta_{40}^{4} - \frac{2}{5} & 0 & 0 \\ 0 & 0 & -\frac{3}{5}\zeta_{40}^{12} + \frac{1}{5}\zeta_{40}^{8} - \frac{4}{5}\zeta_{40}^{4} + \frac{2}{5} & \frac{1}{5}\zeta_{40}^{12} + \frac{3}{5}\zeta_{40}^{8} - \frac{2}{5}\zeta_{40}^{4} + \frac{1}{5} \\ 0 & 0 & \frac{1}{5}\zeta_{40}^{12} + \frac{3}{5}\zeta_{40}^{8} - \frac{2}{5}\zeta_{40}^{4} + \frac{1}{5} & \frac{3}{5}\zeta_{40}^{12} - \frac{1}{5}\zeta_{40}^{8} + \frac{4}{5}\zeta_{40}^{4} - \frac{2}{5} \end{bmatrix},$$

$$\begin{bmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{bmatrix}, \begin{bmatrix} \zeta_{40}^{5} & 0 & 0 & 0 \\ 0 & 0 & \zeta_{40}^{5} & 0 \\ 0 & \zeta_{40}^{5} & 0 & 0 \\ 0 & 0 & 0 & \zeta_{40}^{5} \end{bmatrix}$$

FP12 (Order = 2304)

$$\begin{bmatrix} \zeta_{40}^{10} & 0 & 0 & 0 \\ 0 & -\zeta_{40}^{10} & 0 & 0 \\ 0 & 0 & \zeta_{40}^{10} & 0 \\ 0 & 0 & 0 & -\zeta_{40}^{10} \end{bmatrix}, \begin{bmatrix} \frac{1}{2}\zeta_{40}^{10} - \frac{1}{2} & \frac{1}{2}\zeta_{40}^{10} - \frac{1}{2} & 0 & 0 \\ \frac{1}{2}\zeta_{40}^{10} + \frac{1}{2} & -\frac{1}{2}\zeta_{40}^{10} - \frac{1}{2} & 0 & 0 \\ 0 & 0 & \frac{1}{2}\zeta_{40}^{10} - \frac{1}{2} & \frac{1}{2}\zeta_{40}^{10} - \frac{1}{2} \\ 0 & 0 & \frac{1}{2}\zeta_{40}^{10} + \frac{1}{2} & -\frac{1}{2}\zeta_{40}^{10} - \frac{1}{2} \end{bmatrix}, \begin{bmatrix} \zeta_{40}^{10} & 0 & 0 & 0 \\ 0 & \zeta_{40}^{10} & 0 & 0 \\ 0 & 0 & -\zeta_{40}^{10} & 0 \\ 0 & 0 & 0 & -\zeta_{40}^{10} \end{bmatrix},$$

$$\begin{bmatrix} \frac{1}{2}\zeta_{40}^{10} - \frac{1}{2} & 0 & \frac{1}{2}\zeta_{40}^{10} - \frac{1}{2} & 0 \\ 0 & \frac{1}{2}\zeta_{40}^{10} - \frac{1}{2} & 0 & \frac{1}{2}\zeta_{40}^{10} - \frac{1}{2} \\ \frac{1}{2}\zeta_{40}^{10} + \frac{1}{2} & 0 & -\frac{1}{2}\zeta_{40}^{10} - \frac{1}{2} & 0 \\ 0 & \frac{1}{2}\zeta_{40}^{10} + \frac{1}{2} & 0 & -\frac{1}{2}\zeta_{40}^{10} - \frac{1}{2} \end{bmatrix}, \begin{bmatrix} -1 & 0 & 0 & 0 \\ 0 & \zeta_{40}^{10} & 0 & 0 \\ 0 & 0 & \zeta_{40}^{10} & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & -\zeta_{40}^{10} & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & -\zeta_{40}^{10} \end{bmatrix}$$

FP13 (Order = 320)

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix}, \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix},$$

$$\begin{bmatrix} -\frac{1}{2} & \frac{1}{2} & -\frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2}\zeta_{4} & \frac{1}{2}\zeta_{4} & -\frac{1}{2}\zeta_{4} & -\frac{1}{2}\zeta_{4} \\ -\frac{1}{2}\zeta_{4} & \frac{1}{2}\zeta_{4} & \frac{1}{2}\zeta_{4} & -\frac{1}{2}\zeta_{4} \end{bmatrix}$$

FP14 (Order = 640)

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix}, \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix},$$

$$\begin{bmatrix} -\frac{1}{2} & \frac{1}{2} & -\frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2}\zeta_{40}^{10} & \frac{1}{2}\zeta_{40}^{10} & -\frac{1}{2}\zeta_{40}^{10} & -\frac{1}{2}\zeta_{40}^{10} \\ -\frac{1}{2}\zeta_{40}^{10} & \frac{1}{2}\zeta_{40}^{10} & \frac{1}{2}\zeta_{40}^{10} & -\frac{1}{2}\zeta_{40}^{10} \end{bmatrix}, \begin{bmatrix} 0 & -\zeta_{40}^{10} & 0 & 0 \\ -\zeta_{40}^{10} & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

FP15 (Order = 1280)

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix}, \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix},$$

$$\begin{bmatrix} -\frac{1}{2} & \frac{1}{2} & -\frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2}\zeta_{40}^{10} & \frac{1}{2}\zeta_{40}^{10} & -\frac{1}{2}\zeta_{40}^{10} & -\frac{1}{2}\zeta_{40}^{10} \\ -\frac{1}{2}\zeta_{40}^{10} & \frac{1}{2}\zeta_{40}^{10} & \frac{1}{2}\zeta_{40}^{10} & -\frac{1}{2}\zeta_{40}^{10} \end{bmatrix}, \begin{bmatrix} -\frac{1}{2}\zeta_{40}^{15} + \frac{1}{2}\zeta_{40}^{5} & \frac{1}{2}\zeta_{40}^{15} + \frac{1}{2}\zeta_{40}^{5} & 0 & 0 \\ \frac{1}{2}\zeta_{40}^{15} + \frac{1}{2}\zeta_{40}^{5} & -\frac{1}{2}\zeta_{40}^{15} + \frac{1}{2}\zeta_{40}^{5} & 0 & 0 \\ 0 & 0 & \frac{1}{2}\zeta_{40}^{15} + \frac{1}{2}\zeta_{40}^{5} & \frac{1}{2}\zeta_{40}^{15} - \frac{1}{2}\zeta_{40}^{5} \\ 0 & 0 & -\frac{1}{2}\zeta_{40}^{15} + \frac{1}{2}\zeta_{40}^{5} & -\frac{1}{2}\zeta_{40}^{15} - \frac{1}{2}\zeta_{40}^{5} \end{bmatrix}$$

FP16 (Order = 3840)

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix}, \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix},$$

$$\begin{bmatrix} -\frac{1}{2} & \frac{1}{2} & -\frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2}\zeta_{40}^{10} & \frac{1}{2}\zeta_{40}^{10} & -\frac{1}{2}\zeta_{40}^{10} & -\frac{1}{2}\zeta_{40}^{10} \\ -\frac{1}{2}\zeta_{40}^{10} & \frac{1}{2}\zeta_{40}^{10} & \frac{1}{2}\zeta_{40}^{10} & -\frac{1}{2}\zeta_{40}^{10} \end{bmatrix}, \begin{bmatrix} \frac{1}{2}\zeta_{40}^{10}+\frac{1}{2} & \frac{1}{2}\zeta_{40}^{10}-\frac{1}{2} & 0 & 0 \\ -\frac{1}{2}\zeta_{40}^{10}-\frac{1}{2} & \frac{1}{2}\zeta_{40}^{10}-\frac{1}{2} & 0 & 0 \\ 0 & 0 & \frac{1}{2}\zeta_{40}^{10}+\frac{1}{2} & \frac{1}{2}\zeta_{40}^{10}-\frac{1}{2} \\ 0 & 0 & -\frac{1}{2}\zeta_{40}^{10}-\frac{1}{2} & \frac{1}{2}\zeta_{40}^{10}-\frac{1}{2} \end{bmatrix}$$

FP17 (Order = 3840)

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix}, \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix},$$

$$\begin{bmatrix} -\frac{1}{2} & \frac{1}{2} & -\frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2}\zeta_{40}^{10} & \frac{1}{2}\zeta_{40}^{10} & -\frac{1}{2}\zeta_{40}^{10} & -\frac{1}{2}\zeta_{40}^{10} \\ -\frac{1}{2}\zeta_{40}^{10} & \frac{1}{2}\zeta_{40}^{10} & \frac{1}{2}\zeta_{40}^{10} & -\frac{1}{2}\zeta_{40}^{10} \end{bmatrix}, \begin{bmatrix} \frac{1}{2}\zeta_{40}^{10} & \frac{1}{2} & \frac{1}{2}\zeta_{40}^{10} & -\frac{1}{2} \\ \frac{1}{2}\zeta_{40}^{10} & -\frac{1}{2} & -\frac{1}{2}\zeta_{40}^{10} & -\frac{1}{2} \\ -\frac{1}{2}\zeta_{40}^{10} & \frac{1}{2} & -\frac{1}{2}\zeta_{40}^{10} & -\frac{1}{2} \\ -\frac{1}{2}\zeta_{40}^{10} & -\frac{1}{2} & \frac{1}{2}\zeta_{40}^{10} & -\frac{1}{2} \end{bmatrix}$$

FP18 (Order = 7680)

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix}, \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix},$$

$$\begin{bmatrix} -\frac{1}{2} & \frac{1}{2} & -\frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2}\zeta_{40}^{10} & \frac{1}{2}\zeta_{40}^{10} & -\frac{1}{2}\zeta_{40}^{10} & -\frac{1}{2}\zeta_{40}^{10} \\ -\frac{1}{2}\zeta_{40}^{10} & \frac{1}{2}\zeta_{40}^{10} & \frac{1}{2}\zeta_{40}^{10} & -\frac{1}{2}\zeta_{40}^{10} \end{bmatrix}, \begin{bmatrix} 0 & 0 & -\zeta_{40}^{5} & 0 \\ 0 & 0 & 0 & -\zeta_{40}^{15} \\ \zeta_{40}^{15} & 0 & 0 & 0 \\ 0 & \zeta_{40}^{5} & 0 & 0 \end{bmatrix}$$

FP19 (Order = 7680)

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix}, \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix},$$

$$\begin{bmatrix} -\frac{1}{2} & \frac{1}{2} & -\frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2}\zeta_{40}^{10} & \frac{1}{2}\zeta_{40}^{10} & -\frac{1}{2}\zeta_{40}^{10} & -\frac{1}{2}\zeta_{40}^{10} \\ -\frac{1}{2}\zeta_{40}^{10} & \frac{1}{2}\zeta_{40}^{10} & \frac{1}{2}\zeta_{40}^{10} & -\frac{1}{2}\zeta_{40}^{10} \end{bmatrix}, \begin{bmatrix} \zeta_{40}^{5} & 0 & 0 & 0 \\ 0 & \zeta_{40}^{5} & 0 & 0 \\ 0 & 0 & \zeta_{40}^{5} & 0 \\ 0 & 0 & 0 & -\zeta_{40}^{5} \end{bmatrix}$$

FP20 (Order = 23040)

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix}, \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix},$$

$$\begin{bmatrix} -\frac{1}{2} & \frac{1}{2} & -\frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2}\zeta_{40}^{10} & \frac{1}{2}\zeta_{40}^{10} & -\frac{1}{2}\zeta_{40}^{10} & -\frac{1}{2}\zeta_{40}^{10} \\ -\frac{1}{2}\zeta_{40}^{10} & \frac{1}{2}\zeta_{40}^{10} & \frac{1}{2}\zeta_{40}^{10} & -\frac{1}{2}\zeta_{40}^{10} \end{bmatrix}, \begin{bmatrix} \frac{1}{2}\zeta_{40}^{10}+\frac{1}{2} & 0 & 0 & \frac{1}{2}\zeta_{40}^{10}-\frac{1}{2} \\ 0 & \frac{1}{2}\zeta_{40}^{10}+\frac{1}{2} & \frac{1}{2}\zeta_{40}^{10}-\frac{1}{2} & 0 \\ 0 & -\frac{1}{2}\zeta_{40}^{10}-\frac{1}{2} & \frac{1}{2}\zeta_{40}^{10}-\frac{1}{2} & 0 \\ -\frac{1}{2}\zeta_{40}^{10}-\frac{1}{2} & 0 & 0 & \frac{1}{2}\zeta_{40}^{10}-\frac{1}{2} \end{bmatrix}$$

FP21 (Order = 46080)

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix}, \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix},$$

$$\begin{bmatrix} -\frac{1}{2} & \frac{1}{2} & -\frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2}\zeta_{40}^{10} & \frac{1}{2}\zeta_{40}^{10} & -\frac{1}{2}\zeta_{40}^{10} & -\frac{1}{2}\zeta_{40}^{10} \\ -\frac{1}{2}\zeta_{40}^{10} & \frac{1}{2}\zeta_{40}^{10} & \frac{1}{2}\zeta_{40}^{10} & -\frac{1}{2}\zeta_{40}^{10} \end{bmatrix}, \begin{bmatrix} \frac{1}{2}\zeta_{40}^{10} + \frac{1}{2} & 0 & 0 & \frac{1}{2}\zeta_{40}^{10} - \frac{1}{2} \\ 0 & \frac{1}{2}\zeta_{40}^{10} + \frac{1}{2} & \frac{1}{2}\zeta_{40}^{10} - \frac{1}{2} & 0 \\ 0 & -\frac{1}{2}\zeta_{40}^{10} - \frac{1}{2} & \frac{1}{2}\zeta_{40}^{10} - \frac{1}{2} & 0 \\ -\frac{1}{2}\zeta_{40}^{10} - \frac{1}{2} & 0 & 0 & \frac{1}{2}\zeta_{40}^{10} - \frac{1}{2} \end{bmatrix},$$

$$\begin{bmatrix} \zeta_{40}^{5} & 0 & 0 & 0 \\ 0 & \zeta_{40}^{5} & 0 & 0 \\ 0 & 0 & \zeta_{40}^{5} & 0 \\ 0 & 0 & 0 & -\zeta_{40}^{5} \end{bmatrix}$$

The next 6 groups have projective images which are simple groups. In particular both groups FP22 and FP23 project onto A_5 and FP26 projects onto G_{168}.

FP22 (Order = 240)

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \zeta_{60}^{10} - 1 & 0 \\ 0 & 0 & 0 & -\zeta_{60}^{10} \end{bmatrix}, \begin{bmatrix} \zeta_{60}^{15} & 0 & 0 & 0 \\ 0 & -\frac{1}{3}\zeta_{60}^{15} & \frac{2}{3}\zeta_{60}^{15} & \frac{2}{3}\zeta_{60}^{15} \\ 0 & \frac{2}{3}\zeta_{60}^{15} & -\frac{1}{3}\zeta_{60}^{15} & \frac{2}{3}\zeta_{60}^{15} \\ 0 & \frac{2}{3}\zeta_{60}^{15} & \frac{2}{3}\zeta_{60}^{15} & -\frac{1}{3}\zeta_{60}^{15} \end{bmatrix},$$

$$\begin{bmatrix} -\frac{1}{4}\zeta_{60}^{15} & \frac{1}{2}\zeta_{60}^{14} - \frac{1}{2}\zeta_{60}^{10} - \zeta_{60}^{8} - \frac{1}{2}\zeta_{60}^{6} - \frac{1}{2}\zeta_{60}^{4} + \zeta_{60}^{2} + \frac{3}{4} & 0 & 0 \\ \frac{1}{2}\zeta_{60}^{14} - \frac{1}{2}\zeta_{60}^{10} - \zeta_{60}^{8} - \frac{1}{2}\zeta_{60}^{6} - \frac{1}{2}\zeta_{60}^{4} + \zeta_{60}^{2} + \frac{3}{4} & \frac{1}{4}\zeta_{60}^{15} & 0 & 0 \\ 0 & 0 & 0 & \zeta_{60}^{15} \\ 0 & 0 & \zeta_{60}^{15} & 0 \end{bmatrix}$$

FP23 (Order = 240)

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \zeta_{24}^{4} - 1 & 0 \\ 0 & 0 & 0 & -\zeta_{24}^{4} \end{bmatrix}, \begin{bmatrix} -\frac{1}{2}\zeta_{24}^{6} + \zeta_{24}^{2} & \frac{1}{2} & 0 & 0 \\ \frac{1}{2} & \frac{1}{2}\zeta_{24}^{6} - \zeta_{24}^{2} & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix},$$

$$\begin{bmatrix} -\frac{1}{3}\zeta_{24}^{6} + \frac{2}{3}\zeta_{24}^{2} & 0 & 0 & -\frac{2}{3}\zeta_{24}^{7} + \frac{1}{3}\zeta_{24}^{5} + \frac{1}{3}\zeta_{24}^{3} + \frac{1}{3}\zeta_{24} \\ 0 & \frac{1}{3}\zeta_{24}^{6} - \frac{2}{3}\zeta_{24}^{2} & -\frac{2}{3}\zeta_{24}^{7} + \frac{1}{3}\zeta_{24}^{5} + \frac{1}{3}\zeta_{24}^{3} + \frac{1}{3}\zeta_{24} & 0 \\ 0 & -\frac{2}{3}\zeta_{24}^{7} + \frac{1}{3}\zeta_{24}^{5} + \frac{1}{3}\zeta_{24}^{3} + \frac{1}{3}\zeta_{24} & -\frac{1}{3}\zeta_{24}^{6} + \frac{2}{3}\zeta_{24}^{2} & 0 \\ -\frac{2}{3}\zeta_{24}^{7} + \frac{1}{3}\zeta_{24}^{5} + \frac{1}{3}\zeta_{24}^{3} + \frac{1}{3}\zeta_{24} & 0 & 0 & \frac{1}{3}\zeta_{24}^{6} - \frac{2}{3}\zeta_{24}^{2} \end{bmatrix}$$

FP24 (Order = 1440)

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \zeta_{24}^{4} - 1 & 0 \\ 0 & 0 & 0 & -\zeta_{24}^{4} \end{bmatrix}, \begin{bmatrix} -\frac{1}{2}\zeta_{24}^{6} + \zeta_{24}^{2} & \frac{1}{2} & 0 & 0 \\ \frac{1}{2} & \frac{1}{2}\zeta_{24}^{6} - \zeta_{24}^{2} & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & -1 & 0 \end{bmatrix}$$

$$\begin{bmatrix} -\frac{1}{3}\zeta_{24}^{6} + \frac{2}{3}\zeta_{24}^{2} & 0 & 0 & -\frac{2}{3}\zeta_{24}^{7} + \frac{1}{3}\zeta_{24}^{5} + \frac{1}{3}\zeta_{24}^{3} + \frac{1}{3}\zeta_{24} \\ 0 & \frac{1}{3}\zeta_{24}^{6} - \frac{2}{3}\zeta_{24}^{2} & -\frac{2}{3}\zeta_{24}^{7} + \frac{1}{3}\zeta_{24}^{5} + \frac{1}{3}\zeta_{24}^{3} + \frac{1}{3}\zeta_{24} & 0 \\ 0 & -\frac{2}{3}\zeta_{24}^{7} + \frac{1}{3}\zeta_{24}^{5} + \frac{1}{3}\zeta_{24}^{3} + \frac{1}{3}\zeta_{24} & -\frac{1}{3}\zeta_{24}^{6} + \frac{2}{3}\zeta_{24}^{2} & 0 \\ -\frac{2}{3}\zeta_{24}^{7} + \frac{1}{3}\zeta_{24}^{5} + \frac{1}{3}\zeta_{24}^{3} + \frac{1}{3}\zeta_{24} & 0 & 0 & \frac{1}{3}\zeta_{24}^{6} - \frac{2}{3}\zeta_{24}^{2} \end{bmatrix}$$

FP25 (Order = 10080)

In the following two groups, let $r7 := 2*\zeta_2 8 - \zeta_2 8^7 + 2*\zeta_2 8^9 - 2*\zeta_2 8^1 1$, $p := \zeta_7 + \zeta_7^2 + \zeta_7^4$ and $q := \zeta_7^6 + \zeta_7^5 + \zeta_7^3$.

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \zeta_{28}^{4} & 0 & 0 \\ 0 & 0 & -\zeta_{28}^{2} & 0 \\ 0 & 0 & 0 & \zeta_{28}^{8} \end{bmatrix}, \begin{bmatrix} \frac{p^2}{r7} & \frac{1}{r7} & \frac{1}{r7} & \frac{1}{r7} \\ \frac{1}{r7} & \frac{-p}{r7} & \frac{-p}{r7} & \frac{-q}{r7} \\ \frac{1}{r7} & \frac{-q}{r7} & \frac{-p}{r7} & \frac{-p}{r7} \\ \frac{1}{r7} & \frac{-p}{r7} & \frac{-q}{r7} & \frac{-p}{r7} \end{bmatrix}$$

FP26 (Order = 672)

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \zeta_{28}^{4} & 0 & 0 \\ 0 & 0 & -\zeta_{28}^{2} & 0 \\ 0 & 0 & 0 & \zeta_{28}^{8} \end{bmatrix}, \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix},$$

$$\begin{bmatrix} \frac{1}{r7} & \frac{2}{r7} & \frac{2}{r7} & \frac{2}{r7} \\ \frac{1}{r7} & \frac{\zeta_7^2+\zeta_7^5}{r7} & \frac{\zeta_7^3+\zeta_7^4}{r7} & \frac{\zeta_7+\zeta_7^6}{r7} \\ \frac{1}{r7} & \frac{\zeta_7^3+\zeta_7^4}{r7} & \frac{\zeta_7+\zeta_7^6}{r7} & \frac{\zeta_7^2+\zeta_7^5}{r7} \\ \frac{1}{r7} & , \frac{\zeta_7+\zeta_7^6}{r7} & , \frac{\zeta_7^2+\zeta_7^5}{r7} & , \frac{\zeta_7^3+\zeta_7^4}{r7} \end{bmatrix}$$

FP27 (Order = 103680)

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \zeta_{12}^{2}-1 & 0 \\ 0 & 0 & 0 & -\zeta_{12}^{2} \end{bmatrix}, \begin{bmatrix} \zeta_{12}^{2}-1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \zeta_{12}^{2}-1 & 0 \\ 0 & 0 & 0 & \zeta_{12}^{2}-1 \end{bmatrix},$$

$$\begin{bmatrix} \zeta_{12}^{3} & 0 & 0 & 0 \\ 0 & -\frac{1}{3}\zeta_{12}^{3}+\frac{2}{3}\zeta_{12} & -\frac{1}{3}\zeta_{12}^{3}+\frac{2}{3}\zeta_{12} & -\frac{1}{3}\zeta_{12}^{3}+\frac{2}{3}\zeta_{12} \\ 0 & -\frac{1}{3}\zeta_{12}^{3}+\frac{2}{3}\zeta_{12} & \frac{2}{3}\zeta_{12}^{3}-\frac{1}{3}\zeta_{12} & -\frac{1}{3}\zeta_{12}^{3}-\frac{1}{3}\zeta_{12} \\ 0 & -\frac{1}{3}\zeta_{12}^{3}+\frac{2}{3}\zeta_{12} & -\frac{1}{3}\zeta_{12}^{3}-\frac{1}{3}\zeta_{12} & \frac{2}{3}\zeta_{12}^{3}-\frac{1}{3}\zeta_{12} \end{bmatrix}, \begin{bmatrix} 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix}$$

FP28 (Order = 480)

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \zeta_{120}^{20}-1 & 0 \\ 0 & 0 & 0 & -\zeta_{120}^{20} \end{bmatrix}, \begin{bmatrix} \zeta_{120}^{30} & 0 & 0 & 0 \\ 0 & -\frac{1}{3}\zeta_{120}^{30} & \frac{2}{3}\zeta_{120}^{30} & \frac{2}{3}\zeta_{120}^{30} \\ 0 & \frac{2}{3}\zeta_{120}^{30} & -\frac{1}{3}\zeta_{120}^{30} & \frac{2}{3}\zeta_{120}^{30} \\ 0 & \frac{2}{3}\zeta_{120}^{30} & \frac{2}{3}\zeta_{120}^{30} & -\frac{1}{3}\zeta_{120}^{30} \end{bmatrix}, \begin{bmatrix} \zeta_{120}^{15} & 0 & 0 & 0 \\ 0 & \zeta_{120}^{15} & 0 & 0 \\ 0 & 0 & 0 & \zeta_{120}^{15} \\ 0 & 0 & \zeta_{120}^{15} & 0 \end{bmatrix},$$

$$\begin{bmatrix} -\frac{1}{4}\zeta_{120}^{30} & \frac{1}{2}\zeta_{120}^{28}-\frac{1}{2}\zeta_{120}^{20}-\zeta_{120}^{16}-\frac{1}{2}\zeta_{120}^{12}-\frac{1}{2}\zeta_{120}^{8}+\zeta_{120}^{4}+\frac{3}{4} & 0 & 0 \\ \frac{1}{2}\zeta_{120}^{28}-\frac{1}{2}\zeta_{120}^{20}-\zeta_{120}^{16}-\frac{1}{2}\zeta_{120}^{12}-\frac{1}{2}\zeta_{120}^{8}+\zeta_{120}^{4}+\frac{3}{4} & \frac{1}{4}\zeta_{120}^{30} & 0 & 0 \\ 0 & 0 & 0 & \zeta_{120}^{30} \\ 0 & 0 & \zeta_{120}^{30} & 0 \end{bmatrix}$$

FP29 (Order = 480)

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \zeta_{24}^{4}-1 & 0 \\ 0 & 0 & 0 & -\zeta_{24}^{4} \end{bmatrix}, \begin{bmatrix} -\frac{1}{2}\zeta_{24}^{6}+\zeta_{24}^{2} & \frac{1}{2} & 0 & 0 \\ \frac{1}{2} & \frac{1}{2}\zeta_{24}^{6}-\zeta_{24}^{2} & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{bmatrix},$$

$$\begin{bmatrix} -\frac{1}{3}\zeta_{24}^{6}+\frac{2}{3}\zeta_{24}^{2} & 0 & 0 & -\frac{2}{3}\zeta_{24}^{7}+\frac{1}{3}\zeta_{24}^{5}+\frac{1}{3}\zeta_{24}^{3}+\frac{1}{3}\zeta_{24} \\ 0 & \frac{1}{3}\zeta_{24}^{6}-\frac{2}{3}\zeta_{24}^{2} & -\frac{2}{3}\zeta_{24}^{7}+\frac{1}{3}\zeta_{24}^{5}+\frac{1}{3}\zeta_{24}^{3}+\frac{1}{3}\zeta_{24} & 0 \\ 0 & -\frac{2}{3}\zeta_{24}^{7}+\frac{1}{3}\zeta_{24}^{5}+\frac{1}{3}\zeta_{24}^{3}+\frac{1}{3}\zeta_{24} & -\frac{1}{3}\zeta_{24}^{6}+\frac{2}{3}\zeta_{24}^{2} & 0 \\ -\frac{2}{3}\zeta_{24}^{7}+\frac{1}{3}\zeta_{24}^{5}+\frac{1}{3}\zeta_{24}^{3}+\frac{1}{3}\zeta_{24} & 0 & 0 & \frac{1}{3}\zeta_{24}^{6}-\frac{2}{3}\zeta_{24}^{2} \end{bmatrix}$$

FP30 (Order = 2880)

$$\begin{bmatrix} -\frac{1}{3}\zeta_{24}^{6}+\frac{2}{3}\zeta_{24}^{2} & 0 & 0 & -\frac{2}{3}\zeta_{24}^{7}+\frac{1}{3}\zeta_{24}^{5}+\frac{1}{3}\zeta_{24}^{3}+\frac{1}{3}\zeta_{24} \\ 0 & \frac{1}{3}\zeta_{24}^{6}-\frac{2}{3}\zeta_{24}^{2} & -\frac{2}{3}\zeta_{24}^{7}+\frac{1}{3}\zeta_{24}^{5}+\frac{1}{3}\zeta_{24}^{3}+\frac{1}{3}\zeta_{24} & 0 \\ 0 & -\frac{2}{3}\zeta_{24}^{7}+\frac{1}{3}\zeta_{24}^{5}+\frac{1}{3}\zeta_{24}^{3}+\frac{1}{3}\zeta_{24} & -\frac{1}{3}\zeta_{24}^{6}+\frac{2}{3}\zeta_{24}^{2} & 0 \\ -\frac{2}{3}\zeta_{24}^{7}+\frac{1}{3}\zeta_{24}^{5}+\frac{1}{3}\zeta_{24}^{3}+\frac{1}{3}\zeta_{24} & 0 & 0 & \frac{1}{3}\zeta_{24}^{6}-\frac{2}{3}\zeta_{24}^{2} \end{bmatrix},$$

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \zeta_{24}^{4}-1 & 0 \\ 0 & 0 & 0 & -\zeta_{24}^{4} \end{bmatrix}, \begin{bmatrix} -\frac{1}{2}\zeta_{24}^{6}+\zeta_{24}^{2} & \frac{1}{2} & 0 & 0 \\ \frac{1}{2} & \frac{1}{2}\zeta_{24}^{6}-\zeta_{24}^{2} & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & -1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

4.3 Summary of Group Notation

We provide a quick reference for the groups constructed in the first two sections of this chapter. This section contains all of the notation for the groups and can be used to make the representation tables easier to read.

Imprimitive Monomial Groups:

A monomial group G contains a maximal diagonal subgroup H such that $\mathrm{G}/\mathrm{H} \cong \mathcal{P}_{\mathrm{G}}$ where $\mathcal{P}_{\mathrm{G}}$ is a transitive subgroup of the permutation group S_4. We refer to this group $\mathcal{P}_{\mathrm{G}}$ as the permutation action of G. If G is an infinite monomial group then H must be one of the following groups:

- $\mathbf{H_1^n} = \left\{ \begin{bmatrix} \xi a & 0 & 0 & 0 \\ 0 & a^{-1} & 0 & 0 \\ 0 & 0 & \xi^{-1} a & 0 \\ 0 & 0 & 0 & a^{-1} \end{bmatrix} : a \in \mathrm{C}^*, \xi \in \mathrm{U}_n \right\}$
- $\mathbf{H_2^n} = \left\{ \begin{bmatrix} \xi a & 0 & 0 & 0 \\ 0 & \xi^{-1} b & 0 & 0 \\ 0 & 0 & a^{-1} & 0 \\ 0 & 0 & 0 & b^{-1} \end{bmatrix} : a, b \in \mathrm{C}^*, \xi \in \mathrm{U}_n \right\}$
- $\mathbf{H_3} = \mathrm{diag}(\mathrm{SL}(4, \mathrm{C}))$

Suppose one of the above is the maximal diagonal subgroup of a monomial group G with permutation action $\mathcal{P}_{\mathrm{G}}$. We denote such a group G by

$$\mathbf{G}^{\mathbf{i,n}}_{\mathcal{P}_{\mathrm{G}}}$$

where we delete the integer n when $i = 3$.

Imprimitive Nonmonomial Groups:

We may write such a group G as the union $H \cup \gamma H$ where H is in block diagonal form (so H can be thought of as a subgroup of $GL(2, C) \times GL(2, C)$) and γ is in block off-diagonal form. Since G is imprimitive and nonmonomial, the set of elements that occur in either the upper left hand corner of H or the lower right hand corner of H is equal to a primitive subgroup T of $GL(2, C)$.

Suppose T is projectively finite. Then H has the following properties.

- T projects onto a degree two projective representation of A_4, S_4 or A_5. This permutation group will be denoted by P.
- There exists a normal subgroup N of P such that if $\begin{bmatrix} A & 0 \\ 0 & B \end{bmatrix} \in H$ then $B \in A\tilde{N}$ where $\tilde{N}$ is a subgroup of T which projects onto P. This can be shown using a generalization of Goursat's Lemma.

Given such a subgroup H there are two possible types of groups G. In particular,

- if T projects onto either S_4 or A_5, then G is conjugate to $H \cup \gamma_1 H$ where $\gamma_1 = \begin{bmatrix} 0 & I \\ \xi & 0 \end{bmatrix}$, $\xi \in C^*$.
- if T projects onto A_4 then there are basically two possibilities for G. Either $H \cup \gamma_1 H$ where γ_1 is as above or $H \cup \gamma_2 H$ where $\gamma_2 = \begin{bmatrix} 0 & O_{S_4} \\ \xi O_{S_4} & 0 \end{bmatrix}$.

So given an imprimitive nonmonomial group $G = H \cup \gamma_i H$ where T projects onto the permutation group P and where $\begin{bmatrix} A & 0 \\ 0 & B \end{bmatrix} \in H$ implies $B \in A\tilde{N}$ we denote G by

$$\mathbf{G_i(P, N)}.$$

Suppose T is projectively infinite. Then there exists a nonnegative integer n such that $\mathrm{G} = \mathrm{H} \cup \gamma \mathrm{H}$ where H is one of the following groups and $\gamma = \begin{bmatrix} 0 & \mathrm{I} \\ \mathrm{I} & 0 \end{bmatrix}$. Recall that $\mathrm{G}_n = \{\mathrm{A} \in \mathrm{GL}(2, \mathrm{C}) : \det \mathrm{A}^n = 1\}$.

- $\mathbf{H_1} = \left\{ \begin{bmatrix} \mathrm{B} & 0 \\ 0 & d_{\mathrm{B}}\mathrm{B} \end{bmatrix} : \mathrm{B} \in \mathrm{G}_n, d_{\mathrm{B}} = \frac{1}{\det \mathrm{B}} \right\}$
- $\mathbf{H_2} = \left\{ \begin{bmatrix} \mathrm{B} & 0 \\ 0 & \pm d_{\mathrm{B}}\mathrm{B} \end{bmatrix} : \mathrm{B} \in \mathrm{G}_n, d_{\mathrm{B}} = \frac{1}{\det \mathrm{B}} \right\}$
- $\mathbf{H_3} = \left\{ \begin{bmatrix} \mathrm{A} & 0 \\ 0 & \mathrm{B} \end{bmatrix} : \mathrm{A}, \mathrm{B} \in \mathrm{G}_n, \det \mathrm{AB} = 1 \right\}$

Let n be a nonnegative integer such that $\mathrm{G} = \mathrm{H}_i \cup \gamma \mathrm{H}_i$. Then we denote G by

$$\mathbf{G^i_n}.$$

Primitive Groups:

Suppose G is a finite primitive group. Then G projects onto one of the 30 primitive projective groups listed in [BL2]. For each of these projective groups there is a largest subgroup of $\mathrm{SL}(4, \mathrm{C})$ which projects onto this group. These subgroups of $\mathrm{SL}(4, \mathrm{C})$ are denoted by

$$\mathbf{FP1}, \ldots, \mathbf{FP30}.$$

Suppose G is an infinite primitive group. Then G is a finite extension of its connected component G^0. In this case either G^0 is one of the following semisimple groups or G is a homomorphic image of $\tilde{\mathrm{H}} \times \mathrm{SL}(2, \mathrm{C})$ acting on $\mathrm{V}(1) \otimes \mathrm{V}(1)$ where $\tilde{\mathrm{H}} \in \{A_4^{SL_2}, S_4^{SL_2}, A_5^{SL_2}, (A_4^{SL_2})_2, (S_4^{SL_2})_2, (A_5^{SL_2})_2\}$. A group of this last type will be denoted by $\tilde{\mathrm{H}} \otimes \mathrm{SL}(2, \mathrm{C})$.

- $\mathbf{SL(4, C)}$ acting on its standard representation, V.
- $\mathbf{SL(4, C)}$ acting on the dual, V^*.

- $\mathbf{SL(2,C)}$ acting on $\mathrm{Sym}^3 V$ where V is its standard representation.
- $\mathbf{SP(4,C)}$ acting on its standard representation, V.
- $\mathbf{SO(4,C)} \approx \mathbf{SL(2,C)} \times \mathbf{SL(2,C)}$ acting on $V \otimes V$ where V is the standard representation for $\mathrm{SL}(2,\mathrm{C})$.

Note: We observed that the dimensions of the irreducible components in the decomposition of the G$-$ modules from Chapter 5 where G is either of the above representations of $\mathrm{SL}(4,\mathrm{C})$ are identical. Hence we do not distinguish between them in the tables.

Chapter 5

Representations of the Groups

5.1 Representations of the Imprimitive Groups

5.1.1 Representations of the Imprimitive Monomial Groups

Recall that an infinite monomial group, G, can be written in the form

$$\mathrm{G}^{i,n}_{\mathcal{P}_\mathrm{G}} = \cup_{\tilde{m_\sigma} \in \mathcal{S}} \tilde{m_\sigma} \cdot \mathrm{H}$$

where $\mathcal{P}_\mathrm{G}$ is a transitive subgroup of S_4 and H is an infinite diagonal subgroup of $\mathrm{SL}(4, \mathrm{C})$ with $\dim(\mathrm{H}) = i$ and $[\mathrm{H} : \mathrm{H}^0] = n$. We sometimes refer to this group $\mathcal{P}_\mathrm{G}$ as the permutation action of G. We show here that any representation of such a group can be decomposed into irreducibles using this description of G. We begin with a lemma concerning representations of H which will naturally play a large role in decomposing representations of G.

Lemma 5.1 *Let H be a subgroup of the group of diagonal elements of $GL(m, C)$ for some positive integer m. Then any representation of H is diagonalizable.*

Proof: Let $\rho : \mathrm{H} \to \mathrm{GL}(n, \mathrm{C})$ be a representation of H. Since H is commutative, the elements of $\rho(\mathrm{H})$ may be put simultaneously into Jordan Normal Form. Each element of H is semisimple and this property is preserved by ρ. Therefore the Jordan Normal Form of each element in $\rho(\mathrm{H})$ is diagonal. ∎

Notation and Definitions:

- Let W be a G$-$ module.

- Let $\mathcal{B}_{\mathrm{W}}$ be a basis for W.

- Let $w \in \mathrm{W}$. Then w can be written as the sum $\sum_{v \in \mathcal{B}_{\mathrm{W}}} \alpha_v v$. We define the **support** of w with respect to a basis $\mathcal{B}_{\mathrm{W}}$, denoted $Supp_{\mathcal{B}_{\mathrm{W}}}(w)$, to be the set $\{v \in \mathcal{B}_{\mathrm{W}} : \alpha_v \neq 0\}$. When there is no ambiguity about the basis $\mathcal{B}_{\mathrm{W}}$ to which we are referring this set will be denoted simply by $Supp(w)$.

- Given a subset $\{w_1, \ldots, w_t\}$ of W we will say that this set has property $*\mathrm{S}*$ if $Supp(w_i) \cap Supp(w_j) = \emptyset$ for each $i \neq j$.

- Let H be a subgroup of diag(GL(n, C)) and let W be an H$-$ module. Lemma 5.1 implies that $\mathrm{W} = \bigoplus \mathrm{V}_i$ where each V_i is a one dimensional H$-$ module. Let χ_i be the character associated with V_i. Given such a character χ_i we define the **characteristic subspace** of W associated with χ_i to be the space $\mathrm{U}_i = \bigoplus_{\chi_i = \chi_j} \mathrm{V}_j$.

- Note that an element p in W is a semi-invariant for H if and only if p is in a characteristic subspace for H. Given a basis B for a characteristic subspace of W with the property that $B \subseteq \mathcal{B}_{\mathrm{W}}$ we say that p is a **maximal semi-invariant** for H with respect to the basis, $\mathcal{B}_{\mathrm{W}}$, if $Supp(p) = B$.

- **Remark:** Let $\mathcal{B}_{\mathrm{W}}$ be a basis for W in which H has diagonal form and let $\mathcal{S}$ be a set of coset representatives for $\mathrm{G}/\mathrm{H} \cong \mathcal{P}_{\mathrm{G}}$. We note for future reference that the span of $\{v^{\mathcal{P}_{\mathrm{G}}}\}$ is equal to the span of $\{v^{\mathcal{S}}\}$ for any $v \in \mathcal{B}_{\mathrm{W}}$.

The following collection of lemmas and results lead naturally to the process by which we decompose a given representation of a monomial group into irreducibles.

Lemma 5.2 *Let G be an imprimitive monomial group of degree n with maximal diagonal subgroup H and permutation action $\mathcal{P}_{\mathrm{G}}$. Let W be a representation of G with basis $\mathcal{B}_{\mathrm{W}}$ in which H has diagonal form. Assume also that $\mathcal{P}_{\mathrm{G}}$ acts transitively on $\mathcal{B}_{\mathrm{W}}$. Let $\mathcal{B}_{\mathrm{U}}$ be a basis for a G invariant subspace U of W consisting of maximal semi-invariants of H. If $\mathcal{B}_{\mathrm{U}}$ has property $*S*$, then U is $G-$ irreducible.*

Proof: Suppose that $\tilde{\mathrm{U}} \neq \{0\}$ is a G$-$ submodule of U. Then $\tilde{\mathrm{U}}$ is an H$-$ module. By Lemma 5.1 there is a basis for $\tilde{\mathrm{U}}$ in which H has diagonal form. Therefore there is a u of the form $\sum_{v\in\mathcal{B}_{\mathrm{U}}} c_v v$ which is a semi-invariant for H. Since each v is a maximal semi-invariant for H and since $\mathcal{B}_{\mathrm{U}}$ has property $*$S$*$, $u = c_v v$ for some $v \in \mathcal{B}_{\mathrm{U}}$. Therefore $\tilde{\mathrm{U}} \cap \mathcal{B}_{\mathrm{U}} \neq \emptyset$. We will now show that G acts transitively on the set of lines $\mathcal{L} = \{\mathrm{C}v : v \in \mathcal{B}_{\mathrm{U}}\}$. This will imply that $\mathrm{U} \subseteq \tilde{\mathrm{U}}$. Note that property $*$S$*$ and the transitivity of $\mathcal{P}_{\mathrm{G}}$ imply that $\mathcal{L}$ contains exactly one line from each nonzero H$-$ characteristic subspace of W. Transitivity furthermore implies that for any $v_1, v_2 \in \mathcal{B}_{\mathrm{U}}$, there is a $g \in$ G such that $Supp(v_1^g) \cap Supp(v_2) \neq \emptyset$. Since $\mathcal{B}_{\mathrm{U}}$ has property $*$S$*$ we have that such a g takes the line spanned by v_1 to the line spanned by v_2. ∎

Lemma 5.3 *Let G be an imprimitive monomial group of degree n with permutation action $\mathcal{P}_{\mathrm{G}}$. Let U be a $G-$ module with basis $\mathcal{B}_{\mathrm{U}}$ in which H has diagonal form. Suppose that $\mathcal{B}_{\mathrm{U}}$ is equal $\{v^{\mathcal{P}_{\mathrm{G}}}\}$ for any $v \in \mathcal{B}_{\mathrm{U}}$. Let $\{U_i\}_{i=1}^k$ be the set of distinct $H-$ characteristic subspaces of U with corresponding bases $\mathcal{B}_i \subseteq \mathcal{B}_{\mathrm{U}}$. Assume that U_1 has a basis $\{u_1, \ldots, u_m\}$, $m = dim(U_1)$, of maximal $H-$ semi-invariants so that for any $\gamma \in \mathcal{S}$ such that $u_i^\gamma \in U_1$, u_i^γ is a multiple of u_i. Then $U = \bigoplus_{i=1}^m V_i$ where V_i is an irreducible $G-$ invariant subspace of dimension k.*

Proof: Let $m = \dim(\mathrm{U}_1)$ and let $u_1, \ldots, u_m$ be a basis of U_1 as in the lemma. Let $\mathcal{S}$ be a set of coset representatives for G and for each i let V_i be the span of the set $\{u_i^\gamma : \gamma \in \mathcal{S}\}$. We shall show using Lemma 5.2 that each V_i is an irreducible G$-$ module of dimension k and also that $\mathrm{U} = \bigoplus_{i=1}^m \mathrm{V}_i$.

Since H is normal in G, G permutes the H$-$ characteristic subspaces of U. Therefore each u_i^γ is a maximal H$-$ semi-invariant. Furthermore, if for any $g_1, g_2 \in$ G, $u_i^{g_1}$ and $u_i^{g_2}$ lie in the same H$-$ characteristic subspace, then $u_i^{g_1}$ is a multiple of $u_i^{g_2}$. Indeed if $u_i^{g_1}$ and $u_i^{g_2}$ are in the same H$-$ characteristic subspace then $u_i^{g_2^{-1}g_1}$ and u_i are in the same H$-$ characteristic subspace, namely U_1. Let $\gamma = g_2^{-1}g_1h$ be the coset representative of $g_2^{-1}g_1\mathrm{H}$, then this implies that u_i^γ and u_i both lie in U_1. Therefore, by assumption $u_i^\gamma = cu_i$ for some constant c. Since u_i is an H$-$ semi-invariant we have that $u_i^\gamma = u_i^{g_2^{-1}g_1h} = du_i^{g_2^{-1}g_1}$ for some constant d. That is $du_i^{g_1} = cu_i^{g_2}$. Therefore any two elements of $\mathrm{V}_i = \{u_i^\gamma : \gamma \in \mathcal{S}\}$ either are multiples or have disjoint support. Therefore $V_i \cap \mathrm{U}_j$ has dimension at most one for each U_j. Since $\mathcal{P}_\mathrm{G}$ acts transitively on $\mathcal{B}_\mathrm{U}$ we have that $V_i \cap \mathrm{U}_j$ has dimension exactly one for each j. Therfore $\mathrm{V}_i = \bigoplus_{j=1}^k(\mathrm{V}_i \cap \mathrm{U}_j)$ has dimension k.

We can select a subset $\mathcal{B}_{u_i}$ of $\{u_i^\gamma : \gamma \in \mathcal{S}\}$ consisting of one element from each set $\{u_i^\gamma : \gamma \in \mathcal{S}\} \cap \mathrm{U}_j$. This set $\mathcal{B}_{u_i}$ satisfies property $*$S$*$ and hence Lemma 5.2 implies that V_i is G$-$ irreducible.

Note that for each i, $\mathrm{V}_i \cap \mathrm{U}_1$ is spanned by u_i. Therefore $\mathrm{V}_i \not\subset \sum_{j\neq i}\mathrm{V}_j$. Since each V_i is irreducible, this implies that $\mathrm{V}_i \cap (\sum_{j\neq i}\mathrm{V}_j) = \{0\}$ and therefore $\sum_j \mathrm{V}_j = \bigoplus_j \mathrm{V}_j$. By comparing dimensions we see that $\mathrm{U} = \bigoplus_j \mathrm{V}_j$. ∎

Corollary 5.1 *Let G be an imprimitive monomial group with maximal diagonal subgroup H and permutation action* $\mathcal{P}_\mathrm{G}$. *Let U be a G module with basis* $\mathcal{B}_\mathrm{U} = \{v^{\mathcal{P}_\mathrm{G}}\}$ *for some* $v \in U$, *in which H has diagonal form. Suppose that each H$-$ characteristic subspace of U has dimension one. Then U is irreducible.*

Proof: Let $\mathrm{U}_1, \ldots, \mathrm{U}_k$ be the H$-$ characteristic subspaces of U. Since $\dim(\mathrm{U}_1) = 1$ any two nonzero elements of U_1 are constant multiples. Therefore Lemma 5.3 implies that U is G$-$ irreducible. ∎

The Procedure for Decomposing Representations of a Monomial Group

Let G be a monomial group with maximal diagonal subgroup H and permutation action $\mathcal{P}_\mathrm{G}$. Let W be a G$-$ module. Let $\mathcal{B}_\mathrm{W}$ be a basis for W in which H has diagonal form. By Lemma 5.1 such a basis exists. We decompose W into a sum of irreducible G$-$ invariant subspaces using the following procedure.

Step 1. Write W as $W_1 \oplus \ldots \oplus W_t$ where each W_i equals the span of $\{v^{\mathcal{P}_G}\}$ for some $v \in \mathcal{B}_W$.

Step 2. For each i write W_i as $W_{i,1} \oplus \ldots \oplus W_{i,k}$ where $W_{i,j}$ is a characteristic subspace for H. Note that all of these subspaces of W_i have equal dimension as shown in the proof of Lemma 5.3. Denote this integer by m. Choose a basis, $\mathcal{B}_{i,j}$ for $W_{i,j}$ such that $\mathcal{B}_{i,j} \subseteq \mathcal{B}_W$.

Step 3.

1. In each of the spaces $W_{i,j}$ look for a basis $\{u_1, \ldots u_m\}$ as in Lemma 5.3. If such a basis exists then Lemma 5.3 implies that W_i can be written as the direct sum of m irreducible G$-$ invariant subspaces each equal to the span of $\{\tilde{w}^{\mathcal{S}}\}$ for some $\tilde{w}$ in $W_{i,1}$.
2. If we are not able to find a basis as in 1, then we decompose W_i by hand.

Example: The Decomposition of Sym^2V

Let G be an infinite degree four imprimitive monomial subgroup of SL(4, C). Let V be the usual representation of G with basis $\mathcal{B}_V = \{v_1, v_2, v_3, v_4\}$. Then we can write down the following basis for W $= Sym^2$V.

$$\mathcal{B}_W = \{v_1^2, v_2^2, v_3^2, v_4^2, v_1v_2, v_1v_3, v_1v_4, v_2v_3, v_2v_4, v_3v_4\}$$

Recall that each monomial group can be identified via its maximal diagonal subgroup H and the permutation action $\mathcal{P}_G$ of G. The list of conjugacy classes of such groups is given in Theorem 4.1 on page 38. We decompose Sym^2V under the action of each of the infinite groups in this list. All of the information obtained using this procedure is given in Table 5.1 which can be found at the end of this chapter on page 106.

$\mathcal{P}_G = <(1234)>$. Regardless of the form of the diagonal group H we can write W as the direct sum of the following three G$-$ invariant subspaces.

- $W_1 = span\{v_1^2, v_2^2, v_3^2, v_4^2\}$
- $W_2 = span\{v_1v_2, v_2v_3, v_3v_4, v_1v_4\}$

- $W_3 = span\{v_1v_3, v_2v_4\}$

For each of the following possibilities for H we have one possible group $G = \cup_{\gamma \in \mathcal{S}} \gamma H$ where $\mathcal{S} = \mathcal{S}_1$ of Theorem 4.1. We denote the elements of $\mathcal{S}$ by $\gamma_0 = I, \gamma_1, \gamma_2$ and γ_3.

$\mathbf{H} = \mathbf{H}_1^n$ 1. $n = 1$. Recall that $H_1^1 = \left\{ \begin{bmatrix} a & 0 & 0 & 0 \\ 0 & a^{-1} & 0 & 0 \\ 0 & 0 & a & 0 \\ 0 & 0 & 0 & a^{-1} \end{bmatrix} : a \in C^* \right\}$.

- W_1 has H− characteristic subspaces $W_{1,1} = span\{v_1^2, v_3^2\}$ and $W_{1,2} = span\{v_2^2, v_4^2\}$. One can show that γ_1 and γ_3 do not leave $W_{1,1}$ invariant while for γ_2, $(v_1^2 + v_3^2)^{\gamma_2} = (v_1^2 + v_3^2)$ and $(v_1^2 - v_3^2)^{\gamma_2} = -(v_1^2 - v_3^2)$. In particular, $u_1 = v_1^2 + v_3^2$ and $u_2 = v_1^2 - v_3^2$ satisfy the conditions of Lemma 5.3. Therefore W_1 can be written as the direct sum of two G− invariant subspaces of dimension 2. They are $U_1 = span\{u_1^\gamma : \gamma \in \mathcal{S}\}$ and $U_2 = span\{u_2^\gamma : \gamma \in \mathcal{S}\}$.
- W_2 is an H− characteristic subspace of W. We will decompose W_2 by hand. We begin by looking for a 1 dimensional G− invariant subspace V of W_2. We may assume that V is spanned by an element of the form $w = v_1v_2 + \alpha_1 v_2v_3 + \alpha_2 v_3v_4 + \alpha_3 v_1v_4$, $\alpha_i \in C$. Since W_2 is a characteristic subspace for H the span of w is H− invariant for any choice of α_i. Hence $span\{w\}$ will be G− invariant if and only if $\{\alpha_1, \alpha_2, \alpha_3\}$ is a solution to the system of equations obtained from the following system.

  (a) $w^{\gamma_1} = v_2v_3 + \alpha_1 v_3v_4 - \alpha_2 v_1v_4 - \alpha_3 v_1v_2 = -\alpha_3 w$

  (b) $w^{\gamma_2} = v_3v_3 - \alpha_1 v_1v_4 + \alpha_2 v_1v_2 - \alpha_3 v_2v_3 = \alpha_2 w$

  (c) $w^{\gamma_3} = v_1v_4 - \alpha_1 v_1v_2 - \alpha_2 v_1v_3 + \alpha_3 v_3v_4 = -\alpha_1 w$

  The sets $\{1, 1, -1\}$ and $\{-1, 1, 1\}$ are both solutions of this system. Therefore W_2 has exactly two 1 dimensional G− invariant subspaces. Since $\dim(W_2) = 4$, any G− complement of the sum of these two subspaces must be irreducible. Therefore W_2 decomposes into the direct sum of irreducible subspaces of dimensions 2, 1 and 1.

- W_3 has H$-$ characteristic subspaces $W_{3,1} = span\{v_1v_3\}$ and $W_{3,2} = span\{v_2v_4\}$. Corollary 5.1 of Lemma 5.3 implies then that W_3 is irreducible.

Hence under the action of this group Sym^2V decomposes into the direct sum of four 2 dimensional subspaces and two 1 dimensional subspaces. This will be denoted as follows.

$$\mathbf{Sym^2V \Longrightarrow 2^4, 1^2}$$

2. $n = 2$. Recall that

$$H_1^2 = \left\{ \begin{bmatrix} a & 0 & 0 & 0 \\ 0 & a^{-1} & 0 & 0 \\ 0 & 0 & a & 0 \\ 0 & 0 & 0 & a^{-1} \end{bmatrix} : a \in C^* \right\} \cup \left\{ \begin{bmatrix} -a & 0 & 0 & 0 \\ 0 & a^{-1} & 0 & 0 \\ 0 & 0 & -a & 0 \\ 0 & 0 & 0 & a^{-1} \end{bmatrix} : a \in C^* \right\}.$$

Note that the only place where the group H affects the decomposition is in writing each W_i as a sum of H$-$ characteristic subspaces. Since the H_1^2- characteristic subspaces of each W_i above are the same as the H_1^1- characteristic subspaces the decomposition of Sym^2V under the action of H_1^2 is again given by

$$\mathbf{Sym^2V \Longrightarrow 2^4, 1^2}$$

$\mathbf{H} = \mathbf{H}_2^n$ Recall that $H_2^n = \left\{ \begin{bmatrix} \xi a & 0 & 0 & 0 \\ 0 & \xi^{-1} b & 0 & 0 \\ 0 & 0 & a^{-1} & 0 \\ 0 & 0 & 0 & b^{-1} \end{bmatrix} : a, b \in C^*, \xi \in U_n \right\}.$

1. $n = 1$ **or** 2. For each i the characteristic subspaces of W_i for either of these two groups are identical.

   - W_1 has four H$-$ characteristic subspaces of dimension 1. These are $W_{1,j} = span\{v_j^2\}$ for $i = 1, \ldots 4$. Therefore by Corollary 5.1 of Lemma 5.3, W_1 is irreducible.

   - W_2 is also irreducible by the Corollary 5.1 of Lemma 5.3.

- Consider now W_3. This subspace of Sym^2V is itself an $H-$ characteristic subspace. We will look for a $G-$ invariant subspace of dimension 1. Let $w = v_1v_3 + \alpha v_2v_4$, $\alpha \in C$. Then W_3 has a 1 dimensional $G-$ invariant subspace if and only if there exists an α satisfying

  (a) $w^{\gamma_1} = v_2v_4 - \alpha v_1v_3 = -\alpha w$

  (b) $w^{\gamma_2} = -v_1v_3 - \alpha v_2v_4 = -w$

  (c) $w^{\gamma_3} = -v_2v_4 + \alpha v_1v_3 = \alpha w$

  Any $\alpha \in \{\pm\sqrt{-1}\}$ satisfies these conditions. Therefore W_3 can be written as the sum of two $G-$ invariant subspaces of dimension 1.

Hence under the action of either of these groups we have

$$\mathbf{Sym^2V \Longrightarrow 4^2, 1^2}$$

2. $n \neq 1, 2$. Given i, the characteristic subspaces of W_i for any pair of groups H_2^n and H_2^m where $n, m \notin \{1, 2\}$ are identical. Hence we decompose Sym^2V here under the action of such a group for arbitrary $n > 2$.

   - W_1 again has four $H-$ characteristic subspaces of dimension 1. These are $W_{1,j} = span\{v_j^2\}$ for $i = 1, \ldots 4$. Therefore by Corollary 5.1 of Lemma 5.3 W_1 is irreducible.
   - W_2 is also irreducible by Corollary 5.1 of Lemma 5.3.
   - Consider now W_3. Since $n \neq 1, 2$ the scalars ξ and ξ^{-1} may be different. Therefore contrary to the previous case, W_3 is not a characteristic subspace for H. It can be written as the sum of two 1 dimensional characteristic subspaces and hence is irreducible again by Corollary 5.1. Hence under the action of $G = \cup_{\gamma \in \mathcal{S}} \gamma H$ where H is any of the groups H_2^n $n \neq 1, 2$ we have

$$\mathbf{Sym^2V \Longrightarrow 4^2, 2}$$

$\mathbf{H = H_3}$ Recall that $H_3 = \mathrm{diag}(SL(4, C))$. It is easy to see that each subspace W_i has $\dim(W_i)$ $H-$ characteristic subspaces each of dimension 1. Hence each of the subspaces W_i is irreducible by Corollary 5.1 of Lemma 5.3. Hence the action of this G on Sym^2V is also given by

$$\mathbf{Sym^2V \Longrightarrow 4^2, 2}$$

$\mathcal{P}_G = \mathbf{K}$. Regardless of the form of the diagonal group H, we can write W as the direct sum of the following four G$-$ invariant subspaces.

- $W_1 = span\{v_1^2, v_2^2, v_3^2, v_4^2\}$
- $W_2 = span\{v_1v_2, v_3v_4\}$
- $W_3 = span\{v_1v_3, v_2v_4\}$
- $W_4 = span\{v_1v_4, v_2v_3\}$

For each of the following possibilities for H except for H_1^1 we have one possible group $G = \cup_{\gamma\in\mathcal{S}}\gamma H$ where $\mathcal{S} = \mathcal{S}_3$ of Theorem 4.1. For ease of notation we denote the elements of $\mathcal{S}$ by $\gamma_0 = I$, γ_1, γ_2 and γ_3. Note that when $H = H_1^1$ we get a distinct group when we choose the coset representatives γ from the set $\mathcal{S}_4$ versus $\mathcal{S}_3$ of Theorem 4.1.

$\mathbf{H = H_1^n}$ 1. $n = 1$. Recall that $H_1^1 = \left\{ \begin{bmatrix} a & 0 & 0 & 0 \\ 0 & a^{-1} & 0 & 0 \\ 0 & 0 & a & 0 \\ 0 & 0 & 0 & a^{-1} \end{bmatrix} : a \in C^* \right\}$.

- W_1 has H$-$ characteristic subspaces $W_{1,1} = span\{v_1^2, v_3^2\}$ and $W_{1,2} = span\{v_2^2, v_4^2\}$. Let $\mathcal{S} = \mathcal{S}_3$ from the monomial group list. Once again $u_1 = v_1^2 + v_3^2$ and $u_2 = v_1^2 - v_3^2$ satisfy the hypotheses of Lemma 5.3. Therefore W_1 can be written as the direct sum of two G$-$ invariant subspaces of dimension 2. They are $U_1 = span\{u_1^\gamma : \gamma \in \mathcal{S}\}$ and $U_2 = span\{u_2^\gamma : \gamma \in \mathcal{S}\}$.
- W_2 is itself a characteristic subspace of W and hence any maximal H$-$ semi-invariant $u \in W_2$ which satisfies the conditions of Lemma 5.3 will give us a 1 dimensional G$-$ invariant subspace. One can easily show that $v_1v_2 + v_3v_4$ and $v_1v_2 - v_3v_4$ both satisfy the conditions of this

lemma. Therefore W_2 decomposes into the direct sum of two irreducible subspaces of dimension 1.

- W_3 has $\mathrm{H}-$ characteristic subspaces $\mathrm{W}_{3,1} = span\{v_1v_3\}$ and $\mathrm{W}_{3,2} = span\{v_2v_4\}$. Corollary 5.1 of Lemma 5.3 implies then that W_3 is irreducible.
- W_4 is also an $\mathrm{H}-$ characteristic subspace of W. As in the above decomposition of W_2 one can show that the maximal semi-invariants $v_1v_4 + v_2v_3$ and $v_1v_4 - v_2v_3$ both satisfy the conditions of Lemma 5.3. Hence W_4 also decomposes into the direct sum of two irreducible subspaces of dimension 1.

Hence under the action of this group $Sym^2\mathrm{V}$ decomposes according to

$$\mathbf{Sym^2V} \Longrightarrow \mathbf{2^3}, \mathbf{1^4}$$

Remark: Let $\tilde{\gamma}_i$ denote the element of $\mathcal{S}_4$ with the same permutation action of $\gamma_i \in \mathcal{S}_3$. One can show that for each $v \in \mathcal{B}_{\mathrm{W}}$, $v^{\tilde{\gamma}_i} = v^{\gamma_i}$. This implies that the groups $\cup_{\gamma \in \mathcal{S}_3} \gamma\mathrm{H}$ and $\cup_{\gamma \in \mathcal{S}_4} \gamma\mathrm{H}$ share the same irreducible decomposition of $Sym^2\mathrm{V}$.

2. $n = 2$.

   Again, since the H_1^2- characteristic subspaces of each W_i are the same as the H_1^1- characteristic subspaces the decomposition of $Sym^2\mathrm{V}$ under the action of this group is also given by

$$\mathbf{Sym^2V} \Longrightarrow \mathbf{2^3}, \mathbf{1^4}$$

$\mathbf{H} = \mathbf{H}_2^n$ Recall that $\mathrm{H}_2^n = \left\{ \begin{bmatrix} \xi a & 0 & 0 & 0 \\ 0 & \xi^{-1} b & 0 & 0 \\ 0 & 0 & a^{-1} & 0 \\ 0 & 0 & 0 & b^{-1} \end{bmatrix} : a, b \in \mathrm{C}^*, \xi \in \mathrm{U}_n \right\}.$

1. $n = 1$ or 2. For each i the characteristic subspaces of W_i for either of these two groups are identical. Hence the decompositions of $Sym^2\mathrm{V}$ under the action of these two groups are the same.

- W_1 has four H$-$ characteristic subspaces of dimension 1. These are $W_{1,j} = span\{v_j^2\}$ for $i = 1, \ldots, 4$. Therefore by Corollary 5.1 of Lemma 5.3, W_1 is irreducible.
- W_2 is also irreducible by Corollary 5.1 of Lemma 5.3.
- Consider now W_3. This subspace of Sym^2V is itself an H$-$ characteristic subspace. We will look for a G$-$ invariant subspace of dimension 1. Let $w = v_1v_3 + \alpha v_2v_4$, $\alpha \in$ C. Then W_3 has a 1 dimensional G$-$ invariant subspace if and only if there exists an α satisfying

  (a) $w^{\gamma_1} = v_2v_4 + \alpha v_1v_3 = \alpha w$

  (b) $w^{\gamma_2} = v_1v_3 + \alpha v_2v_4 = w$

  (c) $w^{\gamma_3} = v_2v_4 + \alpha v_1v_3 = \alpha w$

  Any $\alpha \in \{\pm 1\}$ satisfies these conditions. Note that these two elements also satisfy the hypotheses of Lemma 5.3. In any case W_3 can be written as the sum of two G$-$ invariant subspaces of dimension 1.

Hence under the action of either of these groups we have

$$\mathbf{Sym^2V \Longrightarrow 4, 2^2, 1^2}$$

2. $n \neq 1, 2$. Given i, the characteristic subspaces of W_i for any pair of groups H_2^n and H_2^m where $m, n \notin \{1, 2\}$ are identical. Hence we decompose Sym^2V here under the action of such a group for arbitrary $n > 2$.

   - W_1 again has four H$-$ characteristic subspaces of dimension 1. These are $W_{1,j} = span\{v_j^2\}$ for $i = 1, \ldots, 4$. Therefore by Corollary 5.1 of Lemma 5.3, W_1 is irreducible.
   - Since $n \neq 1$ or 2, the scalars ξ and ξ^{-1} may be different. Therefore each of the spaces W_2, W_3 and W_4 has two characteristic subspaces of dimension 1 and hence is also irreducible by Corollary 5.1 of Lemma 5.3.

   Therefore under the action of $G = \cup_{\gamma \in \mathcal{S}} \gamma$H where H is any of the groups H_2^n, $n \neq 1, 2$ we have

$$\mathbf{Sym^2V \Longrightarrow 4, 2^3}$$

$\mathbf{H} = \mathbf{H}_3$ It is easy to see that each subspace W_i has $\dim(\mathrm{W}_i)$ H$-$ characteristic subspaces each of dimension 1. Hence each of the subspaces W_i is irreducible by the corollary to Lemma 5.3. Hence the action of this G on $Sym^2\mathrm{V}$ is also given by

$$\mathbf{Sym^2V} \Longrightarrow \mathbf{4}, \mathbf{2^3}$$

$\mathcal{P}_{\mathrm{G}} =< (1234), (12)(34) >$. Regardless of the form of the diagonal group H we can write W as the direct sum of the following three G$-$ invariant subspaces.

- $\mathrm{W}_1 = span\{v_1^2, v_2^2, v_3^2, v_4^2\}$
- $\mathrm{W}_2 = span\{v_1v_2, v_2v_3, v_3v_4, v_1v_4\}$
- $\mathrm{W}_3 = span\{v_1v_3, v_2v_4\}$

For each of the following possibilities for H we have one possible group $\mathrm{G} = \cup_{\gamma\in\mathcal{S}}\gamma\mathrm{H}$ where $\mathcal{S} = \mathcal{S}_2$ of Theorem 4.1. We denote the elements of $\mathcal{S}$ by $\gamma_0 = \mathrm{I}, \gamma_1, \ldots, \gamma_7$.

$\mathbf{H} = \mathbf{H}_1^n$ 1. $n = 1$. Recall that $\mathrm{H}_1^1 = \left\{ \begin{bmatrix} a & 0 & 0 & 0 \\ 0 & a^{-1} & 0 & 0 \\ 0 & 0 & a & 0 \\ 0 & 0 & 0 & a^{-1} \end{bmatrix} : a \in \mathrm{C}^* \right\}$.

- W_1 has H$-$ characteristic subspaces $\mathrm{W}_{1,1} = span\{v_1^2, v_3^2\}$ and $\mathrm{W}_{1,2} = span\{v_2^2, v_4^2\}$. One can show that $u_1 = v_1^2 + v_3^2$ and $u_2 = v_1^2 - v_3^2$ satisfy the conditions of Lemma 5.3. Therefore W_1 can be written as the direct sum of two G$-$ invariant subspaces of dimension 2. They are $\mathrm{U}_1 = span\{(v_1^2 + v_3^2)^\gamma : \gamma \in \mathcal{S}\}$ and $\mathrm{U}_2 = span\{(v_1^2 - v_3^2)^\gamma : \gamma \in \mathcal{S}\}$.
- W_2 is an H$-$ characteristic subspace of W. In this case we will decompose W_2 by hand. We begin by looking for a 1 dimensional G$-$ invariant subspace of W_2. Let $w = v_1v_2 + \alpha_1 v_2v_3 + \alpha_2 v_3v_4 + \alpha_3 v_1v_4$, $\alpha_i \in \mathrm{C}$. Since W_2 is a characteristic subspace for H the span of w is H invariant for any choice of α_i. Hence $span\{w\}$ will be G$-$ invariant if and only if $\{\alpha_1, \alpha_2, \alpha_3\}$ is a solution to the system of equations obtained from the system $\{w^{\gamma_i} = c_i w : \gamma_i \in \mathcal{S}\}$ for some set of scalars $\{c_i\}$. A computations shows that the sets $\{1, 1, -1\}$ and $\{-1, 1, 1\}$ are the only solutions to

this system. Therefore W_2 has exactly two 1 dimensional G− invariant subspaces. Since $\dim(W_2) = 4$, the G− complement of the sum of these two subspaces must be irreducible. Therefore W_2 decomposes into the direct sum of irreducible subspaces of dimensions 2, 1 and 1.

- W_3 has H− characteristic subspaces $W_{3,1} = span\{v_1 v_3\}$ and $W_{3,2} = span\{v_2 v_4\}$. Corollary 5.1 of Lemma 5.3 implies then that W_3 is irreducible.

Hence under the action of this group Sym^2V decomposes as follows.

$$\mathbf{Sym^2V \Longrightarrow 2^4, 1^2}$$

2. $n = 2$. Recall that

$$\mathrm{H}_1^2 = \left\{ \begin{bmatrix} a & 0 & 0 & 0 \\ 0 & a^{-1} & 0 & 0 \\ 0 & 0 & a & 0 \\ 0 & 0 & 0 & a^{-1} \end{bmatrix} : a \in \mathrm{C}^* \right\} \cup \left\{ \begin{bmatrix} -a & 0 & 0 & 0 \\ 0 & a^{-1} & 0 & 0 \\ 0 & 0 & -a & 0 \\ 0 & 0 & 0 & a^{-1} \end{bmatrix} : a \in \mathrm{C}^* \right\}.$$

Since the H_1^2− characteristic subspaces of each W_i above are the same as the H_1^1− characteristic subspaces the decomposition of Sym^2V under the action of this group is also given by

$$\mathbf{Sym^2V \Longrightarrow 2^4, 1^2}$$

$\mathbf{H = H_2^n}$ Recall that $$\mathrm{H}_2^n = \left\{ \begin{bmatrix} \xi a & 0 & 0 & 0 \\ 0 & \xi^{-1} b & 0 & 0 \\ 0 & 0 & a^{-1} & 0 \\ 0 & 0 & 0 & b^{-1} \end{bmatrix} : a, b \in \mathrm{C}^*, \xi \in \mathrm{U}_n \right\}.$$

1. $n = 1$ or 2. For each i the characteristic subspaces of W_i for either of these two groups are identical.

- W_1 has four H− characteristic subspaces of dimension 1. These are $W_{1,j} = span\{v_j^2\}$ for $i = 1, \ldots 4$. Therefore by Corollary 5.1 of Lemma 5.3 W_1 is irreducible.

- W_2 is also irreducible by Corollary 5.1 of Lemma 5.3.

- W_3 is itself an $H-$ characteristic subspace. We will look for a $G-$ invariant subspace of dimension 1. Let $w = v_1v_3 + \alpha v_2v_4,\ \alpha \in C$. Then W_3 has a 1 dimensional $G-$ invariant subspace if and only if there exists an α satisfying

  (a) $w^{\gamma_1} = v_2v_4 - \alpha v_1v_3 = -\alpha w$

  (b) $w^{\gamma_2} = -v_1v_3 - \alpha v_2v_4 = -w$

  (c) $w^{\gamma_3} = -v_2v_4 + \alpha v_1v_3 = \alpha w$

  (d) $w^{\gamma_4} = -v_2v_4 - \alpha v_1v_3 = -\alpha w$

  (e) $w^{\gamma_5} = v_2v_4 + \alpha v_1v_3 = \alpha w$

  (f) $w^{\gamma_6} = v_1v_3 - \alpha v_2v_4 = w$

  (g) $w^{\gamma_7} = -v_1v_3 + \alpha v_2v_4 = -w$

  These equation imply that $\alpha = -\alpha$ and that $1 = \alpha^2$. Therefore there does not exist a scalar α satisfying these conditions. Hence W_3 is irreducible.

Therefore under the action of either of these groups we have

$$\mathbf{Sym^2V \Longrightarrow 4^2, 2}$$

2. $n \neq 1, 2.$ or $H = H_3$ In any of these cases it is clear that each W_i is composed of $H-$ characteristic subspaces of dimension 1 and hence is irreducible by the Corollary 5.1 of Lemma 5.3. Therefore for each such group, G, we have the decomposition

$$\mathbf{Sym^2V \Longrightarrow 4^2, 2}$$

$\mathcal{P}_G = \mathbf{A}_4$ **or** $\mathbf{S}_4$. Lemma 4.6 of page 36 implies that the only possibility for the diagonal group H in this case is $H = \mathrm{diag}(SL(4, C))$. In either case for $\mathcal{P}_G$ we may write W as the direct sum of the following two $G-$ invariant subspaces.

- $W_1 = span\{v_1^2, v_2^2, v_3^2, v_4^2\}$
- $W_2 = span\{v_1v_2, v_2v_3, v_3v_4, v_1v_4, v_1v_3, v_2v_4\}$

Since the only possibility for H is H_3, each of these subspaces is clearly irreducible by Corollary 5.1 of Lemma 5.3. Therefore we have the following decomposition for Sym^2V under the action of G with $\mathcal{P}_G = A_4$ or S_4.

$$\mathbf{Sym^2V \Longrightarrow 6, 4}$$

Using the same procedure we have decomposed the spaces $\bigwedge^2$V, Sym^3V, and Sym^4V. Table 5.1 lists the dimensions of the irreducible summands in the decompositions of all four vector spaces under the action of each of the infinite monomial groups of degree four. Table 5.1 gives information about the representations of finite monomial groups as well. Note the following remark.

Remark: Any finite monomial group $\tilde{G}$ with permutation action $\mathcal{P}_G$ is a subgroup of an infinite monomial group G with that action. Therefore the $G-$ invariant summands in the decomposition of a $G-$ module W are also $\tilde{G}-$ invariant, although they may no longer be irreducible. Nonetheless, this information will be enough to distinguish the monomial groups from the other groups.

5.1.2 Representations of the Imprimitive Nonmonomial Groups

In this section we consider the representations of imprimitive nonmonomial subgroups of $\mathrm{SL}(4,\mathrm{C})$. Recall that such a group has a normal subgroup $\mathrm{H} \subseteq \mathrm{T} \times \mathrm{T}$ where for $i = 1, 2$, the coordinate projections $\pi(\mathrm{H}) = \mathrm{T}$ and T is a primitive subgroup of $\mathrm{GL}(2,\mathrm{C})$. This subgroup T is either projectively finite or it is equal to $\mathrm{G}_n = \{\mathrm{A} \in \mathrm{GL}(2,\mathrm{C}) : \det(\mathrm{A})^n = 1\}$ for some nonnegative integer n. We will show that when T is projectively finite, it is enough to consider representations of certain finite groups. In this case we will use the computer algebra package *MAGMA* in order to obtain the irreducible decomposition of a representation of such a group. In the case when $\mathrm{T} = \mathrm{G}_n$, we will show that one may make use of the process used to decompose a given $\mathrm{SL}(2,\mathrm{C})-$ module.

Representations of T

The following lemma and corollary concern the representations of the primitive subgroups of $\mathrm{GL}(2,\mathrm{C})$. In particular Lemma 5.4 allows one to list the irreducible G_n modules as one would the irreducible $\mathrm{SL}(2,\mathrm{C})-$ modules.

Lemma 5.4 *Let G be an irreducible algebraic group of degree n. Let $PG = \pi(G)$ where π is the canonical projection from $GL(n,C)$ to $PGL(n,C)$. Let W be an irreducible $G-$ module. Then W is an irreducible $PG-$ module.*

Proof: Note that we make W a PG$-$ module as follows. Given $\overline{g} \in \mathrm{PG}$ and $w \in \mathrm{W}$, we let $\overline{g} \cdot w$ be defined as the span of $\{gw\}$. This action is well defined on PG. Indeed, if $\overline{g_1} = \overline{g_2}$ then $g_1 = cg_2$ for some $c \in Z(\mathrm{GL}(n,\mathrm{C}))$. Therefore $\overline{g_1}w \in span\{g_2w\}$ which implies that $\overline{g_1}w = \overline{g_2}w$. Suppose that W is reducible as a PG$-$ module. Let U be a proper PG$-$ invariant subspace of W. Note that since G is an irreducible group its center $Z(\mathrm{G})$ is scalar. Therefore any vector space can be made into a $Z(\mathrm{G})-$ module via scalar multiplication. In this way U can be lifted to a proper irreducible G$-$ invariant subspace of W which contradicts the irreducibility of W. ∎

Corollary 5.2 *Suppose W is a representation of two groups G_1 and G_2 having the property that $\pi(G_1) = \pi(G_2)$. Then U is a G_1- irreducible subspace of W if and only if U is G_2- irreducible.*

Proof: This is a direct consequence of Lemma 5.4. ∎

Remarks: Corollary 5.2 implies the following two facts which will be used throughout the rest of this section.

1. One can obtain the dimensions of the irreducible components of a T$-$ module for any group T which projects onto P by decomposing this module under the action of any one such group.

2. Any irreducible G_n- module is isomorphic to one of the form $Sym^m\mathrm{C}^2$. This follows from the fact that the irreducible $\mathrm{SL}(2,\mathrm{C})-$ modules are of this form.

The Characters of the Irreducible Representations of $\mathbf{G}_n$

The character of any representation of the group $\mathrm{SL}(2,\mathrm{C})$ is completely determined by its value on the subgroup of diagonal elements. Since G_n is an extension of $\mathrm{SL}(2,\mathrm{C})$ by scalar elements the character of any representation of G_n is also completely determined by its value on the diagonal elements. By remark 2 above any irreducible G_n- module of degree m is isomorphic to $Sym^{m-1}(\mathrm{C}^2)$. This space is denoted by $\mathrm{V}(m)$. We list here the characters of G_n on several of these representations. Such a character will be denoted by χ_m. Let $g = \begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix}$.

1. $\chi_1(g) = ab$

2. $\chi_2(g) = a + b$

3. $\chi_3(g) = a^2 + ab + b^2$

4. $\chi_4(g) = a^3 + a^2b + ab^2 + b^3$, etc.

Note that any module of the form $U \otimes V(m)$ where U is a one dimensional G_n- module is also an irreducible G_n- module of dimension m. The value of a character at $g \in G_n$ on such a space is equal to $\xi\chi_m(g)$ for some scalar $\xi \in U_n$.

Representations of $\mathbf{H} \subseteq \mathbf{T} \times \mathbf{T}$

In Chapter 4 Section 1 we constructed an infinite list of possibilities for this subgroup $H \subset T \times T$ of a nonmonomial imprimitive group. It is clear that one is not able to decompose a given module under the action of each of these groups. The following lemma and its corollary allow us to determine the dimensions of the irreducible components of certain modules under the action of any of the groups via the actions of only a finite list of groups.

Before stating the next lemma, we make the following conventions. We embed $GL(2, C) \times GL(2, C)$ into $GL(4, C)$ by sending (g_1, g_2) to $\begin{bmatrix} g_1 & 0 \\ 0 & g_2 \end{bmatrix}$. This allows us to write $V = C^4$ as a direct sum $V = V_1 \oplus V_2$ where each $V_i \cong C^2$ is invariant under $GL(2, C) \times GL(2, C)$.

Lemma 5.5 *Let H_1 and H_2 be reductive subgroups of $(GL(2, C) \times GL(2, C)) \cap SL(4, C)$ such that $H_2 D = H_1 D$, where $D \subseteq Z(GL(2, C)) \times Z(GL(2, C))$. Let $V = C^4$. Then given a nonnegative integer n, one can write $Sym^n V = \bigoplus W_i$ and $\bigwedge^n V = \bigoplus U_i$ where each W_i and U_i is both an irreducible H_1- module and an irreducible H_2- module.*

Proof: We will prove the result $Sym^n V$. The proof of the case $\bigwedge^n V$ is similar. Since $V = V_1 \oplus V_2$ we may write $Sym^n V = \bigoplus_{a+b=n}(Sym^a V_1 \otimes Sym^b V_2)$ and it is clear that each space $Sym^a V_1 \otimes Sym^b V_2$ is both an H_1- and an H_2- submodule. We shall show that any H_1- invariant subspace of $Sym^a V_1 \otimes Sym^b V_2$ is $H_1 D-$ invariant. A similar argument will show that any H_2- invariant subspace is of $Sym^a V_1 \otimes Sym^b V_2$ is $H_2 D-$ invariant and so any H_1- invariant subspace is H_2- invariant and vice versa. Decomposing each $Sym^a V_1 \otimes Sym^b V_2$ into H_1- irreducible subspaces will yield the lemma.

Let $W \subseteq Sym^a V_1 \otimes Sym^b V_2$ be an H_1- invariant subspace. If $d \in D$, then $d = \begin{bmatrix} c & 0 & 0 & 0 \\ 0 & c & 0 & 0 \\ 0 & 0 & c^{-1} & 0 \\ 0 & 0 & 0 & c^{-1} \end{bmatrix}$ or $\begin{bmatrix} c & 0 & 0 & 0 \\ 0 & c & 0 & 0 \\ 0 & 0 & -c^{-1} & 0 \\ 0 & 0 & 0 & -c^{-1} \end{bmatrix}$ for some $c \in C^*$. If $w_1 \in Sym^a V_1$ and $w_2 \in Sym^b V_2$, then a computation shows that $d(w_1 \otimes w_2) = \pm c^{a-b}(w_1 \otimes w_2)$. Therefore D acts by scalar multiplication on $Sym^a V \otimes Sym^b V$ and so W is $H_1 D-$ invariant. ∎

Corollary 5.3 *Let H be a subgroup of $G_n \times G_n$ containing $SL(2, C) \times SL(2, C)$. Then $Sym^m V$ and $\bigwedge^m V$ are each isomorphic to a direct sum of modules of the form $V(k) \otimes V(n)$ for some pair of nonnegative integers n and k and this decomposition is independent of H.*

Proof: Any irreducible $SL(2, C) \times SL(2, C)$ module is of the form $V(n) \otimes V(m)$ for some pair of nonnegative integers n and m. The result then follows from Lemma 5.5. ∎

Remark: Now suppose that $G = H \cup \gamma H$ is an imprimitive nonmonomial subgroup of $SL(4, C)$. Fix a nonnegative integer n. Let H_n denote one of the groups of the form $\left\{ \begin{bmatrix} A & 0 \\ 0 & \xi_A B \end{bmatrix} : A \in P_n, B \in AF \right\}$ from the list given in Theorem 4.3 on page 56. Let m be any other nonnegative integer and denote by H_m the group $\left\{ \begin{bmatrix} A & 0 \\ 0 & \xi_A B \end{bmatrix} : A \in P_m, B \in AF \right\}$. Lemma 5.5 now implies that given a module of the form $Sym^m V$ or $\bigwedge^m V$, of H_n, the dimensions of the irreducible components in its decomposition as an H_n- module correspond exactly to those in its decomposition as an H_m- module for any nonnegative integer m. Hence we may obtain an accurate list of the dimensions of the irreducible components of such modules for each of these imprimitive groups by decomposing them under the action of only a finite number of groups. A similar fact holds for the cases where T is projectively infinite.

Representations of $G = H \cup \gamma H$

We will use the next two lemmas in order to extract the decomposition of a given $G = H \cup \gamma H-$ module from its decomposition as an $H-$ module.

Lemma 5.6 *Let G be a subgroup of $GL(V)$ and let H be a normal subgroup of G. Assume that $U = W \oplus (\bigoplus_{i=1}^{m} V_i)$ where W and the V_i are irreducible $H-$ modules and that $dim(W) \neq dim(V_i)$ for all i. Then W is $G-$ invariant.*

Proof: Since H $\lhd$ G, gW is an irreducible H$-$ module for all $g \in$ G. Let π_j be the projection onto V$_j$. Since W and V$_j$ are irreducible H$-$ modules, $\pi_j(g\mathrm{W}) = \{0\}$ or V$_j$. Since $\dim(g\mathrm{W}) \neq \dim(\mathrm{V}_j)$ we must have that $\pi_j(g\mathrm{W}) = \{0\}$ for $j = 1, \ldots, m$. Therefore $g\mathrm{W} \subseteq \mathrm{W}$. ∎

Lemma 5.7 *Let $G = H \cup \gamma H$ be an imprimitive nonmonomial subgroup of $SL(4, C)$. Let W be a $G-$ module. Suppose that W_i is an irreducible component in the decomposition of W as an $H-$ module with the property that $\gamma : W_i \to \gamma W_i$ is an $H-$ module isomorphism. Then $U_i = \{w + \gamma w : w \in W_i\}$ and $\overline{U_i} = \{w - \gamma w : w \in W_i\}$ are $G-$ invariant subspaces of W which are isomorphic to W_i as $H-$ modules.*

Proof: U$_i$ is H$-$ invariant since γ is an H$-$ module homomorphism. Furthermore, γ fixes each element of U$_i$. Hence U$_i$ is a G$-$ invariant subspace of W. Let π denote the projection of U$_i$ into W$_i$ defined by $\pi(w + \gamma w) = w$. π is clearly an H$-$ module isomorphism which is onto W$_i$. $\overline{\mathrm{U}_i}$ is treated similarly. ∎

Corollary 5.4 *Let $G = H \cup \gamma H$ be an imprimitive nonmonomial subgroup of $SL(4, C)$. Let W be a $G-$ module. Let $W = \bigoplus_{i=1}^{m} W_i$ be the irreducible decompostion of W as an $H-$ module with $dim(W_i) = d_i$. Suppose $\gamma W_i = W_j$ for some $i \neq j$. Let $U = W_i \oplus W_j$.*

1. *If W_i and W_j are not isomorphic as H modules then U is an irreducible $G-$ module.*
2. *If $\gamma : W_i \to W_j$ is an $H-$ module isomorphism then $U = U_i \oplus U_j$ where $dim(U_i) = dim(U_j) = d_i$ and U_i and U_j are irreducible $G-$ modules.*

Proof: Let G be such a group. Let W be a G$-$ module. Since H is a subgroup of G, W is also an H$-$ module. Let $\mathrm{W} = \bigoplus_{i=1}^{m} \mathrm{W}_i$ be the irreducible decompostion of W as an H$-$ module with $\dim(W_i) = d_i$. Note that also since $H \lhd G$, for any $g \in G$ gW_i is an irreducible H$-$ invariant subspace of W of dimension d_i. Without loss of generality assume $\gamma W_1 = W_2$ and let $U = W_1 \oplus W_2$.

(Proof of 1) Let $W \subset U$ be a proper nonzero G$-$ module and let π_i be the projection of W onto W_i for $i = 1, 2$. If $\pi_1(W) = \{0\}$ then $W \subseteq W_2$. Since W_2 is an H$-$ irreducible module we have that $W = W_2$ which contradicts the fact that W_2 is not G$-$ invariant. Therefore H$-$ irreducibility of W_2 implies that $\pi_1(W) = W_1$. A similar argument shows that $Ker(\pi_1|W) = W_2$. Therefore $\dim(W) = \dim(W_1) + \dim(W_2) = \dim(U)$ contradicting the fact that W is a proper submodule of U. We conclude that U must be G$-$ irreducible.

(Proof of 2) Using the notation of Lemma 5.7 we see that $U = U_1 \oplus \overline{U_1}$ where each component is G$-$ irreducible. ∎

A Procedure for Decomposing Representations of an Imprimitive Nonmonomial Group, G.

Let $G = H \cup \gamma H$ be a nonmonmial imprimitive group. Recall that $H \subseteq T \times T$, where T is a primitive subgroup of GL(2, C) and $\pi_i(H) = T$ for each coordinate projection π_i. Let W be a G$-$ module of the form $Sym^m V$ or $\bigwedge^m V$ for some integer m.

1. Decompose W as an H$-$ module.

   - Suppose that $T = G_n$ for some positive integer n. Given such a group G, the subgroup H is one of the following three groups.

     (a) $H_n^1 = \left\{ \begin{bmatrix} B & 0 \\ 0 & d_B B \end{bmatrix} : B \in G_n, d_B = \frac{1}{\sqrt{\det(B)}} \right\}$

     (b) $H_n^2 = \left\{ \begin{bmatrix} B & 0 \\ 0 & \pm d_B B \end{bmatrix} : B \in G_n, d_B = \frac{1}{\sqrt{\det(B)}} \right\}$

     (c) $H_n^3 = \left\{ \begin{bmatrix} A & 0 \\ 0 & B \end{bmatrix} : A, B \in G_n, \det(AB) = 1 \right\}$

     Let $D = Z(GL(2, C)) \times Z(GL(2, C))$. Note that $H_n^1 D = H_n^2 D =$ $\left\{ \begin{bmatrix} g & 0 \\ 0 & g^{-1} \end{bmatrix} : g \in SL(2, C) \right\} D$. Lemma 5.5 then implies that we may decompose modules of the form $Sym^m V$ and $\bigwedge^k V$ as SL(2, C)$-$ modules in order to obtain the subspaces in their decompositions as either H_n^1- or H_n^2- modules. Similarly Corollary 5.3 of Lemma 5.5 tells us the form of the irreducible H_n^3- modules.

So by the above remarks we may obtain the irreducible submodules in the decompostion of $Sym^n V$ or $\bigwedge^k V$ as an H− module for any H listed above. In addition to this we have the following fact regarding the characters of H− modules. The character of any G− module where G is either $SL(2, C)$ or $SL(2, C) \times SL(2, C)$ is completely determined by its value on the diagonal elements. Since each H_n^i above is simply a diagonal extension of one of these two groups, their characters are also completely determined by their values on diagonal elements.

- Suppose that T is projectively finite. Let P be a fixed projective representation of A_4, A_5 or S_4. Fix a normal subgroup N of P. By the remark following Corollary 5.3 on page 93 if we denote by H_m the group $\left\{ \begin{bmatrix} A & 0 \\ 0 & \xi_A B \end{bmatrix} : A \in P_m, B \in AF \right\}$, where F is the entire lift of N to $SL(2, C)$ and $\xi_A \in C^*$, then the dimensions of the components of an irreducible decomposition of $W = Sym^n V$ as an H_m- module are the same for any non-negative integer m. Hence for each of the 12 collections of groups given in Theorem 4.3 on page 56 we may construct a representative H corresponding to $m = 1$ using Blichfeldt's list of matrix generators of the finite primitive projective groups of degree two. In particular, we may write down a finite list of generating matrices giving a finite group H. In this way we are able to use *MAGMA* program 1 found in Appendix A in order to obtain the decomposion of a given H− module. An example follows shortly.

2. We will then use Lemma 5.6, Lemma 5.7 and its corollariy to obtain the irreducible decomposition of W as a G− module.

EXAMPLE: The decomposition of $Sym^2\mathbf{V}$ under the action of the imprimitive groups having the property that for $i = 1, 2$, $\pi_i(\mathbf{H}) = \mathbf{G}_n$ for some n

Let $G = H \cup \gamma H$ be an imprimitive nonmonomial subgroup of $SL(4, C)$. Assume that H is contained in $T \times T$ where $\pi(T) = G_n$ for some n. Let V be the usual representation of G with basis $\mathcal{B}_V = \{v_1, v_2, v_3, v_4\}$. We can write down the following bases for $W = Sym^2V$.

$$\mathcal{B}_W = \{v_1^2, v_2^2, v_3^2, v_4^2, v_1v_2, v_1v_3, v_1v_4, v_2v_3, v_2v_4, v_3v_4\}$$

We will determine the dimensions of the components of the irreducible decomposition of W as a G− module for each of the groups G listed in Theorem 4.3.

Case 1. Suppose G is equal to $H_n^1 \cup \gamma H_n^1$.

We begin by decomposing W as a representation of H. Note that in this case H is isomorphic to G_n and hence we know the characters of the irreducible representations of H. Let $h \in \mathrm{diag}(H)$. We may write $h = \begin{bmatrix} a & 0 & 0 & 0 \\ 0 & b & 0 & 0 \\ 0 & 0 & \xi a & 0 \\ 0 & 0 & 0 & \xi b \end{bmatrix}$ for some $a, b \in C^*$ and $\xi = \frac{1}{ab} \in U_n$. Note that h is in diagonal form in the above basis for W. Hence we can easily determine the value of the character at h. We then rewrite this character as a sum of characters of the irreducible H− modules. This gives us the irreducible decomposition of Sym^2V as an H− module.

$$\begin{aligned}\chi_W(h) &= a^2 + b^2 + \xi^2a^2 + \xi^2b^2 + ab + \xi a^2 + \xi ab + \xi ab + \xi b^2 + \xi^2 ab \\ &= (a^2 + ab + b^2) + \xi(a^2 + ab + b^2) + \xi^2(a^2 + ab + b^2) + \xi ab \\ &= \chi_3(h) + \xi\chi_3(h) + \xi^2\chi_3(h) + \chi_1(h)\end{aligned}$$

Therefore as a representation of $H = H_n^1$ for arbitrary n, $Sym^2V = \bigoplus_{i=1}^4 W_i$ where $\dim(W_i) = 3$ for $i = 1, 2, 3$ and $\dim(W_4) = 1$. Lemma 5.7 implies that W_4 is G− invariant. One can see that the space W_2 corresponding to the character $\xi(a^2+ab+b^2)$ is G− invariant as well by noticing that $\{v_1v_3, v_1v_4 + v_2v_3, v_2v_4\}$ is a basis for W_2. We see by observing $\chi_W(h)$ that the spaces W_1 and W_3 are given by $span\{v_1^2, v_1v_2, v_2^2\}$ and $span\{v_3^2, v_3v_4, v_4^2\}$ respectively. These two spaces are interchanged by γ. Let $U = W_1 \oplus W_3$. Then U may or may not be reducible depending on the integer n where

$T = G_n$. We apply Corollary 5.4 of Lemma 5.7 to determine the reducibility of this space in each case.

Case 1.(a) Suppose $n = 1$ or 2. Note that $\chi_{W_1}(h) = \chi_{W_3}(h)$ for every $h \in \mathrm{diag}(H)$ and hence for every $h \in H$. Therefore, $\gamma : W_1 \to W_3$ is an $H-$ module isomorphism. Corollary 5.4 of Lemma 5.7 then implies that U decomposes into the sum of two irrecucible $G-$ invariant subspaces each of dimension equal to $\dim(W_1) = 3$. Hence when $n = 1$ or 2 the decomposition of $W = Sym^2V$ is given by

$$\mathbf{Sym^2V \Longrightarrow 3^3, 1}$$

Case 1.(b) Suppose $n \geq 3$. In this case there exists an $h \in H$ such that $\chi_{W_1}(h) \neq \chi_{W_2}(h)$. Therefore Corollary 5.4 of Lemma 5.7 implies that $U = W_1 \oplus W_3$ is irreducible as a representation of G. Hence we have the following decomposition of Sym^2V.

$$\mathbf{Sym^2V \Longrightarrow 6, 3, 1}$$

Case 2. Suppose G is equal to $H_n^2 \cup \gamma H_n^2$.

Note that Lemma 5.5 implies that the dimensions of the irreducible components of the decomposition of Sym^2V as an H_n^2- module correspond to those in its decomposition as an H_n^1- module. Therefore when $n = 1$ or 2

$$\mathbf{Sym^2V \Longrightarrow 3^3, 1}$$

and when $n \geq 3$

$$\mathbf{Sym^2V \Longrightarrow 6, 3, 1}$$

Case 3. Suppose G is equal to $H_n^3 \cup \gamma H_n^3$.

Note that H contains $SL(2, C) \times SL(2, C)$. Corollary 5.3 then implies that the irreducible representations of H are of the form $V(k) \otimes V(l)$. The value of a character on a diagonal element $h = \begin{bmatrix} a & 0 & 0 & 0 \\ 0 & b & 0 & 0 \\ 0 & 0 & c & 0 \\ 0 & 0 & 0 & d \end{bmatrix}$ is equal to the product of $\chi_k(h_1)$ and $\chi_l(h_2)$ where $h_1 = \begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix}$, $h_2 = \begin{bmatrix} c & 0 \\ 0 & d \end{bmatrix}$ and χ_i is the character of G_n on $V(i)$. This gives

$$
\begin{aligned}
\chi_{\mathrm{W}}(h) &= a^2+b^2+c^2+d^2+ab+ac+ad+bc+bd+cd \\
&= (a^2+ab+b^2)+(c^2+cd+d^2)+(a+b)(c+d) \\
&= \chi_3(h)+\chi_3(h)+\chi_2(h)\chi_2(h)
\end{aligned}
$$

Therefore as an H_n^3- module for arbitrary n, $Sym^2\mathrm{V} = \bigoplus_{i=1}^{3} \mathrm{W}_i$ where $\dim(\mathrm{W}_i) = 3$ for $i = 1, 2$ and $\dim(\mathrm{W}_3) = 4$. Lemma 5.7 implies that W_3 is G$-$ invariant. We see by observing $\chi_{\mathrm{W}}(h)$ that the spaces W_1 and W_2 are given by $span\{v_1^2, v_1v_2, v_2^2\}$ and $span\{v_3^2, v_3v_4, v_4^2\}$ respectively and are interchanged by γ. Let $\mathrm{U} = \mathrm{W}_1 \oplus \mathrm{W}_2$. It is clear that for any n the spaces W_1 and W_2 are not isomorphic as H_n^3- modules. Therefore by Corollary 5.4 of Lemma 5.7, $\mathrm{U} = \mathrm{W}_1 \oplus \mathrm{W}_2$ is irreducible as a representation of G. Hence we have the following decomposition of $Sym^2\mathrm{V}$.

$$\mathbf{Sym^2V} \Longrightarrow \mathbf{6}, \mathbf{4}$$

Table 5.2 on page 107 contains the decompositions of several modules under the action of each of the imprimitive nonmonomial groups G where T is projectively infinite.

Representations of $\mathbf{G} = \mathbf{H} \cup \gamma\mathbf{H}$ where $\mathbf{H} \subseteq \mathbf{T} \times \mathbf{T}$, T projectively finite

Remark: Recall by the above remarks that given a fixed projective group P, a fixed subgroup F of SL(2, C), and two nonnegative integers n and m then the dimensions of the irreducible components of the decomposition of a module W under the action of H_n and H_m are the same. Consider then using the decomposition of a given H module in combination with Lemmas 5.6, Lemma 5.7 and Corollary 5.4 in order to obtain the decomposition of this module under the action of $\mathrm{G} = \mathrm{H} \cup \gamma\mathrm{H}$. We see in the above example that in the case when T is projectively infinite we were able to easily determine exactly the action of γ on a certain H$-$ invariant subspace of W including the character of this action. In this case we are obtaining the irreducible decompositions of the H$-$ modules and G$-$ modules via the character tables of these groups which are generated using *MAGMA*. Hence although we will be able to observe the dimensions and characters of the irreducible H$-$ invariant subspaces, we will not be able to decide if two subspaces of equal dimension are exchanged by γ. Nonetheless, we may still apply Lemma 5.7 in order to obtain a finite list of possible dimensions for a group G containing such an H. We will show this in the following example.

EXAMPLE: Possible Decompositions of Sym^2V under the action of the imprimitive groups having the property that $\mathbf{H} = \left\{ \begin{bmatrix} \mathbf{A} & 0 \\ 0 & \xi_{\mathrm{A}}\mathbf{B} \end{bmatrix} : \mathbf{A} \in (\mathbf{A}_4)_n, \mathbf{B} \in \mathbf{A}\{\pm \mathbf{I}\} \right\}$ **for some n.**

We begin by obtaining the generating matrices for the group H. The following generating substitutions for a subgroup of SL(2, C) which projects onto a degree two representation of A_4 are given by Blichfeldt in [BL3].

$$g_1 = \begin{bmatrix} \frac{i-1}{2} & \frac{i-1}{2} \\ \frac{i+1}{2} & \frac{-i-1}{2} \end{bmatrix}, g_2 = \begin{bmatrix} i & 0 \\ 0 & i \end{bmatrix}$$

Note that the smallest group H of the form above is obtained when $n = 1$. That is $\pi(\mathrm{H}) = \mathrm{T}$ where T is a subgroup of SL(2, C) which projects onto A_4. This group T is simply the subgroup of SL(2, C) generated by the matrices that correspond to the above substitutions. Then the set of matrices $\left\{ \begin{bmatrix} g_i & 0 \\ 0 & \pm g_i \end{bmatrix} : g_i \text{ a generator for T} \right\}$ generate the group H. One can construct the character of H on Sym^2V and decompose it via program 1 found in Appendix A. In this case we get the following list of dimensions of irreducible components in this decomposition.

$$\mathbf{Sym^2V \Longrightarrow 1, 3^3}$$

Note that Corollary 5.4 of Lemma 5.7 implies that given any irreducible H− invariant subspace U of dimension d then the space $\mathrm{U} \cup \gamma\mathrm{U}$ is either an irreducible G module of dimension d or $2d$, or it is a reducible G− module which decomposes into two irreducible submodules each of dimension d. Hence we may obtain the following list of possible dimensions in the decomposition of Sym^2V under the action of G where G is any group of the above form, regardless of the integer n.

$$\mathbf{Sym^2V \Longrightarrow 1, 3^3 \text{ or } 1, 3, 6}$$

We obtain the information in Table 5.3 found on page 108 by applying the process shown in the above example to a group of each type listed in Theorem 4.3.

5.2 Represenations of the Primitive Subgroups of SL$(4, \mathbf{C})$

If G is a primitive subgroup of SL$(4, \mathrm{C})$ then either G is a finite group or the connected component of the identity G^0 of G is a semisimple group. When G^0 is irreducible the representation theory for such groups is well known and can be found for example in [FH], [CR], or [DO].

5.2.1 Representations of the Infinite Primitive Groups

$\mathbf{G}^0$ irreducible

Let G be an infinite primitive subgroup of SL$(4, \mathrm{C})$. Then G^0 is equal to an irreducible degree four representation of SL$(2, \mathrm{C})$, SO$(4, \mathrm{C})$, SP$(4, \mathrm{C})$, or SL$(4, \mathrm{C})$. The irreducible modules for each of these groups are known (see for ex.[FH]). For example if $\mathrm{V} \cong \mathrm{C}^4$ then both $\bigwedge^m \mathrm{V}$ and $Sym^m\mathrm{V}$ are irreducible SL$(4, \mathrm{C})-$ modules for any m. Using this information one is able to decompose a given G^0- module, W, using the process shown in the following example. Theorem 4.5 states that if G^0 is equal to SL$(2, \mathrm{C})$, SP$(4, \mathrm{C})$, or SL$(4, \mathrm{C})$ then G is simply a scalar extension of G^0. Lemma 5.4 on page 90 then implies that if $\mathrm{W} = \bigoplus \mathrm{W}_i$ is the irreducible decomposition of W as a G^0- then each W_i is also an irreducible G$-$ module. Similarly, if $\mathrm{G}^0 = \mathrm{SO}(4, \mathrm{C}) \cong \mathrm{SL}(2, \mathrm{C}) \times \mathrm{SL}(2, \mathrm{C})$ then $\mathrm{G} \subseteq \mathrm{H} \cup \mathrm{JH}$ where H is a finite scalar extension of G^0 and $\mathrm{J} = \begin{bmatrix} 0 & \mathrm{I} \\ \mathrm{I} & 0 \end{bmatrix}$. Hence we may apply Corollary 5.4 of Lemma 5.7 on page 94 in addition to Lemma 5.4 in order to obtain the irreducible decomposition of W as a G$-$ module given its decomposition as an SO$(4, \mathrm{C})-$ module.

Example: The decomposition of Sym^2V as an SL$(2, \mathbf{C})-$ module

Recall that any irreducible SL$(2, \mathrm{C})$ module is equal to $\mathrm{V}(m) = Sym^{m-1}\mathrm{C}^2$ for some m. Any character of SL$(2, \mathrm{C})$ is completely determined by its value on diag(SL$(2, \mathrm{C})$). Let $g = \begin{bmatrix} a & 0 \\ 0 & a^{-1} \end{bmatrix} \in$ diag(SL$(2, \mathrm{C})$). We can then compute the value of the character of any irreducible SL$(2, \mathrm{C})-$ module at g. Those for $m = 1, \ldots, 5$ are

1. $\chi_{\mathrm{V}(1)}(g) = 1$

2. $\chi_{\mathrm{V}(2)}(g) = a + a^{-1}$

3. $\chi_{\mathrm{V}(3)}(g) = a^2 + 1 + a^{-2}$

4. $\chi_{\mathrm{V}(4)}(g) = a^3 + a + a^{-1} + a^{-3}$

5. $\chi_{\mathrm{V}(5)}(g) = a^4 + a^2 + 1 + a^{-2} + a^{-4}$

Consider now the $\mathrm{SL}(2, \mathrm{C})-$ module $\mathrm{W} = Sym^2\mathrm{V}$. Let $\mathrm{V} = span\{v_1, v_2, v_3, v_4\}$. Then the following set is a basis for W.

$$\{v_1^2, v_2^2, v_3^2, v_4^2, v_1v_2, v_1v_3, v_1v_4, v_2v_3, v_2v_4, v_3v_4\}$$

Recall that G^0 is the irreducible degree four representation of $\mathrm{SL}(2, \mathrm{C})$. Therefore the image of the above diagonal element $g = \begin{bmatrix} a & 0 \\ 0 & a^{-1} \end{bmatrix}$ is $\begin{bmatrix} a^3 & 0 & 0 & 0 \\ 0 & a & 0 & 0 \\ 0 & 0 & a^{-1} & 0 \\ 0 & 0 & 0 & a^{-3} \end{bmatrix}$. The value of the character of W at g is then equal to

$$a^6 + a^4 + 2a^2 + 2 + 2a^{-2} + a^{-4} + a^{-6}$$

which is equal to $\chi_{\mathrm{V}(6)}(g) + \chi_{\mathrm{V}(2)}(g)$. Therefore $\mathrm{W} \cong \mathrm{V}(6) \oplus \mathrm{V}(2)$ as an $\mathrm{SL}(2, \mathrm{C})-$ module. Table 5.4 on page 109 contains the decompositions of $\bigwedge^2 \mathrm{V}, Sym^2\mathrm{V}, Sym^3\mathrm{V}, Sym^4\mathrm{V}$ and $Sym^6\mathrm{V}$ for each of the possibilities for the connected component G^0 of an infinite primitive group G.

G^0 reducible

Recall from section 4.5 that such a group, G, is equal to the image of $\tilde{\mathrm{H}} \times \mathrm{SL}(2, \mathrm{C})$ acting on $\mathrm{C}^2 \otimes \mathrm{C}^2$ where $\tilde{\mathrm{H}}$ is one of the finite primitive subgroups $A_4^{SL_2}, S_4^{SL_2}, A_5^{SL_2}$ of $\mathrm{SL}(2, \mathrm{C})$ or a degree two scalar extension of one of these groups. As with some of the imprimitive groups we will not be able to distinguish between the groups $\tilde{\mathrm{H}}_1 \times \mathrm{SL}(2, \mathrm{C})$ and $\tilde{\mathrm{H}}_2 \times \mathrm{SL}(2, \mathrm{C})$ when $\tilde{\mathrm{H}}_1$ is a scalar extension of $\tilde{\mathrm{H}}_2$. Therefore we will only consider representations of $\tilde{\mathrm{H}} \times \mathrm{SL}(2, \mathrm{C})$ where $\tilde{\mathrm{H}}$ is equal to $A_4^{SL_2}, S_4^{SL_2}$, or $A_5^{SL_2}$. We have already remarked (see the comments following Theorem 4.4) that in this case the image of $\tilde{\mathrm{H}} \times \mathrm{SL}(2,\mathrm{C})$ in $\mathrm{SL}(4,\mathrm{C})$ lies in $\mathrm{SO}(4,\mathrm{C})$, the image of $\mathrm{SL}(2,\mathrm{C}) \times \mathrm{SL}(2,\mathrm{C})$. Therefore, if W is an $\mathrm{SL}(2,\mathrm{C}) \times \mathrm{SL}(2,\mathrm{C})-$ module constructed from $\mathrm{C}^2 \otimes \mathrm{C}^2$ using tensor products, direct sums, submodules, quotients and duals (for example, $\bigwedge^2 \mathrm{V}$ or $Sym^2\mathrm{V}$) then W is automatically an $\tilde{\mathrm{H}} \times \mathrm{SL}(2,\mathrm{C})-$ module. This gives us the following strategy: to decompose such a W, first decompose it as an $\mathrm{SL}(2,\mathrm{C}) \times \mathrm{SL}(2,\mathrm{C})-$ module and then examine the action of $\tilde{\mathrm{H}} \times \mathrm{SL}(2,\mathrm{C})$ on each $\mathrm{SL}(2,\mathrm{C}) \times \mathrm{SL}(2,\mathrm{C})-$ irreducible factor. the following two lemmas will be the main tools used in this strategy. The first is well known but since we could not find a suitable reference, we include the proof.

Lemma 5.8 *Let k be any algebraically clolsed field of characteristic zero and let G and H be reductive groups. Then any irreducible $G \times H-$ module is of the form $V \otimes W$ where V is an irreducible $G-$ module and W is an irreducible $H-$ module. Conversely, any such tensor product is a $G \times H-$ module.*

Proof: Note that if V is a G$-$ module and W is an H$-$ module then the action of $\mathrm{G} \times \mathrm{H}$ on $\mathrm{V} \otimes \mathrm{W}$ is defined by $(g \times h)(v \otimes w) = gv \otimes hw$. Let U be an irreducible $\mathrm{G} \times \mathrm{H}-$ module. We can consider U as a G$-$ module and as such write $\mathrm{U} = \mathrm{U}_1^{(n_1)} \oplus \ldots \oplus \mathrm{U}_r^{(n_r)}$ where the U_i are irreducible G$-$ modules, pairwise nonisomorphic, and each U_i is repeated n_i times in the direct sum. Since H commutes with G, H leaves each $\mathrm{U}_i^{(n_i)}$ invariant (to see this, let $h \in \mathrm{H}$ and note that for any G$-$ invariant projection π, the map $\pi \circ h$ is a G$-$ morphism). Since U is an irreducible $\mathrm{G} \times \mathrm{H}-$ module, we must have that that $\mathrm{U} = F^{(n)}$ for some irreducible $\tilde{\mathrm{H}}-$ module F. Proposition 2 of [LA] Chapter XVII Section 1 furthermore states that each $h \in \mathrm{H}$ can be associated with a matrix

$$M_h = \begin{bmatrix} b_{11} & \ldots & b_{1n} \\ \vdots & & \vdots \\ b_{n1} & \ldots & b_{nn} \end{bmatrix}$$

where each $b_{ij} \in End_{\mathrm{G}}(F) = k$ (the last equallity is Schur's Lemma). This allows us to associate an H$-$ module M with this representation.

Write $U = F_1 \oplus \ldots \oplus F_n$ where the F_j are isomorphic irreducible G$-$ modules and select bases $\{v_{ij} : i = 1, \ldots, n,\ j = 1, \ldots, \dim(\mathrm{U})\}$ for each summand U such that the matrices of each $g \in$ G with respect to these bases are identical. Let $g \in$ G and let (a_{ij}) be this matrix. Then for each j we have $h(v_{ij}) = \sum_{t=1}^{n} a_{it} v_{tj}$. Therefore

$$(g \times h)(v_{ij}) = \sum_t \sum_s a_{it} b_{sj} v_{st}.$$

This is the same as the representation of G $\times$ H on $F \otimes M$ Finally, it is clear that M must be irreducible since otherwise U would be irreducible.

We now show the converse. If V is an irreducible G$-$ module and W is an irreducible H$-$ module, then we can write:

$$\begin{aligned} \mathrm{V} \otimes \mathrm{W} &= \mathrm{V}^{(m)} \quad \text{as a G$-$ module} \\ &= \mathrm{W}^{(n)} \quad \text{as an H$-$ module} \end{aligned}$$

where m is the dimension of W and n is the dimension of V. If U $\subseteq$ V $\otimes$ W is a G $\times$ H$-$ module, then the first part of this proof shows that U $=$ $\mathrm{V}_1 \otimes \mathrm{W}_1$ where V_1 is an irreducible G$-$ module and W_1 is an irreducible H$-$ module. We can ifentify V_1 with an irreducible submodule of V $\otimes$ W and so it must be isomorphic to V. Similarly, W_1 must be isomorphic to W. Counting dimensions we have that U = V $\otimes$ W. ∎

Lemma 5.9 *Let $W = V(i) \otimes V(j)$ be an irreducible $SL(2, C) \times SL(2, C)-$ module and let $\tilde{H}$ be a primitive subgroup of $SL(2, C)$. Then W is reducible as an $\tilde{H} \times SL(2, C)-$ module if and only if $V(i)$ is reducible as an $\tilde{H}-$ module. Furthermore, if $V(i) = \bigoplus_{k=1}^{n} U_k$ where each U_k is $\tilde{H}-$ invariant then W can be written as the sum, $\bigoplus_{k=1}^{n}(U_k \otimes V(j))$, of $\tilde{H} \times SL(2, C)-$ modules.*

Proof: Let W be as stated. Note that the action of $\mathrm{SL}(2, \mathrm{C}) \times \mathrm{SL}(2, \mathrm{C})$ on W is defined as follows. Let $(g_1, g_2) \in \mathrm{SL}(2, \mathrm{C}) \times \mathrm{SL}(2, \mathrm{C})$ and let $v_1 \otimes v_2 \in$ W. Then $(g_1, g_2) \cdot (v_1 \otimes v_2) = (g_1 v_1 \otimes g_2 v_2)$. Since $\tilde{\mathrm{H}} \times \mathrm{SL}(2, \mathrm{C})$ is a subgroup of $\mathrm{SL}(2, \mathrm{C}) \times \mathrm{SL}(2, \mathrm{C})$ W is an $\tilde{\mathrm{H}} \times \mathrm{SL}(2, \mathrm{C})-$ module via restriction. By Lemma 5.8 any irreducible $\tilde{\mathrm{H}} \times \mathrm{SL}(2, \mathrm{C})-$ module is of the form U $\otimes$ V where U is an irreducible $\tilde{\mathrm{H}}-$ module and V is an irreducible $\mathrm{SL}(2, \mathrm{C})-$ module. This implies that W is reducible as an $\tilde{\mathrm{H}} \times \mathrm{SL}(2, \mathrm{C})-$ module if and only if $\tilde{\mathrm{H}}$ acts reducibly on $\mathrm{V}(i)$. The result then follows from the distributive property of tensor products over direct sums. ∎

Example: The decomposition of $Sym^4\mathbf{V}$ as an $A_4^{SL_2} \times \mathbf{SL(2, C)}-$ module

Let W = Sym^4V. Recall that as in the case where G = $\mathrm{SO}(4, \mathrm{C})$ V $\cong \mathrm{C}^2 \otimes \mathrm{C}^2$. As an $\mathrm{SO}(4, \mathrm{C})-$ module Sym^4V decomposes into the following sum of irreducible submodules.

$$(\mathrm{V}(0) \otimes \mathrm{V}(0)) \oplus (\mathrm{V}(2) \otimes \mathrm{V}(2)) \oplus (\mathrm{V}(4) \otimes \mathrm{V}(4))$$

Table 5.6 provides the necessary information concerning the decompositions of $\mathrm{V}(2), \ldots, \mathrm{V}(6)$ under the action of each of the finite primitive subgroups of $\mathrm{SL}(2, \mathrm{C})$. Since V(2) is irreducible as a $A_4^{SL_2}-$ module and since V(4) is the sum of three $A_4^{SL_2}-$ invariant subspaces of dimensions 1,1 and 3, we have the following decomposition of W as an $\mathrm{H} \times \mathrm{SL}(2, \mathrm{C})-$ module.

$$\mathbf{Sym^4V} \Longrightarrow \mathbf{1, 5^2, 9, 15}$$

5.2.2 Representations of the Finite Primitive Groups

Let G be a finite primitive group. One decomposes a given G$-$ module via the character table of G. We again used *MAGMA* program 1 found in Appendix A to obtain the information found in Table 5.5 which appears in the next couple of pages.

Table 5.1: Decompositions of $\bigwedge^2$ V, Sym^2V, Sym^3V, and Sym^4V under the action of the infinite monomial subgroups of SL(4, C).

- V is the usual representation of the monomial group, G.
- For convenience of notation in the table the group $< (1234) >$ will be denoted by C_4 and the group $< (1234), (12)(34) >$ will be denoted by P_8.

	$\bigwedge^2$ V	Sym^2V	Sym^3V	Sym^4V
$G_{C_4}^{1,n},\ n = 1, 2$	$2^2, 1^2$	$2^4, 1^2$	2^{10}	$2^{14}, 1^7$
$G_{C_4}^{2,n},\ n = 1, 2$	$4, 1^2$	$4^2, 1^2$	4^5	$4^8, 1^3$
$G_{C_4}^{2,n},\ n > 2, G_{C_4}^3$	$4, 2$	$4^2, 2$	4^5	$4^8, 2, 1$
$G_K^{1,n},\ n = 1, 2$	$2, 1^4$	$2^3, 1^4$	2^{10}	$2^{13}, 1^9$
$G_K^{2,n},\ n = 1, 2$	$2^2, 1^2$	$4, 2^2, 1^2$	4^5	$4^5, 2^6, 1^3$
$G_K^{2,n},\ n > 2, G_K^3$	2^3	$4, 2^3$	4^5	$4^7, 2^3, 1$
$G_{P_8}^{1,n},\ n = 1, 2$	$2^2, 1^2$	$2^4, 1^2$	$8, 4^3$	$2^{15}, 1^5$
$G_{P_8}^{2,n},\ n = 1, 2$	$4, 2$	$4^2, 2$	$8, 4^3$	$8, 4^6, 1^3$
$G_{P_8}^{2,n},\ n > 2, G_{P_8}^3$	$4, 2$	$4^2, 2$	$8, 4^3$	$8, 4^4, 2, 1$
$G_{A_4}^3$	6	$6, 4$	$12, 4^2$	$12^2, 6, 4, 1$
$G_{S_4}^3$	6	$6, 4$	$12, 4^2$	$12^2, 6, 4, 1$

Table 5.2: Representations of imprimitive nonmonomial G where T is projectively infinite.

- A star appears in the table below wherever the decomposition is exactly as that immediately above.

G		$\bigwedge^2$ V	Sym^2V	Sym^3V	Sym^4V	Sym^6V
G_n^1	$n=1$	$3, 1^3$	$3^3, 1$	$4^4, 2^2$	$5^5, 3^3, 1$	$7^7, 5^5, 3^3, 1$
	$n=2$	*	*	$8^2, 4$	*	*
	$n=3$	3,2,1	6,3,1	$8, 4^3$	$10^2, 6, 5, 3, 1$	$14^2, 10^2, 7^3, 6, 5, 3, 1$
	$n=4$	*	*	$8^2, 4$	$5^5, 3^3, 1$	$14^2, 10, 7^3, 6, 5^3, 3, 1$
	$n=5,6$	*	*	*	$10^2, 6, 5, 3, 1$	$14^2, 10^2, 7^3, 6, 5, 3, 1$
	$n\geq 7$	*	*	*	*	$14^3, 10^2, 7, 6, 5, 3, 1$
G_n^2	$n=1$	$3, 1^3$	$3^3, 1$	$4^4, 2^2$	$5^5, 3^3, 1$	$7^7, 5^5, 3^3, 1$
	$n=2$	*	*	$8^2, 4$	*	*
	$n=3$	3,2,1	6,3,1	$8, 4^3$	$10^2, 6, 5, 3, 1$	$14^2, 10^2, 7^3, 6, 5, 3, 1$
	$n=4$	*	*	$8^2, 4$	$5^5, 3^3, 1$	$14^2, 10, 7^3, 6, 5^3, 3, 1$
	$n=5,6$	*	*	*	$10^2, 6, 5, 3, 1$	$14^2, 10^2, 7^3, 6, 5, 3, 1$
	$n\geq 7$	*	*	*	*	$14^3, 10^2, 7, 6, 5, 3, 1$
G_n^3	any n	$4, 1^2$	6,4	12,8	16,10,9	30,24,16,14

Table 5.3: Representations of imprimitive nonmonomial G where T is projectively finite

- A # indicates a list of all possible lists of dimensions that can be constructed from the given list of dimensions via combining two irreducible spaces of dimension d into one irreducible space of dimenison $2d$.

	$\bigwedge^2$ V	Sym^2V	Sym^3V	Sym^4V	Sym^6V
$G_i(A_4, I)$	$1^3, 3$ $1, 2, 3$	$1, 3^3$ $1, 3, 6$	4^5 $4^3, 8$ $4, 8^2$	$1^{11}, 3^8$ # #	$1^{18}, 3^{22}$ # #
$G_i(A_4, K)$	$1^2, 4$ $2, 4$	$4, 6$	$4^2, 12$ $8, 12$	$1^4, 3, 4^4, 6^2$ $1^2, 2, 3^3, 4^2, 6, 8$ #	$1^2, 3^2, 4^{10}, 6^6$ $1^2, 3^6, 4^4, 6^4, 8^3$ #
$G_i(A_4, A_4)$	$1^2, 4$ $2, 4$	$4, 6$	$4^2, 12$ $8, 12$	$1^2, 2, 4^2, 6, 8, 9$ #	$1^2, 4^4, 6^4, 8^3, 9^2$
$G_1(S_4, I)$	$1, 2, 3$	$1, 3, 6$	$4, 8^2$ 4,16	$1, 2^5, 3^4, 6^2$ #	$1^4, 2^7, 3^{10}, 6^6$ #
$G_1(S_4, K)$	$2, 4$	$4, 6$	$8, 12$	$2^2, 3, 6^2, 8^2$ #	$2, 4^4, 6^7, 8^3$ #
$G_1(S_4, A_4)$	$2, 4$	$4, 6$	$8, 12$	$4, 6, 9, 16$	$2, 4^2, 6^2, 8^2, 12, 16, 18$ #
$G_1(S_4, S_4)$	$1, 2, 3$	$4, 6$	$8, 12$	$4, 6, 9, 16$	$2, 6^2, 8, 12, 16^2, 18$ #
$G_1(A_5, I)$	$1^3, 3$ $1, 2, 3$	$1, 3^3$ $1, 3, 6$	$4, 8^2$ $4, 16$	$1, 3^3, 5^5$ #	$1, 3^{10}, 4^7, 5^5$ #
$G_1(A_5, A_5)$	$1^2, 4$ $2, 4$	4,6	$8, 12$	$9, 10, 16$	$6, 8, 16, 24, 30$

Table 5.4: Decompositions of $\bigwedge^2$ V, Sym^2V, Sym^3V, Sym^4V, and Sym^6V under the action of the infinite primitive subgroups of SL(4, C).

	$\bigwedge^2$ V	Sym^2V	Sym^3V	Sym^4V	Sym^6V
$G^0 = SL(2, C)$	1,5	3,7	4,6,10	1,5,7,9,13	$3, 7^2, 9, 11, 13, 15, 19$
$G^0 = SL(4, C)$	6	10	20	35	84
$G^0 = SP(4, C)$	1,5	10	20	35	84
$G^0 = SO(4, C)$	3^2	1,9	4,16	1,9,25	1,9,25,49
$A_4^{SL_2} \otimes SL(2, C)$	3^2	1,9	$4, 8^2$	$1, 5^2, 9, 15$	$1, 5^2, 7, 9, 15, 21^2$
$S_4^{SL_2} \otimes SL(2, C)$	3^2	1,9	4,16	1,9,10,15	$1, 7, 9, 10, 15, 21^2$
$A_5^{SL_2} \otimes SL(2, C)$	3^2	1,9	4,16	1,9,25	1,9,21,25,28

Table 5.5: Representations of the finite primitive subgroups of $\mathrm{SL}(4, \mathrm{C})$

	$\bigwedge^2$ V	Sym^2V	Sym^3V	Sym^4V	Sym^6V
FP1	3^2	$1, 9$	4^5	$1^5, 3^4, 9^2$	$1^6, 3^8, 9^6$
FP2	3^2	$1, 9$	$4, 8^2$	$1, 2^2, 3^2, 6, 9^2$	$1^2, 2^2, 3^6, 6, 9^6$
FP3	3^2	1,9	$4, 8^2$	$1, 5^2, 9, 15$	$1, 3, 4, 5^2, 9^3, 12^2, 15$
FP4	3^2	$1, 9$	$4, 16$	$1, 9, 10, 15$	$1, 3, 4, 9^3, 10, 12^2, 15$
FP5	3^2	$1, 9$	$4, 16$	$1, 4, 6^2, 9^2$	$1^2, 3^4, 4, 6^2, 9^6$
FP6	3^2	$1, 9$	$4, 16$	$1, 9, 25$	$1, 9^2, 12^2, 16, 25$
FP7	3^2	$1, 9$	$4, 8^2$	$1, 2^2, 6^2, 9^2$	$1^2, 2^2, 3^4, 6^2, 9^6$
FP8	6	$1, 9$	$4^3, 8$	$1^3, 2, 6^2, 9^2$	$1^4, 2, 6^4, 9^6$
FP9	6	$1, 9$	$4, 8^2$	$1, 2^2, 9^2, 12$	$1^2, 2^2, 6^2, 9^6, 12$
FP10	6	$1, 9$	4,16	$1, 4, 9^2, 12$	$1^2, 4, 6^2, 9^6, 12$
FP11	6	$1, 9$	4,16	$1, 9, 25$	$1, 9^2, 16, 24, 25$
FP12	6	$1, 9$	$4, 16$	$1, 4, 9^2, 12$	$1^2, 4, 6^2, 9^4, 12, 18$
FP13	$1, 5$	5^2	4^5	$1^5, 5^6$	$1^4, 5^{16}$
FP14	$1, 5$	5^2	$4, 8^2$	$1, 2^2, 5^6$	$2^2, 5^{16}$
FP15	$1, 5$	10	$4, 16$	$1, 4, 5^2, 10^2$	$4, 5^4, 10^6$
FP16	$1, 5$	10	20	$5^3, 20$	$4, 5, 10^2, 15, 20^2$
FP17	6	10	4,16	$1, 4, 15^2$	$10^4, 12^2, 20$
FP18	1,5	10	20	5,10,20,	$4, 5, 10^2, 15, 20^2$
FP19	6	10	4,16	$1, 4, 15^2$	$10^4, 20, 24$
FP20	6	10	20	5,30	$10^2, 24, 40$
FP21	6	10	20	5,30	$10^2, 24, 40$
FP22	3^2	$1, 4, 5$	$1, 3^2, 4^2, 5$	$1^2, 3^2, 4^3, 5^3$	$1^3, 3^6, 4^7, 5^7$
FP23	$1, 5$	$3^2, 4$	$4^2, 6^2$	$1^2, 3^2, 4^3, 5^3$	$1, 3^{10}, 4^7, 5^5$
FP24	1,5	10	10^2	$5^2, 8^2, 9$	$5^2, 8^2, 9^2, 10^4$
FP25	6	10	20	14,21	$10^2, 14, 15, 35$
FP26	6	3,7	$4^2, 6^2$	$1, 6^2, 7^2, 8$	$1, 3^4, 6^2, 7^5, 8^3$
FP27	6	10	20	5,30	15,24,45
FP28	6	1,4,5	$1, 4^2, 5, 6$	$1^2, 4^3, 5^3, 6$	$1^3, 4^7, 5^7, 6^3$
FP29	$1^2, 4$	4,6	$4^2, 6^2$	$4, 5^5, 6$	$1, 4^7, 5^5, 6^5$
	1,5	4,6	$4^2, 6^2$	$1^2, 4^3, 5^3, 6$	$1, 4^7, 5^5, 6^5$
FP30	1,5	10	20	$5^2, 9, 16$	$5^2, 9^2, 10^4, 16$

Table 5.6: Representations of the finite primitive subgroups of $\mathrm{SL}(2, \mathrm{C})$

	$Sym^2\mathrm{V}$	$Sym^3\mathrm{V}$	$Sym^4\mathrm{V}$	$Sym^5\mathrm{V}$	$Sym^6\mathrm{V}$
$A_4^{SL_2}$	3	4	5	6	$3, 4$
$S_4^{SL_2}$	3	4	$2, 3$	$2, 4$	$1, 3^2$
$A_5^{SL_2}$	3	2^2	$1^2, 3$	2^3	$1, 3^2$

Chapter 6

Procedure for Determining the Galois Group

In this chapter we state several results concerning the relationship between a linear algebraic group G and the reducibility properties of several G$-$ modules which are constructions of the usual module $\mathrm{V} \cong \mathrm{C}^4$. In particular we will consider the action of such groups on the modules $\bigwedge^2 \mathrm{V}$ and $Sym^m\mathrm{V}$ for small m. These results can be restated to yield relationships between the Galois group of a given fourth order linear differential equation $L(y) = 0$ and factorization properties of equations which are associated with $L(y) = 0$. The equations which are associated with the spaces above are denoted by $L_{\bigwedge^2}(y)$ and $L_{Sym^m}(y)$ respectively and are defined as follows:

Definitions: Let $L(y) = 0$ be a fourth order linear differential with coefficients in the differential field k with algebraicaly closed field of constants, C. Let $\mathrm{V} \cong \mathrm{C}^4$ be the solution space of $L(y) = 0$. Then given a positive integer m one can effectively construct the following linear differential equations. (see for ex. [SU1] and [SI3])

- $L_{\bigwedge^m \mathrm{V}}(y) = 0$ whose solution space is $Gal(L)-$ isomorphic to $\bigwedge^m \mathrm{V}$ and
- $L_{Sym^m\mathrm{V}}(y) = 0$ whose solution space is $Gal(L)-$ isomorphic to $Sym^m\mathrm{V}$

Now let k be a differential field of characteristic zero with algebraically closed field of constants. Then given an order four linear differential equation definied over k

Lemma 3.3 implies that if $L(y)$ is reducible then $Gal(L)$ can be described in terms of algebraic groups of degree less than four. In such a case one should refer to [SU1], [SU2] and [SU3] by M.F. Singer and F. Ulmer. Hence in each of the following results we will assume that the given group or equation is irreducible.

In their totality, the following collection of theorems serve as a decision procedure for determining many properties of the Galois group of a given irreducible linear differential equation of order four. In many cases this procedure allows one to determine the Galois group.

In particular we are able to distinguish the monomial groups from the rest of the groups. Once we have done this we are then able to distinguish the primitive groups from the imprimitive nonmonomial groups. This is done in Theorem 6.1. Within the primitive groups we can distinguish between the finite primitive groups and the infinite primitive groups. Furthermore given that G is a finite primitive group, we can determine which group G is up to projective equivalence. Similarly, given an infinite primitive group G we can determine G^0.

Within the collection of imprimitive nonmonomial groups, $G = H \cup \gamma H,\ H \subseteq T \times T$, we can distiguish between the case when T is projectively infinite and when T is projectively finite. If we assume that T is projectivey infinite, then we are able to differentiate between the cases when $H = \left\{ \begin{bmatrix} A & 0 \\ 0 & B \end{bmatrix} : A, B \in G_n, \det AB = 1 \right\}$ and when H is either $\left\{ \begin{bmatrix} B & 0 \\ 0 & d_B B \end{bmatrix} : B \in G_n, d_B = \frac{1}{\det B} \right\}$ or $\left\{ \begin{bmatrix} B & 0 \\ 0 & \pm d_B B \end{bmatrix} : B \in G_n, d_B = \frac{1}{\det B} \right\}$. When T is projectively finite, that is when G is equal to one of the groups $G_i(P, N)$, we can distinguish between $G_i(P_1, N_1)$ and $G_i(P_2, N_2)$ if either $P_1 \neq P_2$ or $N_1 \neq N_2$ except in the following two cases. We are not able to distinguish beween $G_i(A_4, K)$ and $G_1(S_4, K)$ or between $G_i(A_4, I)$ and $G_1(S_4, I)$.

Within the collection of imprimitive monomial groups, we can distinguish the groups up to their permutation action. Also, for two infinite groups with the same permutation action one can in many cases distinguish between two groups containing maximal diagonal subgroups of different dimensions. We believe one cannot distinguish between finite and infinite monomial groups using representation theory. The reason being that given any representation of the group $\mathrm{GL}(1, \mathrm{C})$ there exists a large enough integer n such that the sets of values of the characters of that representation are equal for $\mathrm{GL}(1, \mathrm{C})$ and for the group of units U_n. Nonetheless, in the monomial case one can distinguish the infinite groups from the finite groups using the procedure for deciding if the logarithmic derivative of a solution is algebraic and possibly using the generalizations of this procedure as well.

In addition to the results given in this thesis, we believe that one may in some cases be able to distinguish even futher between the groups using the orders of one dimensional characters. In particular one should be able to use such characters to distinguish between two groups which are projectively equivalent.

Theorem 6.1 *Let G be an irreducible subgroup of $SL(4, C)$. Let $V \cong C^4$. Either*

1. *one of the following conditions is satisfied in which case G is an imprimitive monomial group.*

   - *$Sym^2 V$ has a 2 dimensional $G-$ invariant subspace.*
   - *$\bigwedge^2 V$ and $Sym^2 V$ both have an irreducible $G-$ invariant subspace of dimension 6.*

2. *or condition 1 does not hold in which case G is either an imprimitive nonmonomial group or a primitive group. Furthermore, G is a primitive group if and only if one of the following is true.*

   - *$\bigwedge^2 V$ has no $G-$ invariant subspace of dimension 2.*
   - *$\bigwedge^2 V$ has two one dimensional $G-$ invariant subspaces and $Sym^3 V$ has a $G-$ invariant subspace of dimension 6.*

Proof: Since G is an irreducible group it acts either primitively, imprimitively and monomially, or imprimitively and nonmonomially on V. Tables 5.1, 5.2, 5.3, 5.4, and 5.5 give the decompositions of $\bigwedge^2$ V and Sym^mV for small m under the action of all of the irreducible subgroups of $\mathrm{SL}(4, \mathrm{C})$. ∎

In the following, the term factor signifies right factor.

Corollary 6.1 *Let $L(y) = 0$ be an irreducible order four linear homogeneous differential equation with coefficients in k and unimodular differential Galois group. Either*

1. *one of the following conditions is satisfied in which case $Gal(L)$ is an imprimitive monomial group.*
   - *$L_{Sym^2}(y)$ has an order 2 factor.*
   - *$L_{\bigwedge^2}(y)$ and $L_{Sym^2}(y)$ both have an irreducible factor of order 6.*
2. *or, condition 1 does not hold in which case $Gal(L)$ is either an imprimitive non-monomial group or a primitive group. Furthermore, $Gal(L)$ is a primitive group if and only if one of the following is true.*
   - *$L_{\bigwedge^2}(y)$ does not have an order 2 factor.*
   - *$L_{\bigwedge^2}(y)$ has two order 1 factors and $L_{Sym^3}(y)$ has an order 6 factor.*

Proof: This result clearly follows from theorem 6.1 given the definitions of $L_{\bigwedge^2}(y)$ and $L_{Sym^m}(y)$. ∎

Theorem 6.2 *Let G be a primitive subgroup of $SL(4, C)$. Let $V \cong C^4$. Then G is a finite group if and only if one of the following conditions holds.*

1. *$Sym^3 V$ has a $G-$ invariant subspace of dimension 1.*
2. *$Sym^3 V$ has a $G-$ invariant subspace of dimension 8 and $Sym^6 V$ has a $G-$ invariant subspace of dimension 2,3, or 4.*
3. *$Sym^3 V$ is $G-$ irreducible and $Sym^4 V$ is $G-$ reducible.*
4. *$Sym^3 V$ is $G-$ reducible and $Sym^2 V$ is $G-$ irreducible.*
5. *None of the above hold and there exists a $k \neq 5, 7$, or 21 such that $Sym^6 V$ has more than one $G-$ invariant subspace of dimension k.*

Proof: See Tables 5.5 and 5.4. ∎

Corollary 6.2 *Let $L(y) = 0$ be an irreducible order four linear homogeneous differential equation with coefficients in k and unimodular differential Galois group $Gal(L)$. Assume $Gal(L)$ is a primitive group. Then $Gal(L)$ is a finite group if and only if one of the following conditions holds.*

1. *$L_{Sym^3}(y)$ has an order 1 factor.*
2. *$L_{Sym^3}(y)$ has an order 8 factor and $L_{Sym^6}(y)$ has a factor of order 2,3, or 4.*
3. *$L_{Sym^3}(y)$ is irreducible and $L_{Sym^4}(y)$ is reducible.*
4. *$L_{Sym^3}(y)$ is reducible and $L_{Sym^2}(y)$ is irreducible.*
5. *None of the above hold and there exist a $k \neq 5, 7,$ or 21 such that $L_{Sym^6}(y)$ has more than one factor of order k.*

Proof: Same as Corollary 6.1 of Theorem 6.1 ∎

Theorem 6.3 *Let G be an infinite primitive subgroup of $SL(4, C)$. Let $V \cong C^4$. Then exactly one of the following holds.*

1. *$\bigwedge^2 V$ is $G-$ irreducible, in which case $G = SL(4, C)$.*
2. *$\bigwedge^2 V$ has a one dimensional $G-$ invariant subspace and $Sym^2 V$ is*
   - *$G-$ reducible, in which case $G^0 = SL(2, C)$ or*
   - *$G-$ irreducible, in which case $G^0 = SP(4, C)$.*
3. *None of the above hold (equivalently $Sym^2 V$ has a $G-$ invariant subspace of dimension 1) and one of the following holds.*
   - *$Sym^3 V$ has a $G-$ invariant subspace of dimension 8, in which case $G = A_4^{SL_2} \otimes SL(2, C)$.*

- *Above does not hold and $Sym^4 V$ has a $G-$ invariant subspace of dimension 10, in which case $G = S_4^{SL_2} \otimes SL(2, C)$.*
- *Neither of the above hold and $Sym^4 V$ has a $G-$ invariant subspace of dimension 21, in which case $G = A_5^{SL_2} \otimes SL(2, C)$.*
- *$G^0 = SO(4, C)$.*

Proof: Table 5.4 gives the result. ∎

Corollary 6.3 *Let $L(y) = 0$ be an irreducible order four linear homogeneous differential equation with coefficients in k and unimodular differential Galois group $Gal(L)$. Assume $Gal(L)$ is an infinite primitive group. Then exactly one of the following holds.*

1. *$L_{\bigwedge^2}(y)$ is irreducible, in which case $G = SL(4, C)$.*
2. *$L_{\bigwedge^2}(y)$ has an order 1 factor and $L_{Sym^2}(y)$ is*
   - *reducible, in which case $Gal(L)^0 = SL(2, C)$ or*
   - *irreducible, in which case $Gal(L)^0 = SP(4, C)$.*
3. *None of the above hold (equivalently, $L_{Sym^2}(y)$ has an order 1 factor) and one of the following holds:*
   - *$L_{Sym^3}(y)$ has an order 8 factor, in which case $(L) = A_4^{SL_2} \otimes SL(2, C)$.*
   - *Above does not hold and $L_{Sym^4}(y)$ has an order 10 factor, in which case $Gal(L) = S_4^{SL_2} \otimes SL(2, C)$.*
   - *Neither of the above hold and $L_{Sym^4}(y)$ has an order 21 factor, in which case $Gal(L) = A_5^{SL_2} \otimes SL(2, C)$.*
   - *$Gal(L)^0 = SO(4, C)$.*
4. *None of the above hold, in which case $Gal(L) = SL(4, C)$. (Equivalently, $L_{\bigwedge^2}(y)$ is irreducible.)*

Proof: Clear. ∎

Theorem 6.4 *Let G be an finite primitive subgroup of $SL(4, C)$. Let $V \cong C^4$. Then one of the following holds.*

1. *$\bigwedge^2 V$ has a $G-$ invariant subspace of dimension 1, in which case G can be determined via Flow Chart A.*

2. *$\bigwedge^2 V$ has a $G-$ invariant subspace of dimension 3, in which case G can be determined via Flow Chart B.*

3. *Neither of the above hold or $\bigwedge^2 V$ is $G-$ irreducible, in which case G can be determined via Flow Chart C.*

Proof: Refer to Table 5.5. ∎

Corollary 6.4 *Let $L(y) = 0$ be an irreducible order four linear homogeneous differential equation with coefficients in k and unimodular differential Galois group $Gal(L)$. Assume $Gal(L)$ is a finite primitive group. Then one of the following holds.*

1. *$L_{\bigwedge^2}(y)$ has an order 1 factor, in which case $Gal(L)$ can be determined via Flow Chart A.*

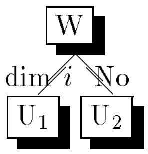
2. *$L_{\bigwedge^2}(y)$ has an order 3 factor, in which case $Gal(L)$ can be determined via Flow Chart B.*

3. *Neither of the above hold or $L_{\bigwedge^2}(y)$ is irreducible, in which case $Gal(L)$ can be determined via Flow Chart C.*

Proof: Clear. ∎

Explanation of the flow charts.

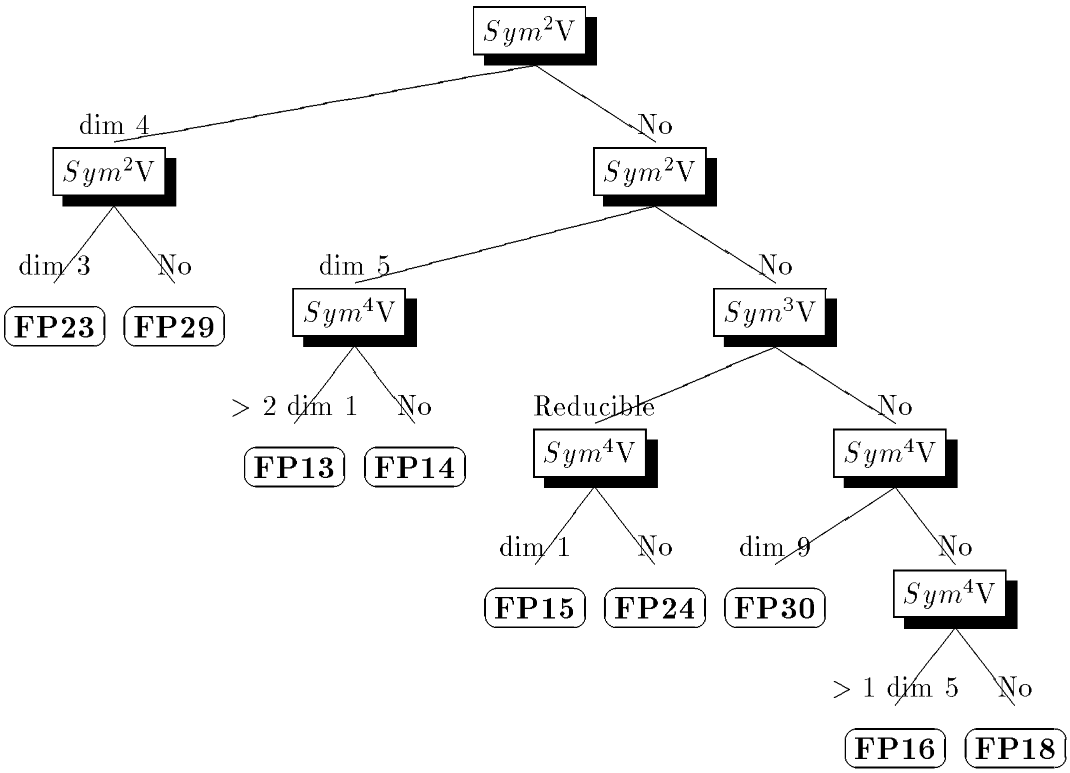
A branching of the following type



means that if W has an invariant subspace of dimension i then consider the space U_1, if not consider the space U_2. In general W, U_1, and U_2 are constructions of $V \cong C^4$. Sometimes the phrase "dim i" is replaced by "irr dim i" (respectively "$> k$ dim i"). These apply when W has an irreducible invariant subspace of dimension i (respectively, when W has an invariant subspace that is the sum of at least k invariant subspaces of dimension i.)

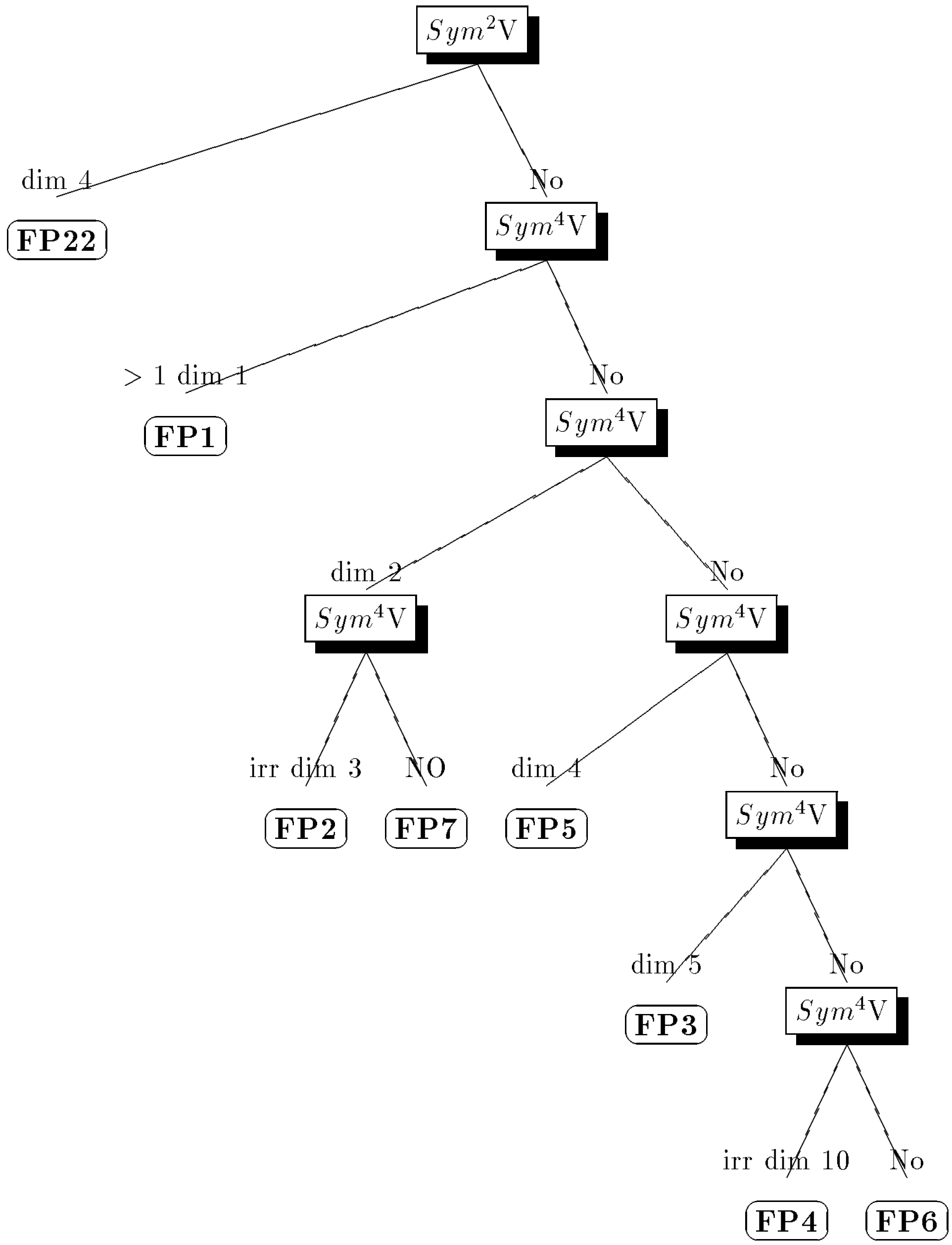
Flow Chart A

$\bigwedge^2$ V has a dim 1 invariant.



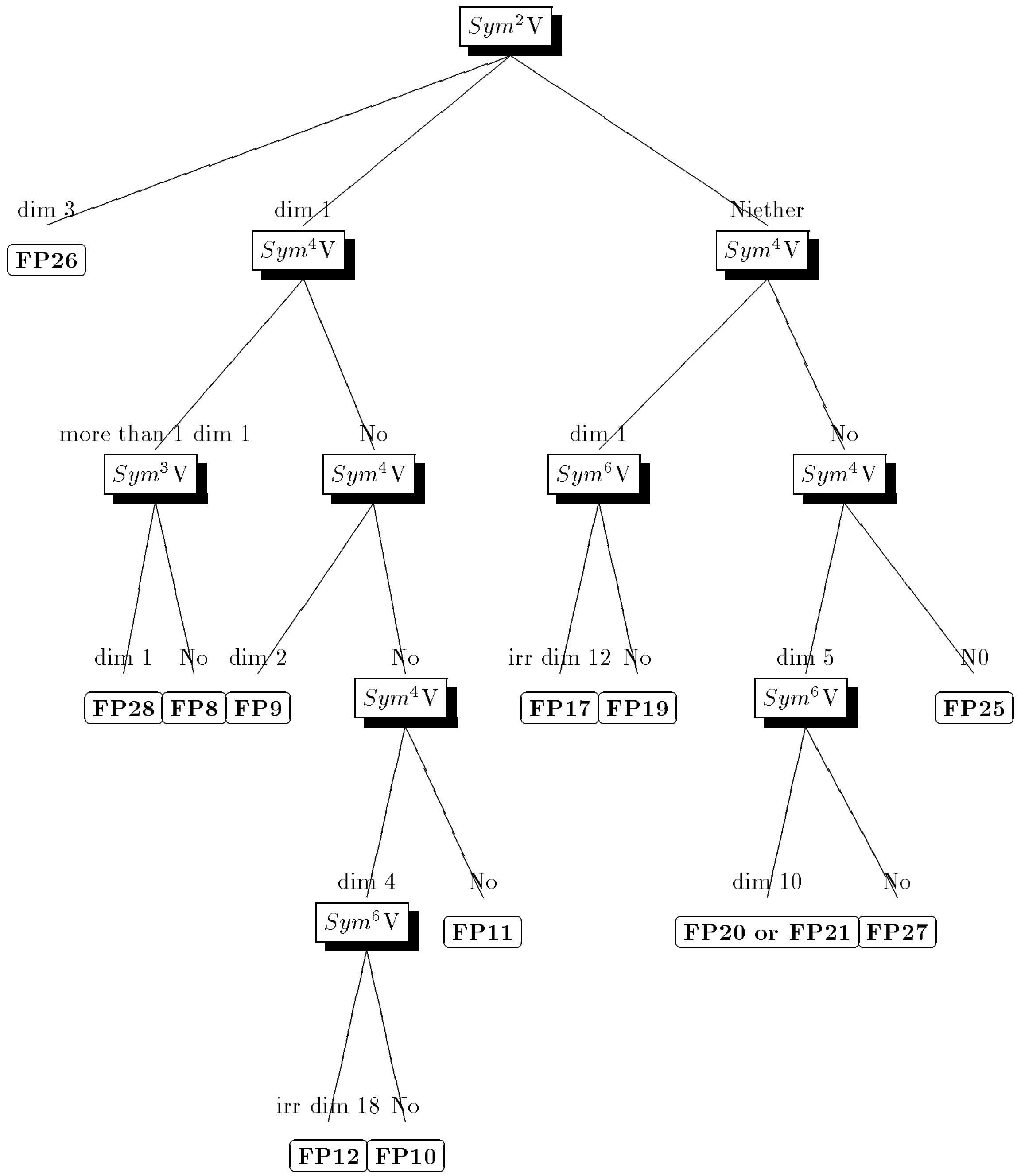
Flow Chart B

$\bigwedge^2$V has an dim 3 invariant.



Flow Chart C

$\bigwedge^2$ V is irreducible.



In addition to constructing a decision procedure using the irreducible decompositions of the group modules, one may also use the tables from Chapter 5 to determine very specific results. For example,

Theorem 6.5 *Let $L(y) = 0$ be an irreducible order four linear homogeneous differential equation with coefficients in k and unimodular differential Galois group $Gal(L)$. Then $Gal(L)$ projects onto G_{168} if and only if $L_{\bigwedge^2}(y)$ is irreducible and $L_{Sym^2}(y)$ has an irreducible factor of order 7.*

Proof: Clear, given the information in all of the tables. ∎

Theorem 6.6 *Let G be an infinite imprimitive monomial subgroup of $SL(4, C)$ with permutation action $\mathcal{P}_{\mathrm{G}}$. Let $V \cong C^4$. Then exactly one of the following holds.*

1. $\mathcal{P}_{\mathrm{G}}$ **is conjugate to** $<(1234)>$ *if any of the following hold.*

   - *$Sym^4 V$ has exactly seven 1 dimensional $G-$ invariant subspaces.*
   - *$Sym^4 V$ has exactly three 1 dimensional $G-$ invariant subspaces and $\bigwedge^2 V$ has no $G-$ irreducible dimension 2 invariant subspace.*
   - *$Sym^4 V$ has exactly one 1 dimensional $G-$ invariant subspace and exactly 1 two dimensional $g-$ invariant subspace, while $Sym^3 V$ has no $G-$ irreducible subspace of dimension 8.*

2. $\mathcal{P}_{\mathrm{G}}$ **is conjugate to K** *if any of the following hold.*

   - *$\bigwedge^2 V$ has four 1 dimensional $G-$ invariant subspaces.*
   - *$Sym^4 V$ has exactly three 1 dimensional $G-$ invariant subspaces and a $G-$ irreducible 2 dimensional invariant subspace.*
   - *$Sym^4 V$ has exactly one 1 dimensional $G-$ invariant subspace and more that one $G-$ invariant subspace of dimension 2.*

3. $\mathcal{P}_{\mathrm{G}}$ **is conjugate to** $<(1234), (12)(34)>$ *if any of the following hold.*

- *$Sym^4 V$ has exactly five 1 dimensional $G-$ invariant subspaces.*
- *$Sym^3 V$ has a $G-$ irreducible dimension 8 invariant subspace.*

4. $\mathcal{P}_{\mathrm{G}}$ **is conjugate to $\mathbf{A}_4$ or $\mathbf{S}_4$** *if*

   - *$\bigwedge^2 V$ is $G-$ irreducible.*

Proof: Table 5.1 gives the result. ∎

Corollary 6.5 *Let $L(y) = 0$ be an irreducible order four linear homogeneous differential equation with coefficients in k and unimodular differential Galois group $Gal(L)$. Assume $Gal(L)$ is an infinite imprimitive monomial group with permutation action $\mathcal{P}_{\mathrm{G}}$. Then exactly one of the following holds.*

1. $\mathcal{P}_{\mathrm{G}}$ **is conjugate to** $<(1234)>$ *if any of the following hold.*

   - *$L_{Sym^4}(y)$ has exactly seven order 1 factors.*
   - *$L_{Sym^4}(y)$ has exactly three order 1 factors and $L_{\bigwedge^2}(y)$ has no irreducible order 2 factor.*
   - *$L_{Sym^4}(y)$ has exactly one order 1 factor and exactly one order 2 factor while $L_{Sym^3}(y)$ has no irreducible order 8 factor.*

2. $\mathcal{P}_{\mathrm{G}}$ **is conjugate to K** *if any of the following hold.*

   - *$L_{\bigwedge^2}(y)$ has four order 1 factors.*
   - *$L_{Sym^4}(y)$ has exactly three order 1 factors and an irreducible order 2 factor.*
   - *$L_{Sym^4}(y)$ has exactly one order 1 factor and more than one order 2 factors.*

3. $\mathcal{P}_{\mathrm{G}}$ **is conjugate to** $<(1234),(12)(34)>$ *if any of the following hold.*

   - *$L_{Sym^4}(y)$ has exactly five order 1 factors.*
   - *$L_{Sym^3}(y)$ has an irreducible order 8 factor.*

4. $\mathcal{P}_{\mathrm{G}}$ **is conjugate to $\mathbf{A}_4$ or $\mathbf{S}_4$** *if*

- $L_{\bigwedge^2}(y)$ *is irreducible.*

Proof: Clear. ∎

Theorem 6.7 *Let $G = H \cup \gamma H$ be an imprimitive nonmonomial subgroup of $SL(4, C)$ with $H \subsetneq T \times T$ and $\pi_i(H) = T$ for $i = 1, 2$. Then T is projectively infinite if and only if $Sym^6 V$ has an irreducible $G-$ invariant subspace of order a multiple of 7. Furthermore, if this condition is satisfied then there exists a nonnegative integer n such that G is conjugate to G_n^1 or G_n^2 if and only if any one of the following* **equivalent conditions** *holds. Otherwise $G = G_n^3$ for some n.*

- *$\bigwedge^2 V$ has an odd number of one dimensional $G-$ invariant subspaces.*
- *$Sym^2 V$ has a 1 dimensional $G-$ invariant subspace.*
- *$Sym^3 V$ has either a 2 dimensional or 4 dimensional $G-$ invariant subspace.*
- *$Sym^4 V$ has a 1 dimensional $G-$ invariant subspace.*
- *$Sym^6 V$ has either a 7 dimensional or a 1 dimensional $G-$ invariant subspace.*

Proof: Table 5.2 and Table 5.3 give the decompositions of the modules $\bigwedge^2$ V as well as Sym^2V $\ldots$, Sym^4V and Sym^6V under the action of the imprimitive nonmonomial subgroups of SL(4, C). ∎

Corollary 6.6 *Let $L(y) = 0$ be an irreducible order four linear homogeneous differential equation with coefficients in k and unimodular differential Galois group $Gal(L)$. Assume $Gal(L) \cong H \cup \gamma H$ is an imprimitive nonmonomial group with $H \subsetneq T \times T$ and $\pi_i(H) = T$ for each $i = 1, 2$. Then T is projectively infinite if and only if $L_{Sym^6}(y)$ has an irreducible factor of order a multiple of 7. Furthermore, if this condition is satisfied then there exists a nonnegative integer n such that $Gal(L)$ is conjugate to G_n^1 or G_n^2 if and only if any one of the following* **equivalent conditions** *holds. Otherwise $Gal(L) = G_n^3$ for some n.*

- $L_{\bigwedge^2}(y)$ *has an odd number of order 1 factors.*
- $L_{Sym^2}(y)$ *has an order 1 factor.*
- $L_{Sym^3}(y)$ *has either an order 2 or order 4 factor.*
- $L_{Sym^4}(y)$ *has an order 1 factor.*
- $L_{Sym^6}(y)$ *has either an order 7 or an order 1 factor.*

Proof: Clear. ∎

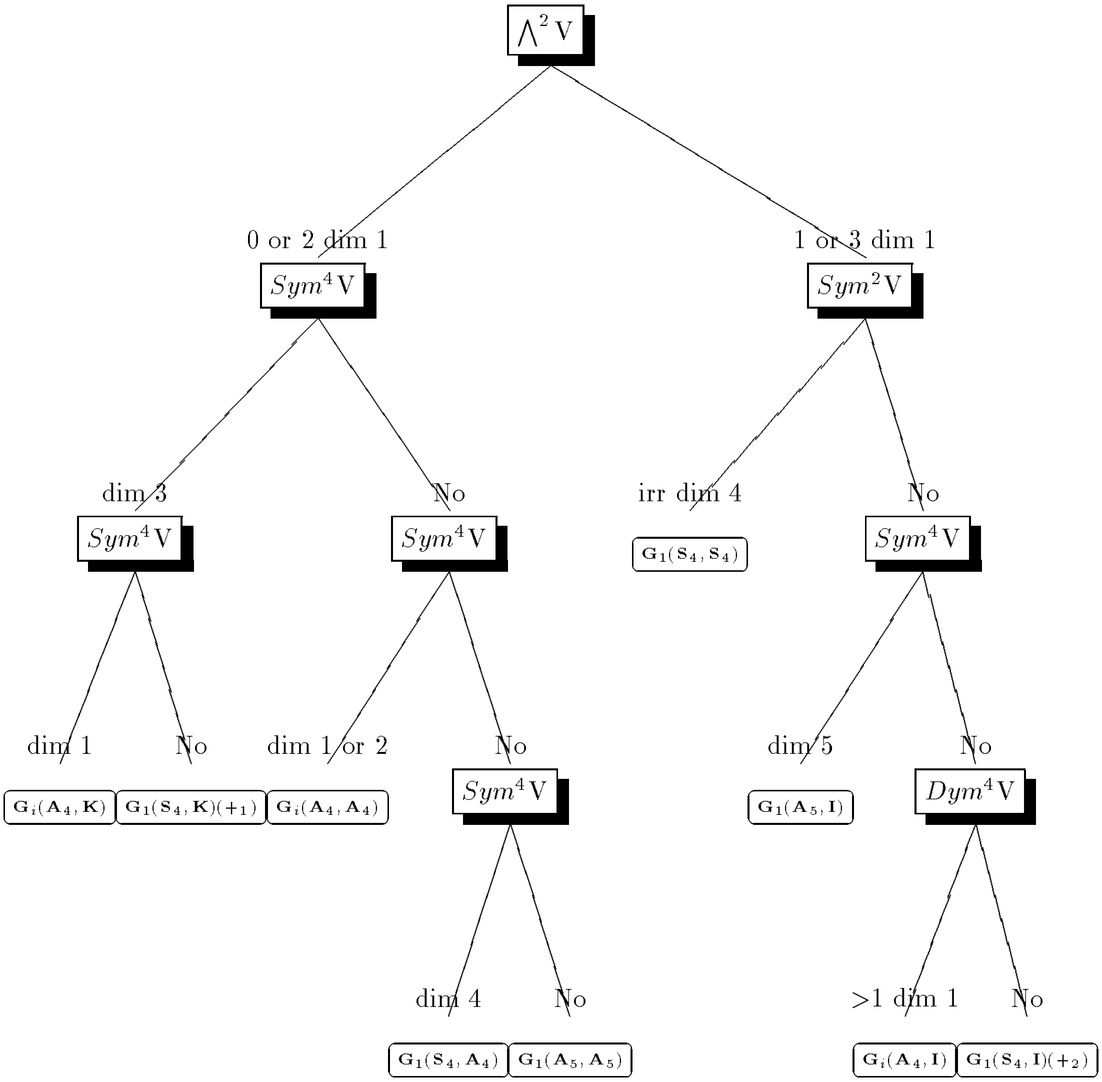
Theorem 6.8 *Let $G = H \cup \gamma H$ be an imprimitive nonmonomial subgroup of $SL(4, C)$ with $\pi_i(H) = T$ where T is projectively finite. Let $V \cong C^4$. Then one can identify the group G via Flow Chart D.*

Proof: Refer to Table 5.3. ∎

Corollary 6.7 *Let $L(y) = 0$ be an irreducible order four linear homogeneous differential equation with coefficients in k and unimodular differential Galois group $Gal(L)$. Assume $Gal(L)$ is an imprimitive nonmonomial group equal to $H \cup \gamma H$ where $\pi_i(H) = T$ and T is projectively finite. Then $Gal(L)$ can be determined via Flow Chart D.*

Proof: Clear. ∎

Flow Chart D



Notes:

($+_1$) It is most likely that the group is $G_1(S_4, K)$ but it may also be $G_i(A_4, K)$

($+_2$) It is most likely that the group is $G_1(S_4, I)$ but it may also be $G_i(A_4, I)$

Theorem 6.9 *Let $L(y) = 0$ be an irreducible order four linear homogeneous differential equation with coefficients in k and unimodular differential Galois group $Gal(L)$. Then $L(y) = 0$ is* **not** *solvable in terms of solutions of lower order equations if and only if $L_{Sym^4}(y)$ is irreducible.*

Proof: Theorem 5.1 from [SI2] found on page 7 of this thesis implies that $L(y) = 0$ is not solvable in terms of solutions to equations of lower order if and only if the Lie algebra associated with the connected component G^0 if $Gal(L)$ is both simple and has no faithful representation of degree less 4. The only algebraic subgroups of $\mathrm{SL}(4, \mathrm{C})$ satisfying both of these conditions are the infinite primitive groups G where G^0 is either $\mathrm{SL}(4, \mathrm{C})$ or $\mathrm{SP}(4, \mathrm{C})$. By observing the tables one sees that these are the only subgroups of $\mathrm{SL}(4, \mathrm{C})$ which act irreducibly on $Sym^4\mathrm{V}$. ∎

Theorem 6.10 *Let $L(y) = 0$ be an irreducible order four linear homogeneous differential equation with coefficients in k and unimodular differential Galois group $Gal(L)$. Then $L(y) = 0$ has only liouvillian solutions if and only if one of the following conditions is satisfied.*

1. *$L_{Sym^4}(y)$ has an irreducible factor of even order k where $k \neq$ 6,10, or 16.*
2. *$L_{Sym^4}(y)$ has more than one order 1 factor.*
3. *For some $k \neq 5, 10$, $L_{Sym^4}(y)$ has an even number of irreducible factors of order k.*
4. *For each k, $L_{Sym^4}(y)$ has at most one factor of order k and there is an $m \neq 7$ such that $L_{Sym^6}(y)$ has more than one irreducible factor of order m.*
5. *None of the above hold and $L_{Sym^6}(y)$ has no irreducible factor of order equal to a multiple of 7.*

Proof: Theorem 2.1 implies that $L(y) = 0$ has only liouvillian solutions if and only if the connected component of the identity $Gal(L)^0$ is a solvable group. In the order four case $Gal(L)^0$ is solvable if and only if $Gal(L)$ is a finite group, a monomial group, or an imprimitive nonmonomial group $H \cup \gamma H$ where $H \subseteq T \times T$ and T is projectively finite. The result then follows from the information found in the tables from Chapter 5. ∎

Appendix A

MAGMA Programs

Program 1 - "Programrun.mag"

Given a file containing the matrix generators of a finite group, $G \subset SL(4, C)$, this program returns the decompositions of the $G-$ modules $\bigwedge^2 V$, $Sym^2V, \ldots, Sym^nV$ for a positive integer n into sums of irreducible $G-$ invariant subspaces. The output is given as a list of pairs [i,j] where i is the number of the irreducible character appearing in the decomposition and j is its multiplicity.

```
//load("groupfile.mag"); //Load the file containing the generators.
//SetLogFile("output.mag"); //We keep track of the output here.
//Compute a faithful permutation representation of the group.
//Try to get a lower degree permutation representation to input into
//Programsmallrep.mag.
//————————- The above commands are done by hand ———
load "Programsmallrep.mag";
//The command permrep() calls this program which finds the subgroup of G which
gives a permutation representation of G of lowest degree .
S := Subgroups(L);
numb:=permrep(L,S);
Pf,PG,PK:=CosetAction(L,S[numb]`subgroup);
//We compute the character table for G.
R:=CharacterRing(PG);
table:=Basis(R);
print table;
d:=1;
i:=1;
//We pick out the first instance of an irreducible character of degree four.
while d lt 4 do
i:=i+1;
```

```
d:=Degree(table[i]);
end while;
load "Programdecomps.mag";
```
//The command decomps() calls this program which constructs the characters associated with the modules $\bigwedge^2$V, Sym^2V $\ldots$, Sym^nV and decomposes them with respect to the character table of G.

```
load "Programniceoutput.mag";
```
//This program simply gives us the output from the program "Programdecomps.mag" in a more readable form.

```
while Degree(table[i]) eq 4 do;
usualchar:=table[i];
if IsFaithful(usualchar) eq true then
U:=CoefficientField(usualchar);
decompalt,decompsyms:=decomps(PG,table,5,usualchar,U);
nicealtlist,nicesymslist:=niceoutput(decompalt,decompsyms);
end if;
i:=i+1;
end while;
UnsetLogFile();
```

Program 2 - "Programsmallrep.mag"

// sgs is the list of subgroups of G.

```
function permrep(g,sgs);
minpermdeg:=Order(g);
minclassnumber:=#sgs;
for i in 1..#sgs do
if Index(g,sgs[i]`subgroup) lt minpermdeg then
f,L,K:=CosetAction(g,sgs[i]`subgroup);
if Order(K) eq 1 then
minpermdeg:= Index(g,sgs[i]`subgroup);
minclassnumber:=i;
print minclassnumber;
end if;
end if;
end for;
return minclassnumber;
end function;
```

Program 3 - "Programdecomps.mag"

// This program will take as input:

//G = a small permutation representation of a finite group.

//symnumber = the largest symmetric power that you would like to decompose.

//table = character table of the group G.

//usualchar = the usual character of G.

//U = the universe needed for creating altchar.

// It will output:

// decompalt = the multiplicities of the irreducible characters in the

// decomposition of the 2nd alternating power.

// decompsyms = the multiplicities for the wanted symmetric powers.

```
decomps:=function(G, table, symnumber, usualchar,U);
c:=NumberOfClasses(G);
f:=PowerMap(G);
```
//f(i,2)=j where $x_i^2 \in$ jth class of G.

// Create altchar (the charater of $\bigwedge^2$ V).

```
altchar:=[U | ];
for i:= 1 to c do;
altchar[i]:=1/2*((usualchar[i])^2-usualchar[f(i,2)]);
end for;
altchar:=R ! altchar;
decompalt,extrachar:=Decomposition(table,altchar);
print "alternating character:";
print "";
print decompalt;
print "";
```

// Create a list of symchars (the characters of Sym^mV).

// We must first create the characters needed for the recursive formula.

```
negchar:=R!0;
trivchar:=R!1;
dualchar:=GaloisConjugate(usualchar,-1);
```

```
recurlist:=[negchar,negchar,trivchar,usualchar];
symchars:=[usualchar];
for i:= 2 to symnumber do;
newchar:=(usualchar*recurlist[i+2]) - (altchar*recurlist[i+1]) +
(dualchar*recurlist[i])-recurlist[i-1];
symchars[i]:=newchar;
recurlist[i+3]:=newchar;
end for;
// Decompose all symmetric powers.
decompsyms:=[* *];
for i:=1 to symnumber do;
tempsymlist:=[* *];
for j:=1 to #table do;
tempsymlist[j]:=InnerProduct(table[j], symchars[i]);
end for;
Append( decompsyms,tempsymlist);
end for;
print "";
print "symmetric decomps.";
print "";
print decompsyms;
print "";
return decompalt,decompsyms;
end function;
```

Program 4 - "Programniceoutput.mag"

```
function niceoutput(altlist,symlists);
print "In the following, ordered pairs are
[ith charater, its multiplicity]";
```

```
altmultlist:=[* *];
symsmultlist:=[* *];
symtemplist:=[* *];

print "decomposition of the 2nd alt.  power";
for i in 1.#altlist do;
if altlist[i] ne 0 then
Append( altmultlist,[i,altlist[i]]);
end if;
end for;
print altmultlist;

for i in 1..#symlists do;
print "";
print "decomposition of the ";print i;print "th symmetric power";
print "";
for j in 1..#altlist do;
if symlists[i,j] ne 0 then
Append( symtemplist,[j,symlists[i,j]]);
end if;
end for;

print symtemplist;
Append( symsmultlist,symtemplist);
symtemplist:=[* *];
end for;
return altmultlist, symsmultlist; end function;
```

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