On the Infinitesimal Geometry of Integrable Systems

BY

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Consider a complex analytic symplectic manifold M, a holomorphic Hamiltonian vector field X_H on M and a non-equilibrium integral curve $\Gamma \subset M$ of X_H. At the infinitesimal level the position of Γ relative to nearby orbits can be measured by means of a flat connection on a holomorphic symplectic vector bundle over Γ, with rank depending on the number of known independent holomorphic integrals. Obstructions to the existence of additional independent meromorphic integrals are then reflected in the monodromy group of this connection. This fact has been used extensively in the past decade for establishing the nonintegrability of many specific systems (e.g. see [B-C, B-C-R, F-I, I, K-S, M-S, N, Y1, Zi] and references therein).

The key idea is that r independent meromorphic integrals of X_H (which may be dependent along Γ) must induce r independent rational invariants for the monodromy group Mon (see §1); one then attempts to understand what restrictions this imposes on the group (see §2). The program was initiated by S.L. Ziglin in the paper [Zi], which introduces the relevant connection (in different language), constructs the induced Mon-invariants, and concentrates on the case in which Mon possesses both n independent rational invariants and a "non-resonant" element.

The purpose of this work is twofold: to present a more detailed version of Ziglin theory than appears in the original paper [Zi], so as to make the methods available to a wider audience; and to recast the ideas in group theoretic and geometric terms. The latter program has significant conceptual advantages; e.g., when formulated in terms of linear algebraic groups the number of independent rational Mon-invariants can be seen in terms of maximal orbit dimension of the Zariski closure $\overline{Mon}$ (Theorem 2.10), and the existence of a non-resonant element of Mon becomes equivalent to $\overline{Mon}$ having maximal rank (Theorem 2.14). Indeed, one can now understand Ziglin's 'Theorem

[1]Research supported in part by Natural Sciences and Engineering Research Council of Canada Grant OGP0008507.

[2]Research supported in part by NSF Grant 90-24624.

2' [Zi, p. 182] in terms of maximal tori and permutations of weight spaces thereof (Theorem 2.15 and Proposition 2.17). The linear algebraic group approach first appears in [C-R, M-S], but the development here is in greater depth. In particular, at points where we do not add detail to [Zi] we offer alternate proofs in terms of these objects. The necessary background for understanding linear algebraic groups is included in §2.

Although Ziglin formulated his results in terms of monodromy groups, it became clear quite recently (see [C-R] and [M-S]) that much more could be achieved, under mild restrictions, by replacing these with differential galois groups. Indeed, in this framework the problem of deciding which groups are "Ziglin", in the case of two degrees of freedom, is decidable [C-R]. In fact, a good deal of this program has now been extended to three degrees of freedom (see [C-R-S]) but is not discussed here. An outline of differential galois theory is presented in §3.

In §4 we offer a class of examples amenable to Ziglin analysis. Indeed, every example encountered by the authors in the literature has either been in this form, or could easily be cast in this form. A specific example involving two degrees of freedom is worked out in detail. This is intended to serve as a model for readers interested in applying the techniques. We note that this example can also be treated using the methods of [Y2]. Those readers interested in other examples that arise in a biological context and which use the full power of the galois group/Kovacic algorithm approach as given in §4 are referred to [R-S].

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§1 Integrals for the Monodromy of the Reduced Variational Equations

This chapter is an expanded version of a part of Ziglin's elegant theory on non-integrability of Hamiltonian systems, that first appeared in [Zi]. It is hoped that a balanced compromise between added detail and length of exposition has been attained.

§1.1 Invariant Subbundles

Let $X = \langle v(x), \frac{\partial}{\partial x}\rangle$ be a holomorphic vector field on a complex d-dimensional manifold M, and let $t \mapsto \Phi_t(p)$ be its associated flow, namely the solution of the system of differential equations

$$\dot{x} = v(x) \tag{1.1}$$

passing through p at time $t = 0$. As is well known, the map $(t, p) \mapsto \Phi_t(p)$ is always defined on some neighborhood $U \times V$ of $(0, p)$ in $\mathbb{C} \times M$. This leads to the concept of the (complete) *orbit* through p, namely the component Γ of p in the topology on M having for basis the sets $\{\Phi_t(p) \mid t \in U\}$. It is easy to verify that if p is not a point of equilibrium, then Γ is a Riemann surface maximal relative to the property: for each $q \in \Gamma$ there is an arc $\gamma : [0, 1] \to \mathbb{C}$ along which the map $t \to \Phi_t(p)$ can be analytically continued so that $\Phi_{\gamma(1)}(p) = q$. Note that the natural inclusion $i : \Gamma \to M$, though an immersion, is generally not a proper map, so that it is necessary to distinguish between Γ as defined above and the subset $i(\Gamma)$ of M.

For the rest of the chapter we deal with a (general but) fixed non-equilibrium orbit Γ. In this case X can be thought of as a section of the tangent bundle $T\Gamma$. Its dual is the closed form dt whose integral defines t as a multivalued function on Γ. Clearly t is a local inverse to $t \mapsto \Phi_t(p)$ and thus a local parameter for Γ up to an additive constant. For an arbitrary function f on Γ one has $\dot{f} = \frac{df}{dt} = X(f)$.

Let $T_\Gamma M$ be the restriction (strictly speaking the pull-back i^*TM) of the tangent bundle of M to Γ. The Lie derivative $\mathcal{L}_X$ is a derivation on the space of sections of TM which induces a well defined derivation D on the sections of $T_\Gamma M$ by extension and restriction. The relation

$$D(fu) = \dot{f}u + fDu \tag{1.2}$$

holds for all holomorphic functions f and sections u. The null sections of D coincide with those of the flat connection $\nabla := D \otimes dt$ and are also the solutions to the variational equations

$$\dot{\xi} = v_x(x(t))\xi$$

of X along Γ.

A subbundle E of $T_\Gamma M$ is *invariant* if D "preserves" the sheaf $\mathcal{O}(E)$ of holomorphic sections of E in the obvious sense. In this case D defines uniquely a derivation on E, as well as one on the quotient bundle $N := T_\Gamma M/E$ (still called D), which commutes with the projection $T_\Gamma M \to N$. A section of any of these bundles in the kernel of D is called *parallel.* For any local frame $\{e_j\}$ one has

$$De_j(t) = \Sigma A_{ij}(t)e_i(t)$$

for an appropriate matrix valued function $A(t)$ on Γ. In terms of A a section $\Sigma y^i e_i$ is parallel if and only if the coordinate vector y satisfies the linear differential equation

$$\dot{y} + A(t)y = 0. \tag{1.3}$$

§1.2 Hamiltonian Reduction

We now specialize to the case where M is a symplectic manifold with canonical form Ω and $X = X_{h_1}$ is symplectic with Hamiltonian h_1. The sign convention we are using here is such that

$$\flat(X_{h_1}) = dh_1,$$

where $\flat : TM \to T^*M$ is the "musical " isomorphism obtained by contraction of Ω with its *first* variable:

$$\langle \flat(v), w\rangle = \Omega(v, w)$$

for all v, w, in the same fiber of TM. A basic well known property of Hamiltonian flows is the invariance of the symplectic product, whose infinitesimal version is $\mathcal{L}_{X_{h_1}}\Omega = 0$. This, coupled with the identity

$$\mathcal{L}_X(\Omega(Y, Z)) = \mathcal{L}_X(\Omega)(Y, Z) + \Omega(\mathcal{L}_X Y, Z) + \Omega(Y, \mathcal{L}_X Z),$$

valid for arbitrary vector fields X, Y, Z, yields the fundamental formula

$$\mathcal{L}_{X_{h_1}}(\Omega(Y,Z)) = \Omega(\mathcal{L}_{X_{h_1}}Y, Z) + \Omega(Y, \mathcal{L}_{X_{h_1}}Z). \tag{1.4}$$

To use this we need the following straightforward result from symplectic linear algebra, whose proof will be left as an exercise to the reader.

Lemma 1.5. *Let E be a subspace of a symplectic vector space V, and let $E^{\perp}$ be its (symplectic) orthogonal. Then the quotient $N = E/(E \cap E^{\perp})$ carries a unique symplectic structure such that the canonical map $E \to N$ is a symplectic morphism.*

A vector bundle version of the lemma would require constancy of the fiber dimension of $E \cap E^{\perp}$. Here we will only need the particular case where $E^{\perp} := \cup\{(E_p)^{\perp} \mid p \in \Gamma\}$ is a subset of E. In what follows we let the notation $u \cdot v$ represent the symplectic product $\Omega(u,v)$ on any vector space or bundle.

Proposition 1.6. *Let $E^{\perp} \subset E$, where the latter is an invariant sub-bundle of $T_{\Gamma}M$. Then $E^{\perp}$ is also such, and the quotient $N = E/E^{\perp}$ is a symplectic bundle whose natural symplectic form is invariant under D: for any sections u, v of N one has*

$$\frac{d}{dt}(u \cdot v) = Du \cdot v + u \cdot Dv \tag{1.7}$$

Proof: Since $p \mapsto dim(E_p)^{\perp} = codim(E_p)$ is constant on Γ, $E^{\perp}$ is clearly smooth. Also note that for the particular case $E = N = T_{\Gamma}M$, (1.7) is simply a reformulation of (1.4). This allows us to establish invariance of $E^{\perp}$ as follows: if u, v, are sections of $E^{\perp}$ and E respectively, then

$$Du \cdot v = \frac{d}{dt}(u \cdot v) - u \cdot Dv = 0$$

by orthogonality and invariance of E. Thus Du is orthogonal to all sections of E, i.e., a section of $E^{\perp}$. Finally, the formula (1.7) follows from the fact that both D and the symplectic form on N are defined by passage to the quotient from the corresponding objects in $T_\Gamma M$, where, as already mentioned, (1.7) does hold. $\square$

The vector bundles $T_\Gamma M$ and N are examples of what one might call a *symplectic differential bundle*, that is, a quintuple $(N, \Gamma, X, D, \Omega)$, where $N \to \Gamma$ is a symplectic bundle over a Riemann surface endowed with a nowhere vanishing vector field X, and a derivation D on the sections of N such that the identities (1.2) and (1.7) hold. Of interest to us is that such a situation gives rise to the so called *monodromy representation* $\rho = \rho^N : \pi_1(\Gamma, p) \to Sp(N_p)$ defined as follows: for $[\gamma] \in \pi_1(\Gamma, p)$ and $v \in N_p$ let $\rho([\gamma])v$ be the value obtained by analytically continuing along γ^{-1} the germ of the parallel section with initial value v. That ρ is well defined follows from the principle of analytic continuation and the fact that solutions to linear equations can always be continued along arbitrary paths on Γ. The linearity of $\rho([\gamma])$ follows from the fact that γ can be covered by a finite number of coordinate patches with local parameter t, on which the equations have the form (1.3). That $\rho([\gamma])$ is symplectic follows locally from (1.7) applied to parallel sections u, v, and globally by a finite cover argument as above.

Remark 1.8. The subgroup $\rho(\pi_1(\Gamma, p))$ of $Sp(N_p)$ is called the monodromy group of N: it only depends trivially on the base point p, in the sense that if c is an arc joining p to q then the standard isomorphism $[\gamma] \mapsto [c]^{-1}[\gamma][c]$ from $\pi_1(\Gamma, p)$ to $\pi_1(\Gamma, q)$ induces an isomorphism of monodromy groups and any two such differ by an inner automorphism of $Sp(N_p)$.

In what follows we will assume that a number of holomorphic independent integrals $h_1, h_2, \ldots, h_k$ in involution are known in a full neighborhood of Γ, so that

$$dh_1 \wedge \ldots \wedge dh_k \neq 0. \tag{1.9}$$

More specifically, we will demand that the dh_i be linearly independent at *each point* of Γ.

Remark 1.10. The closed set where the k-form $\alpha_{\mathbf{h}}$ on the left hand side of (1.9) vanishes is clearly a union of orbits. Thus it suffices to assume $\alpha_{\mathbf{h}} \neq 0$ at a single point of Γ for this to be true at each point in a full neighborhood of it. Since these integrals will be kept fixed throughout our discussion, we may assume them to be globally defined — by discarding the complement of their domain of definition, if necessary. Similarly, we will assume (1.9) to hold everywhere on M.

To make use of the last proposition in the following context, let $M_{\mathbf{c}} = \mathbf{h}^{-1}(\mathbf{c})$ be the level surface of the map $\mathbf{h} = (h_1, \ldots, h_k) : M \to \mathbb{C}^k$ through some point of Γ. Since the h_i Poisson commute, $\mathbf{h}$ is constant on Γ so that $\Gamma \subset M_{\mathbf{c}}$. Let $E = T_\Gamma M_{\mathbf{c}} = i^* T M_{\mathbf{c}}$ be its tangent bundle along Γ. We will abuse – and thereby simplify the notation, by viewing the X_{h_i} as sections of $T_\Gamma M$.

Proposition 1.11. *The following hold:*

(1) *$E^\perp$ is the distribution spanned by the X_{h_i}.*

(2) *$E^\perp \subset E$.*

(3) *The X_{h_i} are parallel.*

(4) *E and $E^\perp$ are invariant.*

Proof: For each $p \in \Gamma$ the first three items are established by symplectic linear algebra involving $T_p M$. We suppress the label p from all pertinent objects.

(1) We clearly have $E = \{v \in T_\Gamma M \mid \langle dh_i, v \rangle = 0,\ i = 1, \ldots, k\}$, so that the annihilator E^0 of E in $T^* M$ is spanned by the dh_i. Under the isomorphism $\flat : TM \to T^* M$ the spaces $E^\perp$ and E^0 correspond to one another, and $\flat(X_{h_i}) = dh_i$ by definition. This establishes (1).

(2) We have $X_{h_i} \cdot X_{h_j} = \Omega(X_{h_i}, X_{h_j}) = \langle \flat(X_{h_i}), X_{h_j} \rangle = \langle dh_i, X_{h_j} \rangle = X_{h_j}(h_i) = \{h_i, h_j\} = 0$. Thus $E^\perp$ is self-orthogonal (involutive), and therefore a subspace of its own annihilator $(E^\perp)^\perp = E$.

(3) We have

$$DX_{h_i} = \mathcal{L}_{X_{h_1}} X_{h_i} = [X_{h_1}, X_{h_i}] = -X_{\{h_1, h_i\}} = 0.$$

(4) By (1) and (3) $E^\perp$ is spanned by parallel sections and is therefore invariant in view of (1.2). Finally the invariance of $E = (E^\perp)^\perp$ follows from that of $E^\perp$ as in the proof of Proposition 1.6.

□

For easy reference we summarize the results of the last two propositions in

Proposition 1.12. *Let $h_1, \ldots, h_k$ be global holomorphic functions such that $\alpha_{\mathbf{h}} = dh_1 \wedge \ldots \wedge dh_k$ vanishes nowhere on M. Let $E = T_\Gamma M_{\mathbf{c}}$ where $\Gamma \subset M_{\mathbf{c}} = \mathbf{h}^{-1}(\mathbf{c})$. Then E is the annihilator of $\{dh_1, \ldots, dh_k\}$, $E^\perp \subset E$ is the vector bundle spanned by $X_{h_1}, \ldots, X_{h_k}$, and the quotient $N = E/E^\perp$ is a symplectic differential bundle of dimension $dim(M) - 2k$.*

We call the quotient N the *totally reduced normal bundle.* Its parallel sections are solutions to the so called NVE (normal variational equations), a terminology that we shall follow here too. We remark that the case $k = 1$ has been extensively studied in the literature.

§1.3 Ziglin's Lemma

Let $\mathcal{O}$ be the ring of germs of holomorphic functions at $0 \in \mathbb{C}^d$, and let $\mathcal{M}$ be its quotient field. For any coordinate system $x = (x_1, \ldots, x_d), \mathcal{O}$ can and will be identified with the ring of convergent power series $\Sigma c_\alpha x^\alpha$.

A d-tuple of non-negative integers $s = (s_1, \ldots, s_d)$ determines an order function $\nu = \nu^s : \mathcal{O} \to \mathbf{Z} \cup \{+\infty\}$ by declaring the ν-order of a non-zero monomial cx^α to be $\langle s, \alpha \rangle$, and extending it to all functions $f = \Sigma c_\alpha x^\alpha$ by setting $\nu(f) = \min\{\nu(x^\alpha) \mid c_\alpha \neq 0\}$ if $f \neq 0$, and $\nu(0) = +\infty$. The map $\nu : (\mathcal{O}_*, \times) \to (\mathbf{Z}, +)$ is clearly a morphism of semigroups, and for that reason it extends uniquely to a map $\nu : \mathcal{M} \to \mathbf{Z} \cup \{+\infty\}$ which is a group homomorphism from $(\mathcal{M}_*, \times)$ to $(\mathbf{Z}, +)$. It has the additional property:

$$\nu(f + g) \geq \min \{\nu(f), \nu(g)\}.$$

For later use it will be useful to observe that any monomial x^α is a $\nu(x^\alpha)$-eigenvector of the (weighted) Euler field $E := \Sigma s_i x_i \, \partial/\partial x_i$. E is semisimple on the polynomial ring $\mathbb{C}[x]$, and its kernel is the $\mathbb{C}$-algebra generated by the x_i of order 0. An element $f \in \mathcal{M}_*$ is *(quasi-)homogeneous* of degree m iff $E(f) = mf$. A moment's thought will convince the reader that every holomorphic function can be written as a (convergent) series $f = f_m + h.o.t.$ of homogeneous components where $m = \nu(f)$, and "h.o.t." refers to higher $\nu-order$ terms. Following Ziglin we call f_m the *junior term* of f and denote it f^0. The relation $fg = (f_m + h.o.t.)(g_k + h.o.t.) = f_m g_k + h.o.t.$ implies that the map $f \mapsto f^0$ is a semigroup endomorphism of $(\mathcal{O}_*, \times)$, and for that reason, again, it extends uniquely to an endomorphism of $(\mathcal{M}_*, \times)$, with the property that $f = f^0 + h.o.t.$ If $f = u/v \in \mathcal{M}_*$ then one has $f^0 = u^0/v^0$.

Remark 1.13. When the s_i are all positive then f^0 is rational for any $f \in \mathcal{M}$. The converse also holds.

Using the Euler operator we can now extend both ν and the "junior term" map to the space of germs of meromorphic differential forms

$$\mathcal{M}(\Lambda^*) = \bigoplus_{p=0}^{d} \mathcal{M}(\Lambda^p).$$

We first extend to the eigenspaces of $\mathcal{L}_E$ by declaring a form $\alpha \neq 0$ to be homogeneous of order $\nu(\alpha)$ if $\mathcal{L}_E(\alpha) = \nu(\alpha)\alpha$. From

$$\mathcal{L}_E(\alpha \wedge \beta) = \mathcal{L}_E(\alpha) \wedge \beta + \alpha \wedge \mathcal{L}_E(\beta)$$

it follows that the wedge product of homogeneous forms is homogeneous of order the sum of orders of the factors, *unless the product vanishes*, in which case the order is declared to be $+\infty$. Since $\mathcal{L}_E$ commutes with d, one has $\mathcal{L}_E(d\alpha) = \nu(\alpha)d\alpha$ so that $\nu(d\alpha) = \nu(\alpha)$ (unless α is closed). In particular $\mathcal{L}_E(dx_i) = s_i dx_i$ so that $\mathcal{L}_E$ is semisimple on the subalgebra of forms with polynomial coefficients spanned by the dx_i. As before, any holomorphic form can be written as a convergent series of homogeneous tems. If $\alpha = \alpha_m + h.o.t.$ with $\alpha_m \neq 0$ we define the order and junior term as $\nu(\alpha) = m$ and $\alpha^0 = \alpha_m$ respectively. The obvious multiplicative properties $\nu(f\alpha) = \nu(f) + \nu(\alpha)$ and $(f\alpha)^0 = f^0\alpha^0$ for holomorphic non-zero f, α, permit the extension of both maps to $\mathcal{M}(\Lambda^*)$. We leave it to the reader to work out the details. The essential facts are summarized in

Proposition 1.14. *Let $f \in \mathcal{M}$ and $\alpha, \beta \in \mathcal{M}(\Lambda^*)$. Then the following hold:*

1. $\nu(\alpha + \beta) \geq \; min\, \{\nu(\alpha), \nu(\beta)\}$.

2. *If α, β, are non-vanishing then $\alpha \wedge \beta = \alpha^0 \wedge \beta^0 + \gamma$, where $\nu(\gamma) > \nu(\alpha) + \nu(\beta)$ so that*

$$(\alpha \wedge \beta)^0 = \begin{cases} \alpha^0 \wedge \beta^0 & \text{if } \alpha^0 \wedge \beta^0 \neq 0 \\ \gamma^0 & \text{otherwise.} \end{cases}$$

3. $(f\alpha)^0 = f^0\alpha^0$*, and* $\nu(f\alpha) = \nu(f) + \nu(\alpha)$.

4. *If $\alpha_1 := d\alpha - d\alpha^0$ then $\nu(\alpha_1) > \nu(\alpha)$. Thus $(d\alpha)^0 = d\alpha^0$ unless α^0 is closed. In all cases $\nu(d\alpha) \geq \nu(\alpha)$.*

□

The following is a standard result in the elementary theory of fields. For details on derivation extensions see [La, Ch. 10, §7].

Proposition 1.15. *Let $L \subset K$ be a field extension of* 0*-characteristic fields. Then any derivation on L extends to a derivation on K.*

Proof: By the maximal principle it clearly suffices to show that for $t \in K$, any derivation D on L extends to some derivation $\tilde{D}$ on $L(t)$. When t is transcendental over L the extension can be done by assigning an arbitrary value $q \in L(t)$ to $\tilde{D}(t)$. One simply uses the standard rules of differentiation and checks that the result is a well defined derivation.

On the other hand if t is algebraic over L the extension is unique: the value of $\tilde{D}(t)$ is determined by the minimal equation $p(t) = 0$. That this value yields a derivation is a matter of computation. (Characteristic zero is needed only here.) □

Proposition 1.16. *In $\mathbb{C}(x)$ functional and algebraic independence are equivalent.*

Proof: That $P(u_1, \ldots, u_n) = 0$ implies dependence of the $\{du_i\}$ is immediate from $\Sigma \frac{\partial}{\partial u_i} P(u_1, \ldots, u_n) du_i = d(P(u_1, \ldots, u_n)) = 0$. Conversely, if the u_i are algebraically independent then the operators $\partial/\partial u_k$ are well defined on $\mathbb{C}(u)$ and by the last proposition they extend to derivations D_k on $\mathbb{C}(x)$ such that $D_k(u_j) = \delta_{kj}$. It is now straightforward that the vector fields $X_k := \Sigma D_k(x_i)\partial/\partial x_i$ satisfy $\langle df, X_k \rangle = D_k f$ for any $f \in \mathbb{C}(x)$. Thus contraction with X_k of any relation of the form $\Sigma f_j du_j = 0$ gives $f_k = 0$. □

Remark 1.17. The identification of vector fields with derivations on $\mathbb{C}(x)$ is *false* in $\mathcal{M}$. Indeed in this case the space of derivations is infinite dimensional over $\mathcal{M}$. (Use Proposition 1.15 to extend *arbitrarily* from $\mathbb{C}(x)$ to $\mathcal{M}$). Yet the space of germs of meromorphic fields is spanned by $\{\partial/\partial x_i\}$. What makes vector fields special is that the derivations they define are *local.*

Let K be a subfield of $\mathcal{M}$ containing $\mathbb{C}$. As is standard in analytic geometry, we write $\dim\,(K) = \dim_{\mathcal{M}} \text{span}\,\{df \mid f \in K\}$. Note that $0 = \dim\,(\mathbb{C}) \leq \dim\,(K) \leq d = \dim\,(\mathcal{M})$.

Theorem 1.18 (Ziglin's Lemma). *Let $(s_1, \ldots, s_d)$ be a fixed d-tuple of non-negative integers. For a field K as above let K^0 be the field spanned by $\{f^0 \mid f \in K\}$. If $K^0 \subset \mathbb{C}(x)$ then* trans $\deg_{\mathbb{C}} K^0 \geq$ *dim (K). In fact for $\{f_1, \ldots, f_n\} \subset K$ independent there exist polynomials $\{P_i(z_1, \ldots, z_i) \mid i = 1, \ldots, n\}$ such that if $g_i := P_i(f_1, \ldots, f_i)$ then $g_1^0, \ldots, g_n^0$ are algebraically independent.*

Proof: (After [Zi, Ku]). Suppose f is such that $df \neq 0$ but $f^0 = c \in \mathbb{C}$. Then $P_1(z_1) = z_1 - c$ takes care of the case $n = 1$. We may therefore assume $f_1^0, \ldots, f_{n-1}^0$ to be independent by replacing $\{f_i\}_{i=1}^{n-1}$ by $\{P_i(f_1, \ldots, f_i)\}_{i=1}^{n-1}$. For any n-tuple $\mathbf{g} = (g_1, \ldots, g_n)$ of functions define the form

$$\omega(\mathbf{g}) := \frac{dg_1}{g_1} \wedge \ldots \wedge \frac{dg_n}{g_n},$$

and the integer

$$\mu(\mathbf{g}) := \nu(\omega(\mathbf{g})),$$

and observe that the n-tuple $\mathbf{g}^0 := (g_1^0, \ldots, g_n^0)$ is independent iff $\mu(\mathbf{g}) = 0$. Indeed one clearly has

$$\omega(\mathbf{g}) = \frac{dg_1^0}{g_1^0} \wedge \ldots \wedge \frac{dg_n^0}{g_n^0} + \omega_1 = \omega(\mathbf{g}^0) + \omega_1,$$

with $\nu(\omega_1) \geq 1$, so that

$$\mu(\mathbf{g}) = \begin{cases} \mu(\mathbf{g}^0) & \text{if } \omega(\mathbf{g}^0) \neq 0 \\ \nu(\omega_1) & \text{otherwise}\ , \end{cases}$$

from which the stated property follows.

Now suppose that $\mu(\mathbf{f}) > 0$, i.e. that the f_i^0 are dependent. Since $K^0 \subset \mathbb{C}(x)$, they are also algebraically dependent by Prop. 1.16. Let $P(z_1, \ldots, z_n)$ be the minimal polynomial of f_n^0 in $\mathbb{C}[z_1, \ldots, z_{n-1}][z_n] = \mathbb{C}[z_1, \ldots, z_n]$. We endow this ring with the (unique) order function $\tilde{\nu}$ such that the order of monomials are preserved when pulled back by $\mathbf{f}^0$. Thus $\tilde{\nu}(z_i) = \nu(f_i^0)$. It is crucial to observe that the orders of ($\tilde{\nu}$-)homogeneous polynomials are also preserved, unless pulled back to zero. It follows that each homogeneous component of P annhiliates $\mathbf{f}^0$ as well, so that we may and will assume P to be homogeneous. Let $g_i = f_i$ for $i \neq n$, and $g_n = P(f_1, \ldots, f_n) = \mathbf{f}^* P$. For any monomial $c_\alpha z^\alpha$ appearing in the expansion of P one has $c_\alpha \mathbf{f}^\alpha = c_\alpha (\mathbf{f}^0)^\alpha + \varphi_\alpha$ with $\nu(\varphi_\alpha) > \nu((\mathbf{f}^0)^\alpha) = \tilde{\nu}(z^\alpha) = \tilde{\nu}(P)$. Thus $P(\mathbf{f}^0) = 0 \Rightarrow g_n = \Sigma \varphi_\alpha$ so that

$$\nu(g_n) > \tilde{\nu}(P). \tag{1.19}$$

On the other hand the $\tilde{\nu}$-homogeneity of P and the fact that $\partial P/\partial z_n$ cannot vanish (otherwise $f_1^0, \ldots, f_{n-1}^0$ would be dependent) implies $\tilde{\nu}(P) = \tilde{\nu}(z_n \partial P/\partial z_n)$. Since P is the minimal polynomial of f_n^0 one must have $(\partial P/\partial z_n)(\mathbf{f}^0) \neq 0$. As a result we obtain

$$\nu(f_n^0 (\partial P/\partial z_n)(\mathbf{f}^0)) = \tilde{\nu}(P). \tag{1.20}$$

Now simple calculation gives

$$\omega(\mathbf{g}) = \omega(\mathbf{f}) \frac{f_n (\partial P/\partial z_n)(\mathbf{f})}{g_n},$$

so that

$$\begin{aligned} \mu(\mathbf{g}) &= \mu(\mathbf{f}) + \nu(f_n (\partial P/\partial z_n)(\mathbf{f})) - \nu(g_n) \\ &= \mu(\mathbf{f}) + \nu(f_n^0 (\partial P/\partial z_n)(\mathbf{f}^0)) - \nu(g_n) \\ &< \mu(\mathbf{f}) \qquad\qquad \text{(by (1.19) and (1.20)).} \end{aligned}$$

Replacing $\mathbf{f}$ by $\mathbf{g}$ we iterate this procedure until $\mu(\mathbf{g}) = 0$, in which case we are done. $\square$

An important application is

Proposition 1.21. *Let G be a group acting linearly on a vector space F, and assume there is a linear function $0 \neq \xi \in F^*$ which belongs to the field K_F of G-invariant rational functions. Then $S := \{\xi\}^0 = \{v \in F \mid \langle \xi, v\rangle = 0\}$ is G-invariant and* $\text{trans deg}_{\mathbb{C}} K_S \geq \text{trans deg}_{\mathbb{C}} K_F - 1$.

Proof: The invariance of S is obvious. Now extend ξ to a system of (linear) coordinates (ξ, y) and consider the order function associated with $s = (1, 0, \dots, 0)$. Note that in these coordinates $S = \{(0, y)\}$. Let $\xi, f_1, f_2, \dots, f_n$ be an independent subset of K_F. By Ziglin's Lemma we may assume that $\xi, f_1^0, \dots, f_n^0$ are independent. Replacing f_j by $\xi^{k_j} f_j$ does not destroy independence, so that by picking $k_j = -\nu(f_j)$ we may further assume $\nu^s(f_j) = 0$. Thus f_j is of the form

$$f_j = \frac{a_j(y) + \xi b_j(\xi, y)}{c_j(y) + \xi d_j(\xi, y)}$$

with a_j, b_j, c_j, d_j polynomials, and $f_j^0 = a_j(y)/c_j(y)$ in the rational field of S. (Cautionary note: there is generally no *natural inclusion* of the rational field of a linear subspace into the field of an ambient space containing it. In this situation, however, the inclusion $\mathbb{C}(y) \subset \mathbb{C}(\xi, y)$ is "geometric": it does not arise from the coordinates, but from the pull-back action of the projection $F \ni v \mapsto v - \langle \xi, v\rangle w \in S$ where $\langle \xi, w\rangle = 1$.)

To finish the proof we need to show that the f_j^0 are G-invariant under the induced action on S. For $g \in G$ let $\tilde{g}$ denote its representative matrix relative to the y coordinates on S. With some fixed v_g we have $g \cdot (\xi, y) = (\xi, \tilde{g} y + \xi v_g)$. Thus

$$\begin{aligned} \frac{a_j(y) + \xi b_j(\xi, y)}{c_j(y) + \xi d_j(\xi, y)} &= g^* f_j \\ &= \frac{a_j(\tilde{g} y + \xi v_g) + \xi b_j(\xi, \tilde{g} y + \xi v_g)}{c_j(\tilde{g} y + \xi v_g) + \xi d_j(\xi, \tilde{g} y + \xi v_g)} \end{aligned}$$

$$= \frac{a_j(\tilde{g}y) + \xi(\ldots)}{c_j(\tilde{g}y) + \xi(\ldots)}.$$

It follows that $a_j(y)/c_j(y) = a_j(\tilde{g}y)/c_j(\tilde{g}y)$ so that $f_j^0 = (\tilde{g})^* f_j^0$. □

We shall give an alternate group-theoretic proof of the above result in Chapter 2 following Theorem 2.10. A straightforward induction gives the following generalization.

Corollary 1.22. *Let $E \subset F$ be the annihilator of a set $\xi_1, \ldots, \xi_k$ of independent linear G-invariant functions on F. Then E is G-invariant and* $\text{trans deg}_{\mathbb{C}} K_E \geq \text{trans deg}_{\mathbb{C}} K_F - k$.

§1.4 Integrals of the NVE

We now pass to a manifold M and give an invariant definition of the order and junior term of $f \in \mathcal{M}_p$. Let $\mathcal{J}_p$ be the maximal ideal $\{f \in \mathcal{O}_p \mid f(p) = 0\}$. For $f \in \mathcal{O}_*$ write $\nu_p(f)$ for its vanishing order at p, namely the unique $m \in \mathbf{Z}_+$ such that $f \in \mathcal{J}_p^m \backslash \mathcal{J}_p^{m+1}$, and $\nu_p(0) = +\infty$. We remark that for a system of coordinates x centered at p, ν_p is nothing but the order ν^s of the last section with $s = (1, \ldots, 1)$. As usual, the notation will not distinguish between $f \in \mathcal{M}_p$ and its representative in $\mathcal{M}$.

We define the (invariant) junior term of f as a symmetric homogeneous form $f^0 = f_p^0$ of degree $\nu_p(f)$ on T_pM by the two step procedure:

1. Pick a coordinate system $x = (x_1, \ldots, x_d)$ centered at p and write $f = f_m + h.o.t.$, where $m = \nu_p(f)$ and f_m is homogeneous of degree m in the variables x.

2. For $v \in T_pM$, set $f^0(v) = f_m(\xi)$ where $\xi = \langle v, dx \rangle$ is the coordinate vector of v.

For a second coordinate system y we have $x = \varphi(y) = \varphi\prime(0)y + h.o.t.$, and $f(\varphi(y)) = f_m(\varphi\prime(0)y) + h.o.t.$ so that the new homogeneous lowest order component is $f_m \circ \varphi\prime(0)$. If v is represented in the new coordinates by the vector η we clearly have $\xi = \varphi\prime(0)\eta$, so that $f_m(\xi) = f_m(\varphi\prime(0)\eta) = f_m \circ \varphi\prime(0)(\eta)$. This shows that f^0 is well defined by the above prescription.

Example 1.23. Recall our standing assumption (1.9) – see Remark 1.10 for notation – that $\alpha_{\mathbf{h}}(p) \neq 0$. We have

$$h_i^0 = \begin{cases} c_i & \text{if } h_i(p) = c_i \neq 0 \\ dh_i & \text{if } h_i(p) = 0. \end{cases}$$

The example shows that a simple normalization of the h_i will insure that the (linear) functions h_i^0 are independent on T_pM. Note that this is a very special case of Ziglin's Lemma. For convenience we will add to our assumptions concerning the h_i the requirement that they vanish on Γ – by subtracting appropriate constants, if necessary.

The map $f \mapsto f^0$ extends to a group homomorphism $(\mathcal{M}_*, \times) \to (\mathcal{R}_*, \times)$, where $\mathcal{R} = \mathcal{R}_p$ is the rational function field of T_pM. This follows by passage to the quotient fields from the relation $(fg)^0 = f^0 g^0$, itself a consequence of the identity $fg = (f_m + \cdots)(g_k + \cdots) = f_m g_k + \cdots$.

Remark 1.24. Note that for global meromorphic f the map $p \mapsto \nu_p(f)$ is generically zero and therefore discontinuous – unless identically zero. Similar consideration apply to the family of rational functions $\{f_p^0 \mid p \in M\}$.

For the next result recall the subbundle $E^\perp$ of $T_\Gamma M$ spanned by the fields $X_j := X_{h_j}$. The bundle $F := T_\Gamma M/E^\perp$ contains the reduced bundle N as a subbundle and will be a useful intermediary for its study. We shall see below that N can be obtained from F in the same way E was obtained from $T_\Gamma M$.

Proposition 1.25. *Let $f \neq 0$ be a common integral for the fields X_j and let Γ be an orbit of X_1. Then $p \mapsto v_p(f)$ is constant on Γ, and $(p, v) \mapsto f_p^0(v)$ is a meromorphic function on $T_\Gamma M$ and an integral for the variational flow. For fixed $p \in \Gamma$ the function f_p^0 belongs to the rational field $\mathcal{R}_p(F)$ –canonically* *embedded in $\mathcal{R}_p(TM)$ by pull-back along the projection $T_pM \to F_p$. Moreover f_p^0 is invariant under the action of the monodromy group $\rho(\pi_1(\Gamma, p))$ (of either bundle).*

Proof: Since the h_i are in involution they extend by the Darboux Theorem to a set of canonical coordinates (τ, h, u, v) so that $\Omega = d\tau \wedge dh + du \wedge dv$. In such coordinates (which we shall center at p), $\frac{\partial f}{\partial \tau_i} = X_i(f) = 0$. We claim that near $(\tau, h, u, v) = 0, f = \varphi/\psi$, with φ, ψ, holomorphic functions in the variables (h, u, v) alone. Indeed this is clear if f is holomorphic. Otherwise write

$$f = \frac{\tilde{\varphi}(\tau, h, u, v)}{\tilde{\psi}(\tau, h, u, v)}$$

in a rectangular connected neighborhood $U \times V$ of 0 with $\tau \in U$ and $(h, u, v) \in V$. Since $\tilde{\psi} \neq 0$, f is holomorphic on some open set $U_1 \times V_1 \subset U \times V$ and independent of τ there. Therefore

$$f(\tau, h, u, v) = \frac{\tilde{\varphi}(a, h, u, v)}{\tilde{\psi}(a, h, u, v)}$$

for some fixed $a \in U_1$ and $(h, u, v) \in V_1$. The desired conclusion holds on $U_1 \times V_1$ with $\varphi(h, u, v) = \tilde{\varphi}(a, h, u, v)$ and $\psi(h, u, v) = \tilde{\psi}(a, h, u, v)$, and by analytic continuation also on $U \times V$.

Let us now write $x = (\tau, h, u, v)$ and $\xi = (\xi_\tau, \xi_h, \xi_u, \xi_v)$ for the associated coordinates on TM. The flow equations for X_1 are $\dot{x}_j = \delta_{1j}$, and the variational flow on TM is described by the additional equations $\dot{\xi} = 0$, so that sections of the form $\xi(t) = const.$ are parallel. Note that the time variable associated with X_1 is $t = x_1 = \tau_1$, a local parameter for Γ, and that coordinates for $T_\Gamma M$ are (t, ξ). The system (τ, h, u, v) is a simultaneous flow box for all the X_j. Thus $E^\perp = span\ \{X_1, \ldots X_k\} = \{(\xi_\tau, 0, 0, 0)\}$, so that $\{(\xi_h, \xi_u, \xi_v)\}$ constitutes an appropriate coordinate representation for the fibers of the quotient F, and $\mathcal{R}_pF$ is the subfield $\mathbb{C}(\xi_h, \xi_u, \xi_v)$ of rational functions independent of ξ_τ in $\mathcal{R}_p(T_\Gamma M) = \mathbb{C}(\xi_\tau, \xi_h, \xi_u, \xi_v)$. Writing

$$f = \frac{\varphi_m(h,u,v) + h.o.t.}{\psi_n(h,u,v) + h.o.t.}$$

we have according to our two step prescription

$$f_p^0(\xi) = \frac{\varphi_m(\xi_h, \xi_u, \xi_v)}{\psi_n(\xi_h, \xi_u, \xi_v)}. \tag{1.26}$$

Since the right hand side is constant under the variational flow, and only depends on the variables of F, all the assertions of the proposition are immediate consequences of this equation. □

Our next task is to show that the existence of additional integrals imposes severe restrictions on the monodromy. Specifically, we assume the existence of meromorphic functions $f_j, j = k+1, \ldots, k+r$, satisfying the following conditions:

1. They Poisson commute with the h_i:

$$\{f_j, h_i\} = 0,\ i = 1, \ldots, k,\ j = k+1, \ldots, k+r.$$

2. The h_i and the f_j are (jointly) functionally independent:

$$dh_1 \wedge \ldots \wedge dh_k \wedge df_{k+1} \wedge \ldots \wedge df_{k+r} \neq 0. \tag{1.27}$$

3. They are globally defined. (By previous convention this means: in a full neighborhood of Γ.)

Note that our standing assumption concerning the holomorphic integrals in involution $h_1, \ldots, h_k$, is still in force. A few comments will clarify the situation:

- We *do not require* the f_j to be in involution among themselves.

- The pole divisor D_j of f_j is a codimension 1 analytic variety invariant under the X_{h_1} flow, so that $D_j \cap \Gamma \neq \emptyset \Rightarrow \Gamma \subset D_j$. We allow $D_j \cap \Gamma \neq \emptyset$. In other words, we allow the f_j to blow up along Γ.
- Even if f_{j_0} is holomorphic, it may still not qualify as one of the h_i for the particular orbit in consideration. For that we need $df_{j_0} \wedge dh_1 \wedge \ldots \wedge dh_k(p) \neq 0$ at some point $p \in \Gamma$. Note however that due to (1.27), this inequality will hold for most other orbits. It is well known that integrable systems will generally contain degenerate orbits along which the integrals are dependent.

An additional distinction between the h_i and f_j is – one might say – "subjective": it has to do with our state of knowledge, particularly in applications of the theory. Whereas the h_i represent known objects that can be used to perform Hamiltonian reduction, compute monodromy and the like, the f_j play the role of putative integrals whose existence will be used to derive special properties of the monodromy representation. For the next result however, the holomorphic character of the h_i plays no special role, so we lump them together with the f_j by setting $f_i := h_i, i = 1, \ldots, k$ and $n := k + r$, the total number of functionally independent integrals available.

Proposition 1.28. *Let p be an arbitrary but fixed point of Γ. The monodromies of the differential bundles $T_\Gamma M$ and F based at p have fixed (rational) fields of transcendence degree at least n over $\mathbb{C}$. In fact one can find n polynomials $P_i(z_1, \ldots, z_i)$ such that the junior terms of the functions $g_i = P_i(f_1, \ldots f_i)$ are algebraically independent on T_pM.*

Proof: For any p the global field $\mathcal{M}(M)$ is a subfield of $\mathcal{M}_p$. A coordinate system x centered at p identifies the latter with $\mathcal{M}$. Likewise the integrals of X_1 are mapped onto a field $K \subset \mathcal{M}$, and $\mathcal{R}(T_pM)$ onto $\mathbb{C}(x)$. The result follows from Prop.1.25 and Prop.1.18 with $s = (1, \ldots, 1)$ (cf. Remark 1.13). $\square$

Theorem 1.29. *The monodromy at any point $p \in \Gamma$ of the totally reduced normal bundle $N = E/E^{\perp}$ has a fixed field of transcendence degree at least r, the number of meromorphic "unreduced" integrals $f_{k+1}, \ldots, f_{k+r}$.*

Proof: E_p is clearly the annihilator of $\xi_h = (dh_1, \ldots, dh_k) = (h_1^0, \ldots, h_k^0)$ in F_p. Since we may assume that the set of functions $\{h_i^0, f_j^0\}$ is algebraically independent on F_p, the result follows from Cor. 1.22. □

References for §1.

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§2 Group-Invariant Rational Functions

§2.1 Ziglin Groups

Let V be a complex vector space of dimension $m \geq 1$. We denote by $S = S(V)$ the $\mathbb{C}$-algebra of polynomial functions on V (i.e., the subalgebra of all functions $f : V \to \mathbb{C}$ generated by the constant functions and the elements in the dual space V^*), and let $L = L(V)$ be the associated field of rational functions on V (i.e., the quotient field of S). By choosing a basis we can identify V with $\mathbb{C}^m$, S with the polynomial algebra $\mathbb{C}[x_1, \ldots, x_m]$ in m-variables, and L with the field $\mathbb{C}(x_1, \ldots, x_m)$ of rational functions on $\mathbb{C}^m$.

For a subgroup $G \subset Gl(V)$ and $g \in G$, the assignment $(g, f) \mapsto g \cdot f := f \circ g^{-1}$ defines a left action of G on elements f in S or L. Letting $P = S, L$, we denote by $P^G \subset P$ the $\mathbb{C}$-algebra of G-invariant elements in P (i.e. those $f \in P$ satisfying $g \cdot f = f$ for all $g \in G$). We identify the constant polynomials with the complex field $\mathbb{C}$ and thereby regard $\mathbb{C} \subset L^G$ as a field extension. Our study of this situation is motivated by Theorem 1.29 on taking $V = N_p$ and $G = \rho(\pi_1(\Gamma, p))$ (see Remark 1.8).

For $r \geq 1$ we say a group G is *r-Ziglin* if trans $\deg_{\mathbb{C}}(L^G) \geq r$, and *exactly* r-Ziglin when equality holds. If trans $\deg_{\mathbb{C}}(L^G) = 0$, then G is declared to be *not Ziglin* .

Proposition 2.1.

(a) trans $\deg_{\mathbb{C}}(L^G) \leq m = \dim(V)$.

(b) *G is r-Ziglin iff all subgroups of G are r-Ziglin.*

(c) *The field extension $L^G \subset L^H$ is algebraic whenever H is a subgroup of G of finite index. In this case G is (exactly) r-Ziglin iff H is (exactly) r-Ziglin.*

Proof: (a) This follows from the equivalence of algebraic and functional independence over $\mathbb{C}$ for subsets of L as stated in Proposition 1.16.
(b) If H is a subgroup of G, then $L^G \subset L^H$, hence trans $\deg_{\mathbb{C}}(L^H) \geq$ trans $\deg_{\mathbb{C}}(L^G)$, and the result follows. (Note that by definition the transcendence degrees are equal iff the field extension $L^G \subset L^H$ is algebraic.)
(c) Let $g_1 = \mathrm{id}, g_2, \ldots, g_n$ be a complete set of n distinct coset representatives of G/H, choose $f \in L^H$, and consider the polynomial $p(y) = \prod_{j=1}^n (y - g_j \cdot f)$. For each $g \in G$ let the action of G on the coefficients of $p(y)$ be denoted $p_g(y) = \prod_{j=1}^n (y - (gg_j) \cdot f)$. The $\{gg_j\}$ is again a complete set of coset representatives, so there is a permutation σ of $\{1, \ldots, n\}$ and elements $h_j \in H$ so that $gg_j = g_{\sigma(j)} h_j$. Since $h_j \cdot f = f$, we have $p_g(y) = p(y)$, hence the coefficients of $p(y)$ are in L^G. As $g_1 =$ id , f is a root of $p(y)$ and is thus algebraic over L^G. □

Examples 2.2. (a) Any finite group G is m-Ziglin. Indeed, the subgroup consisting of the identity alone has this property, and the result then follows from Proposition 2.1(c).

(b) Any subgroup of $Gl(V)$ that admits $V\backslash\{0\}$ as an orbit is not Ziglin. Hence $Gl(V), Sl(V)$, and, when V is symplectic, $Sp(V)$ are not Ziglin.
(c) The group $\left\{\begin{pmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{pmatrix} \mid \lambda \in \mathbb{C}\backslash\{0\}\right\}$ preserves x_1x_2, hence is 1-Ziglin.

§2.2 Linear Algebraic Groups

Ziglin groups are best studied in the context of linear algebraic groups. In this section we collect some background results necessary for the later sections. For more detailed introductions to both algebraic geometry and algebraic groups see [S, pp. 1-11], [Ca, Chapter 1], [KAP] , and [R, 2].

Let V and W respectively be m and n-dimensional complex vector spaces (we could replace $\mathbb{C}$ by any algebraically closed field of characteristic zero). For a set of polynomials $P \subset S(V)$ the zero set $\mathbf{V}(P) = \{v \in V \mid p(v) = 0$ for all $p \in P\}$ is called an *affine (algebraic) variety*. The map $f : V \to W$ is said to be *polynomial* if for some basis $\{e_j\}$ of W we have $f(v) = \sum_{j=1}^{n} f_j(v)e_j$ with $f_j \in S(V)$. It is easy to check that this definition is independent of the basis chosen for W. More generally, for an affine variety $X \subset V$ the map $f : V\backslash X \to W$ is *rational* if for some basis $\{e_j\}$ of W we have $f(v) = \sum_{j=1}^{n} f_j(v)e_j$ with $f_j = (p_j/q_j) \in L(V)$ for polynomials p_j, q_j, such that the zero sets satisfy $\bigcup_{j=1}^{n} \mathbf{V}(\{q_j\}) \subset X$. Again, independence of the basis of W is readily checked. The composition of polynomial maps is polynomial, and using the following proposition it is easy to show that the composition of rational maps is rational (see the references cited above). This result is adapted from [KAP, p. 27].

Proposition 2.3. *Let $X \subset V$ and $Z \subset W$ be affine varieties and $f : V\backslash X \to W$ a rational map. Then $f^{-1}(Z) \subset V\backslash X$ is an affine subvariety of $V\backslash X$, i.e., the intersection of $V\backslash X$ with an affine variety in V.*

Proof: $Z = \mathbf{V}(P)$ for some collection of polynomials $P \subset S(W)$. Define P^* to be the numerators of the rational maps in the set $f^*P = \{p \circ f | p \in P\}$.

Then $f^{-1}(Z) = (V\backslash X) \cap \mathbf{V}(P^*)$. $\square$

If $Y \subset V$ and $Z \subset W$ are affine varieties and $f : V \to W$ is a polynomial mapping such that $f(Y) \subset Z$, then $f \mid Y : Y \to Z$ is said to be a *polynomial mapping* between these affine varieties. When $X \subset V$ is an affine variety and $f : V\backslash X \to W$ is rational with $f((V\backslash X) \cap Y) \subset Z$, then $f \mid (V\backslash X) \cap Y$ is said to be a *rational map* from Y to Z. Note that any polynomial map is rational.

We say an affine variety $X \subset V$ is *reducible* if $X = X_1 \cup X_2$, where X_1 and X_2 are non-empty affine varieties, neither containing the other, and *irreducible* otherwise.

We now define the dimension of an affine variety. The motivation comes from thinking of X as a d-dimensional submanifold of V wherein the coordinate functions of a chart provide d independent $\mathbb{C}$-valued functions. To give the precise definition first assume X is irreducible. Then the ring $\mathbb{C}[X] = S(V)/J(X)$ is an integral domain since $J(X) = \{p \in S(V) \mid p(x) = 0$ for all $x \in X\}$ can be shown to be a prime ideal in $S(V)$ (see [H, Proposition C, p. 8]). $\mathbb{C}[X]$ is called the *coordinate ring* of X and is easily identified with the ring of polynomial functions $p : X \to \mathbb{C}$. The quotient field $\mathbb{C}(X)$ of $\mathbb{C}[X]$ can in turn be identified with the rational functions $f : X \to \mathbb{C}$ (where it is understood that for $f = p/q$ in lowest terms the function is defined only on $X\backslash\mathbf{V}(\{q\})$). By viewing $\mathbb{C}$ as constant functions on X the *function field* $\mathbb{C}(X)$ of X can be regarded as a field extension of $\mathbb{C}$. The *dimension* of X, denoted $\dim(X)$, is defined to be the transcendence degree of this extension. For example, the dimension of a point is 0, and $\dim(V) = m$. When X is not irreducible $\dim(X)$ is defined as the maximal dimension of the irreducible components of X.

Proposition 2.4. *Let $X_1 \subset X_2$ be irreducible affine varieties in V with X_1 properly contained in X_2. Then* $\dim(X_1) < \dim(X_2)$.

Proof: See [H, p. 25].

For sets of polynomials $P \subset S(V)$ the zero sets $\mathbf{V}(P) \subset V$ form a basis of closed sets for the *Zariski topology* on V. The Zariski topology on an affine variety $X \subset V$ is the corresponding relative topology.

Proposition 2.5. *A rational map $f : Y \to Z$ of affine varieties $Y \subset V$ and $Z \subset W$ is continuous in the induced Zariski topologies on these sets.*

Proof: This follows from Proposition 2.3. □

For a subset $A \subset X$ of an affine variety $X \subset V$ we denote the closure in the Zariski topology by $\overline{A}$. A *locally closed* subset of a topological space is one which is the intersection of an open set with a closed set. All open and closed subsets of V are locally closed since V itself is both open and closed. If $X \subset V$ is an affine variety and $A \subset X$ is locally closed we define the *dimension* of A to be that of $\overline{A} \subset X$.

Theorem 2.6. *Let the affine variety $X \subset V$ have the relativized Zariski topology. Then:*

(a) *X is T_1, i.e., singleton subsets of V are closed.*

(b) *X is Noetherian, i.e., any sequence $X_1 \supset X_2 \supset X_3 \supset \ldots$ of distinct closed subsets of X must be finite.*

(c) *If X is irreducible, then any non-empty (relatively) open set $U \subset X$ is Zariski dense, i.e., $\overline{U} = X$, hence $dim(U) = dim(X)$.*

(d) *When X is irreducible any two non-empty open subsets of X have non-empty intersection. In particular, X is not Hausdorff.*

(e) *$Gl(V)$ is a Zariski open subset of $L(V, V)$, the space of linear transformations of V into V, of dimension m^2.*

Proof: See [S, pp. 3-4] for (a)-(d). (e) is obvious. □

Assuming the relative Zariski topology on $Gl(V)$, a closed subgroup $G \subset Gl(V)$ is called a *linear algebraic group.* For linear algebraic groups $G \subset Gl(V)$ and $G' \subset Gl(W)$ we define a *morphism (isomorphism)* $\varphi : G \to G'$ to be a rational map that is also a group homomorphism (isomorphism). In more concrete terms, for $\dim(V) = m$ and $\dim(W) = n$, φ is given by n^2 rational functions $\varphi_{ij} = p_{ij}(x_{11}, \ldots, x_{mm})/q(x_{11}, \ldots, x_{mm})$ where the p_{ij} and q are polynomials and q vanishes nowhere on G, and $\varphi(g_1 \cdot g_2) = \varphi(g_1) \cdot \varphi(g_2)$, $\varphi(id_m) = id_n$. For example, the map $\varphi(g) = (g^{-1})^t$, where t denotes transpose, is a morphism of $Gl(V)$ to $Gl(V)$ as can be seen from the usual way of writing the entries of the inverse of a matrix as quotients of cofactor and determinants.

Note that a closed subgroup of a linear algebraic group is again a linear algebraic group. We refer to arbitrary subgroups (not necessarily closed) of $Gl(V)$ as *abstract groups.* Some examples of linear algebraic groups are $Sl(n, \mathbb{C})$, $SO(n, \mathbb{C})$, $Sp(n, \mathbb{C})$, the group $T(n, \mathbb{C})$ of upper triangular matrices, and any finite subgroup of $Gl(n, \mathbb{C})$.

We remark that any linear algebraic group over $\mathbb{C}$ can be retopologized so as to become a complex Lie group of the same dimension (see [O-V, p. 100]). This suggests a strong parallel between the structure of these two types of groups. In particular, every linear algebraic group has an associated Lie algebra which reflects properties of the group.

Proposition 2.7. *For any linear algebraic group $G \subset Gl(V)$ the following assertions hold:*

(a) *The mappings $g \to g^{-1}, g \to gh$, and $g \to hg$ for fixed $h \in G$ are rational.*

(b) *There is a unique irreducible component G^0 of the identity $e \in G$.*

(c) *G^0 is a normal subgroup of finite index in G. The cosets of G^0 are connected as well as irreducible components of G, hence are all distinct.*

(d) *Each closed subgroup of finite index in G contains G^0.*

Proof: (a) is clear. For (b) - (d) see [H, p. 53]. $\square$

Proposition 2.8. *For any abstract subgroup $G \subset Gl(V)$ we have:*

(a) *$\overline{G}$ is a linear algebraic subgroup of $Gl(V)$.*

(b) *For $S = S(V)$ and $L = L(V)$ we have $S^G = S^{\overline{G}}$ and $L^G = L^{\overline{G}}$.*

Proof: For (a) see [H, p. 54]. For (b) take $f = p/q \in L^G$ (with $q = 1$ when $f \in S$) and $v \in V$. Then the polynomial $P_v(g) = (g \cdot p)(v)q(v) - (g \cdot q)(v)p(v)$ vanishes on G and hence on $\overline{G}$. Since v was arbitrary the result follows. $\square$

§2.3 Linear Algebraic Ziglin Groups

We continue with the notation of the previous section.

Proposition 2.9. *For an abstract subgroup $G \subset Gl(V)$ and any integer $1 \leq r \leq m = dim(V)$ the following statements are equivalent.*

(a) *G is (exactly) r-Ziglin;*

(b) *$\overline{G}$ is (exactly) r-Ziglin;*

(c) *$(\overline{G})^0$ is (exactly) r-Ziglin.*

Proof: (a) and (b) are equivalent by Proposition 2.8, and (b) and (c) are equivalent by Proposition 2.1 (c) and Proposition 2.7 (c). $\square$

Since $(\overline{G})^0$ is connected Proposition 2.9 reduces the study of Ziglin groups to that of connected linear algebraic Ziglin groups.

Theorem 2.10. *Let $G \subset Gl(V)$ be a connected linear algebraic group and $1 \leq r \leq m = dim(V)$. Then the following assertions hold.*

(a) *G is r-Ziglin iff the maximal orbit dimension (of the action of G on V) is at most $(m-r)$.*

(b) *G is exactly r-Ziglin iff the maximal orbit dimension in (a) is $(m-r)$ iff all orbits within some Zariski dense open subset of V have dimension $(m-r)$.*

(c) *If G admits an orbit of dimension $(m-r)$, and if $\{f_j\}_{j=1}^{r} \subset L^G$ is a functionally independent set, then this set must be a transcendence basis for the field extension $\mathbb{C} \subset L^G$.*

(d) *L^G is algebraically closed in L, i.e., $\alpha \in L$ is algebraic over L^G iff $\alpha \in L^G$.*

(e) *For any abstract subgroup $H \subset G$ the field extension $L^G \subset L^H$ (of Proposition 2.1 (c)) is algebraic iff $L^G = L^H$. In particular, G and H are both exactly r-Ziglin for some r iff $L^G = L^H$.*

With regard to (a) and (b) we note that the orbits of G are locally closed (see [H, p. 60]), hence the dimension of these sets can be defined (see the discussion in Section 2.2 between Proposition 2.5 and Theorem 2.6).

Proof: We first note that a result of Rosenlicht [R,1] implies

(1) trans $\deg_{\mathbb{C}}(L^G) = m - r$, where r is the maximal dimension of an orbit of G in V, and

(2) the union of orbits of maximal dimension contains a Zariski open set.

(a) Recall from Proposition 1.16 the equivalence of algebraic independence over $\mathbb{C}$ and functional independence for subsets of $L(V)$. Thus, when G is r-Ziglin the G-orbit of any $p \in V$ is contained in the intersection of the level sets of at least r independent rational functions, hence the dimension of this orbit is at most $(m-r)$ (e.g. see [O-V, p. 90]). The converse follows from the above cited result of Rosenlicht.

(b) The two forward implications follow from Rosenlicht's result with density of the open set coming from Theorem 2.6 (c). Conversely, when there is such an open set the level set argument used in establishing (a) shows that G is at most r-Ziglin. Now, if G is exactly $(r-\ell)$-Ziglin for some $\ell \geq 1$, then the forward implication gives a second dense open set in which all G-orbits have dimension $(m-r)+\ell$, which gives a contradiction by Theorem 2.6(d). Thus G is exactly r-Ziglin.

(c) If G is exactly $(r+\ell)$-Ziglin with $\ell \geq 1$, then by (a) the maximal orbit dimension is $m-(r+\ell) < (m-r)$, a contradiction.

(d) (The proof is courtesy of M. Putcha.) Let $\alpha = \alpha_1, \alpha_2, \ldots, \alpha_s$ be the roots in L of the minimal polynomial $p(x) \in L^G[x]$ of $\alpha \in L^H$. For $j = 1, \ldots s$ set $G_j = \{g \in G \mid g \cdot \alpha = \alpha_j\}$. Each G_j is closed and G is the disjoint union of the G_j. Since G is connected and $G_1 \neq \emptyset$, we have $G = G_1$ so $g \cdot \alpha = \alpha$ for all $g \in G$. Therefore $\alpha \in L^G$.

(e) The forward implication follows from (d); the reverse is obvious.

□

As an immediate application of Theorem 2.10 we have the following alternate proof of Proposition 1.21: Let F have dimension n and let $r =$ trans $\deg_{\mathbb{C}}(K_F)$. Theorem 2.10 implies that the maximal orbit dimension in F is $(n-r)$. Therefore the maximal orbit dimension in S is at most $(n-r)$. Theorem 2.10 now implies that S is $(n-1)-(n-r) = (r-1)$ Ziglin and so trans $\deg_{\mathbb{C}}(K_S) \geq (r-1)$.

Examples 2.11.

(a) Any linear algebraic group $G \subset Gl(V)$ of dimension at most $(m-r)$, where m=dim(V), is r-Ziglin. Indeed, G^0 is r-Ziglin by Theorem 2.10 (a), whence G is r-Ziglin by the equivalences given in Proposition 2.9.

(b) The connected 1-Ziglin subgroups of $Sl(2,\mathbb{C}) = Sp(1,\mathbb{C})$ consist of the identity subgroup and all conjugates of $\left\{\begin{pmatrix} 1 & 0 \\ a & 1 \end{pmatrix} \mid a \in \mathbb{C}\right\}$ and $\left\{\begin{pmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{pmatrix} \mid \lambda \in \mathbb{C}\backslash\{0\}\right\}$. These are precisely the connected subgroups of $Sl(2,\mathbb{C})$ of dimension at most 1. The identity subgroup has dimension 0, and the other two subgroups have dimension 1, hence are 1-Ziglin by (a) above. To see that this covers all cases note that $Sl(2,\mathbb{C})$ is not 1-Ziglin by Example 2.2 (b). Any proper subgroup of $Sl(2,\mathbb{C})$ has dimension ≤ 2 and so is solvable (see [H, Ex. 4, p. 137]). The Lie-Kolchin Theorem (see §3.3) implies that such a group is conjugate to a subgroup of $G = \left\{\begin{pmatrix} \lambda & 0 \\ \mu & \lambda^{-1} \end{pmatrix} \mid \lambda \in \mathbb{C}\backslash\{0\}, \mu \in \mathbb{C}\right\}$. If $v = (1,0)$, $\overline{G \cdot v}$ has dimension 2 so G itself is not a 1-Ziglin group by Theorem 2.10(a). Any proper connected subgroup of G can be seen to be of the above form.

(c) The 1-Ziglin subgroups of $Gl(2,\mathbb{C})$ are those than can be conjugated to a subgroup of one of the following groups:

(1) $T(n) = \left\{\begin{pmatrix} \alpha & 0 \\ \delta & \beta \end{pmatrix} \mid \alpha^n = 1\right\}$ (preserves x^n).

(2) $T(m,n) = \left\{\begin{pmatrix} \alpha & 0 \\ 0 & \beta \end{pmatrix} \mid \alpha^m \beta^n = 1\right\}$ (preserves $x^m y^n$).

(3) $DP(n) = \left\{\begin{pmatrix} \alpha & 0 \\ 0 & \beta \end{pmatrix} \mid (\alpha\beta)^n = 1\right\} \cup \left\{\begin{pmatrix} 0 & \gamma \\ \delta & 0 \end{pmatrix} \mid (\delta\gamma)^n = 1\right\}$ (preserves $(xy)^n$).

(4) the projectively finite groups Z (preserves $\dfrac{\Pi g^* x}{\Pi g^*(x - \lambda y)}$ for suitable

$\lambda \in \mathbb{C}\backslash\{0\}$, where the products are taken over $g \in Z/(Z \cap K_0)$ with $K_0 = \{c \cdot \text{ id } \mid c \in \mathbb{C}\backslash\{0\})$.

A proof can be constructed from (b) above using arguments as in [KAP, p. 32]. For details see [C-R].

§2.4 Symplectic Ziglin Groups

In this section V denotes a complex symplectic vector space of dimension $(2n) \geq 2$, ω is the symplectic form, and $G \subset Sp(V)$ is a linear algebraic group.

A *torus* is a connected diagonalizable linear algebraic subgroup of $Gl(V)$. A torus $T \subset G$ is *maximal* if $\dim(T) \geq \dim(T')$ for any torus $T' \subset G$, and all such tori are conjugate [H, p. 135]. The *rank* of G, denoted rank (G), is defined as the dimension of any maximal torus. (These definitions do not depend on V being symplectic.) We need the following facts concerning tori in $Sp(V)$.

Proposition 2.12.

(a) *Any torus $T \subset Sp(V)$ can be diagonalized with a symplectic basis. More precisely, there is a symplectic basis of V relative to which each $t \in T$ has matrix representation $diag(\lambda_1, \ldots, \lambda_n, \lambda_1^{-1}, \ldots, \lambda_n^{-1})$.*

(b) *The diagonal subgroup $D = \{\ diag\,(\lambda_1, \ldots, \lambda_n, \lambda_1^{-1}, \ldots, \lambda_n^{-1})\} \subset Sp(n, \mathbb{C})$ has rank$(D) = n$.*

(c) *D is exactly n-Ziglin, and if $x_i, i = 1, \ldots, 2n$, denote the standard coordinate functions on $\mathbb{C}^{2n}$, then the n functions $z_j = x_j x_{n+j},\ j = 1, \ldots, n$, constitute a transcendence basis for $(L(\mathbb{C}^{2n}))^D$.*

(d) *Any linear algebraic subgroup $G \subset Sp(V)$ satisfies rank $(G) \leq n$ = rank $(Sp(V))$.*

(e) *Any torus $T \subset Sp(V)$ contains a Zariski dense cyclic subgroup.*

Proof:

(a) Given any basis $\{e_j\}_{j=1}^{2n}$ diagonalizing T, the nondegeneracy of ω guarantees $a \;\; j \geq 2$ such that $\omega(e_1, e_j) \neq 0$, and by rescaling, if necessary, we can assume $\omega(e_1, e_j) = 1$. But then $\text{span}\{e_1, e_j\} = E \subset V$ is T-invariant and $T \mid E$ is diagonalized by the symplectic basis $\{e_1, e_j\}$. For $t \in T$ we have $1 = \omega(e_1, e_j) = (t^*\omega)(e_1, e_j) = \lambda_1\lambda_j$, hence the matrix representation of $t \mid E$ is diag $(\lambda_1, \lambda_1^{-1})$.

By standard symplectic algebra we can write $V = E \oplus E^\perp$ where $E^\perp$ is also T-invariant. But T also acts on $E^\perp$ as a torus (to verify this see [H, Exercise 2, p. 137] and use the fact that restrictions of semisimple operators are again semisimple). Thus the desired basis can be constructed inductively.

(b) Obvious.

(c) The maximal orbit dimension is n, and the n indicated invariants are independent by the equivalence given in Proposition 1.16. The result then follows from Theorem 2.10(c).

(d) Follows from (a) and (c) above.

(e) See [H, Exercise 7, p. 108].

□

An element $g \in L(V, V)$ is *semisimple* if it is diagonalizable, *unipotent* if 1 is its only eigenvalue. The Jordan-Chevalley Theorem [H, p. 96] states that any $g \in Gl(V)$ can be written uniquely as $g = g_s g_u = g_u g_s$ where g_s is semisimple and g_u is unipotent. Moreover, for any linear algebraic subgroup $G \subset Gl(V)$ we have $g \in G$ iff $g_s, g_u \in G$ (see [H, p. 99]). Lastly, any semisimple element of G is contained in a torus $T \subset G$ (see [O-V, Corollary 1, p. 128]).

The spectrum of any $g \in Sp(V)$ has the form $\{\lambda_1, \ldots, \lambda_n, \lambda_1^{-1}, \ldots \lambda_n^{-1}\}$ (see [A-M, p. 169]). The element g is *resonant* if there are integers $m_j \in \mathbb{Z}$, not all zero, so that $\Pi_{j=1}^n \lambda_j^{m_j} = 1$; g is *nonresonant* if this condition is not satisfied. Note that a nonresonant element must be diagonalizable, hence semisimple, since in this case all the eigenvalues will be distinct.

Proposition 2.13. *Let $D = \{diag(\lambda_1, \ldots, \lambda_n, \lambda_1^{-1}, \ldots, \lambda_n^{-1})\}$ be the diagonal subgroup of $Sp(n, \mathbb{C})$. Then the following assertions hold:*

(a) *A closed subgroup $T \subset D$ is proper iff there are integers $m_j \in \mathbb{Z}$, not all zero, such that $\Pi_{j=1}^n \lambda_j^{m_j} = 1$ for all elements in T.*

(b) *Suppose $T \subset D$ is closed. Then $T = D$ iff T contains a nonresonant element.*

Proof:

(a) T must be the intersection of the kernels of some collection of characters of D ([H, Proposition 16.1, p. 103]), and so when T is proper there must be a character which is trivial on T but not on D. This character (in fact all characters of D) must have the form $\Pi_{j=1}^{2n} x_j^{r_j}, r_j \in \mathbb{Z}$ ([$O - V$, Theorem 4, p. 113]), and therefore has value $\Pi_{j=1}^n \lambda_j^{m_j}$ on diag $(\lambda_1, \ldots, \lambda_n, \lambda_1^{-1}, \ldots, \lambda_n^{-1}) \in D$, where $m_j = r_j - r_{n+j}$, $j = 1, \ldots, n$. Since the character is not trivial (on D), not all m_j vanish.

(b) This follows from (a).

□

Theorem 2.14. *Let $G \subset Sp(V)$ be a linear algebraic group. Then*

(a) *G has only resonant elements iff rank $(G) < n$;*

(b) *G has a nonresonant element iff rank $(G) = n$. Moreover, the Zariski closure of the subgroup generated by such an element is a maximal torus $T \subset G$, and any maximal torus $T \subset G$ contains such an element.*

Proof: By Proposition 2.12 (d) it suffices to prove (b). Suppose $t \in G$ is nonresonant, choose a symplectic base diagonalizing t (see Proposition 2.12), and let T be the Zariski closure of the group generated by t. Then Proposition 2.13 (b) shows that $T = D$, the diagonal subgroup of $Sp(n, \mathbb{C})$. By (b) and (d) of Proposition 2.12 the torus T is n-dimensional, hence of maximal rank; in particular, rank $(G) = n$.

Conversely, assume rank $(G) = n$ and $T \subset G$ is a maximal torus. Then by Proposition 2.12 (e) there is an element $t \in T$ that generates a Zariski dense subgroup of T. This element must be nonresonant by Proposition 2.12 (a) and Proposition 2.13 (a). $\square$

Theorem 2.15. *For a connected linear algebraic group $G \subset Sp(V)$ of rank n the following two statements are equivalent:*

(a) *G is n-Ziglin; and*

(b) *G is an n-dimensional torus.*

Proof: $(a) \Rightarrow (b)$. Choose a maximal torus $T \subset G$ and diagonalize T with a symplectic basis (Proposition 2.12 (a)), hence now think of $G \subset Sp(n, \mathbb{C})$. We have $L^G \subset L^T$ (see Proposition 2.1), whereas trans $\deg_{\mathbb{C}}(L^G) \geq n =$ trans $\deg_{\mathbb{C}}(L^T)$ by assumption and Proposition 2.12(c). But the field extension $L^G \subset L^T$ is then algebraic, hence $L^G = L^T$ by Theorem 2.10 (e).

From Proposition 2.12 (c) we know that the functions $z_j = x_j x_{n+j}$, for $j = 1, \ldots, n$, form a transcendence base for the field extension $\mathbb{C} \subset L^T =$

L^G since G is n-Ziglin by hypothesis. Since $g \cdot (x_j x_{n+j}) = x_j x_{n+j}$, unique factorization then implies that either $g \cdot x_j = c_g x_j$ and $g \cdot x_{n+j} = c_g^{-1} x_{n+j}$, or $g \cdot x_j = c_g x_{n+j}$ and $g \cdot x_{n+j} = c_g^{-1} x_j$, for each $g \in G$. The group G then acts by permutation on the coordinate axes, and, being connected by hypothesis, must therefore fix those axes. But then $G = T$.

$(b) \Rightarrow (a)$ This follows from Proposition 2.12 (a) and (c). □

Let $X \subset V$ be a linear subspace and $H \subset Gl(V)$ a linear algebraic group. Then X is a *weight space* for H if there is a rational character (i.e., a morphism) $\sigma : H \to \mathbb{C}/\{0\}$ so that $h \cdot x = \sigma(h)x$ for all $h \in H$ and $x \in X$. Elements of X are called *weight vectors* of H.

Proposition 2.16. *Any n-dimensional torus $T \subset Sp(V)$ admits $2n$ one-dimensional weight spaces which coincide with the eigenspaces of any non-resonant element $t \in T$. V is the direct sum of these spaces.*

Proof: See Proposition 2.12 (a). □

The following result is a restatement of Proposition 1 of [M-S] and generalizes Theorem 2 of [Zi].

Proposition 2.17. *Let $G \subset Sp(V)$ be a linear algebraic group that is n-Ziglin and of rank n. Then the following two results hold.*

(a) *G^0 is an n-dimensional torus and is the unique maximal torus in G; and*

(b) *G permutes the weight spaces of G^0 in pairs, hence $[G : G^0] \leq n!2^n$.*

Proof:

(a) Any maximal torus $T \subset G$, being connected, must lie in G^0, and so G^0 is also of rank n. But G^0 is n-Ziglin, by Proposition 2.9(c), and is therefore and $n-$torus by Theorem 2.15. Since dim $(T) = n =$ dim (G^0), we must have $T = G^0$ by Proposition 2.4.

(b) Let $\{e_j\}_{j=1}^{2n}$ be a symplectic basis diagonalizing the torus G^0 as in Proposition 2.12(a). Then for $g \in G,\ h \in G^0$, and a weight vector $w \in V$ we have $h \cdot w = \sigma(h)w$, and $g^{-1}hg = h' \in G^0$. Thus $h \cdot (g \cdot w) = (hg) \cdot w = (gh')w = g \cdot (h' \cdot w) = g(\sigma(h')w) = \sigma(h')(g \cdot w)$, and hence G permutes the eigenspaces of G^0 (see also [H, p. 188]). That it does so in pairs follows from $1 = \omega(e_j, e_{n+j}) = (g^*\omega)(e_j, e_{n+j})$. It is then immediate that $[G : G^0] \leq n!2^n$.

$\square$

The following corollary is from [I].

Corollary 2.18. *Let dim $(V) = 2$ and assume $G \subset Sp(V)$ is a 1-Ziglin linear algebraic group containing a nonresonant element t. Then for any $g \in G$ the commutator $(t, g) = tgt^{-1}g^{-1}$ satisfies $(t, g) =$ id or $(t, g) = t^2$, with only $(t, g) =$id possible if g does not admit $\pm i$ as eigenvalues. As a consequence, if for such a nonresonant t there is some $g \in G$ so that id $\neq (t, g) \neq t^2$, then G is not 1-Ziglin.*

Proof: Picking a symplectic basis that diagonalizes t, Proposition 2.17 implies we can assume that $V = \mathbb{C}^2$, $t = \begin{pmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{pmatrix}$, and either $g = \begin{pmatrix} \eta & 0 \\ 0 & \eta^{-1} \end{pmatrix}$ or $g = \begin{pmatrix} 0 & \eta \\ -\eta^{-1} & 0 \end{pmatrix}$. Note that G has rank 1 by Theorem 2.14 (b). The result now follows by direct computation. $\square$

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§3 Transition to the Differential Galois Group and Differential Galois Theory

§3.1 Basic Differential Galois Theory

As general references for this chapter we recommend [S, 2] and [V].

A field k with a distinguished derivation $a \mapsto a'$, with $a, a' \in k$, is a *differential field*; an extension field $k \subset K$ which is also a differential field is a *differential extension* of k if the derivation on K restricts to that on k. When k is a differential field the subfield Const $(k) := \{a \in k : a' = 0\}$ is the *field of constants* (of the derivation). In this paper all fields have characteristic zero.

As examples of differential fields consider $\mathbb{C}(x)$ with derivation d/dx, and the field of meromorphic functions $\mathcal{M}$ on a compact Riemann surface together with any nonvanishing vector field on the surface acting as a derivation on $\mathcal{M}$. In both cases the field of constants is identified with $\mathbb{C}$.

Suppose k is a differential field, K_1, K_2 are differential extensions of k and $\sigma : K_1 \to K_2$ is a field isomorphism (resp. embedding) commuting with the derivations on K_1, K_2 and restricting to the identity on k. Then σ is a *differential k-isomorphism* (resp. *k-embedding*). When $K_1 = K_2$ (with the same derivation) σ is a *differential k-automorphism.*

Now let k be a differential field and let

$$y' + Ay = 0 \tag{3.1}$$

be a linear differential equation, where A is an $n \times n$ matrix with entries in k and $y = (y_1, ., y_n)^t$. A differential extension K of k is a *Picard-Vessiot extension* (or P-V *extension*) of k associated with (3.1) if: $K = k(y_{11}, \dots, y_{1n}, y_{21}, \dots, y_{nn})$, where $\{(y_{i1}, \dots, y_{in})^t\}_{i=1}^n$ is a *fundamental set of solutions* of (3.1), i.e., a collection of solutions of (3.1) linearly independent over Const (k); and Const (K) = Const (k). Note from the form of (3.1) that K is again a differential field. The *(differential) galois group* $G(K/k)$ of (3.1) is the group of k-automorphisms of K.

One can show that the set of solutions of (3.1) forms a vector space over Const(k) of dimension at most n [KAP, p. 28], and that whenever Const(k) is algebraically closed a P-V extension for (3.1) exists and is unique up to a differential k-isomorphism [KAP, p. 22]. In particular, $G(K/k)$ is well-defined up to isomorphism. For these reasons one speaks of "the" P-V extension and galois group of (3.1). Note that when $k = \mathbb{C}(x)$ the existence of a P-V extension can be deduced from the classical existence theory of ordinary differential equations.

The foregoing definitions apply to n^{th} order linear differential equations through the standard observation that an n^{th} order equation

$$L(y) = y^{(n)} + a_{n-1}y^{(n-1)} + \cdots + a_0 y = 0 \tag{3.2}$$

can be written as in (3.1) by taking

$$A = \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \ddots & \cdots & & \ddots & \\ \ddots & & & & 1 \\ a_0 & a_1 & & \cdots & a_{n-1} \end{pmatrix}.$$

The associated P-V extension is then of the form $k\langle y_1, \ldots, y_n\rangle := k(y_1, \ldots, y_n, y_1', \ldots y_n', \ldots, y_n^{(n-1)})$, where $y_1, \ldots, y_n$ is a fundamental set of solutions of (3.2).

For the remainder of §3.1 we let $k \subset K$ denote the P-V extension of (3.1) and $G := G(K/k)$ the associated galois group. We will study the structure of G and associated subgroups thereof.

For any field $k \subset L \subset K$ define $L^\vee := \{\sigma \in G : \sigma(a) = a$ for all $a \in L\}$, and for any subgroup $H \subset G$ define $H^\vee := \{a \in K : \sigma(a) = a$ for all $\sigma \in H\}$. Note that $L^\vee$ ('L-check') is a subgroup of G, and $H^\vee$ ('H-check') is a differential subfield of K. L (resp. H) is *closed* if $L^{\vee\vee} = L$ (resp. $H^{\vee\vee} = H$). One easily has the following fundamental aspect of galois theory.

Proposition 3.3. *Any checked object is closed, i.e.* $L^\vee = ((L^\vee)^\vee)^\vee$ *and* $H^\vee = ((H^\vee)^\vee)^\vee$. *Moreover, checking gives a one-to-one correspondence between closed subgroups of G and closed subfields of K.*

To give an alternate characterization of closed objects observe that G acts linearly on the n-dimensional k-vector space of solutions of (3.1), and so by fixing a basis $Y = (y_1, \ldots y_n)$ of solutions we obtain a representation

$$\lambda : G \to Gl(n, C), \quad C = Const(k). \tag{3.4}$$

In fact this representation is faithful [KAP, Lemma 5.4, p. 34], and we thereby identify G with a subgroup of $Gl(n, C)$. The resulting Zariski topology on G is easily seen to be independent of the basis Y used in (3.4).

Proposition 3.5. *The galois group G is Zariski closed when identified (as in (3.4)) with a subgroup of $Gl(n, C)$. Moreover, any closed subgroup $H \subset G$ is also Zariski closed.*

Proof: The second statement follows from the first since if H is closed then $H = G(K/H^{\vee})$ is a P-V extension of $H^{\vee}$. We will prove the first statement under the assumptions that $k = C(x)$, $x' = 1$, and that $K = C(x, y_{11}, y_{12}, y_{21}, y_{22})$ is the P-V extension of a 2×2 first order system (see [KAP, p. 36], for the general case). We follow the exposition in [KOV] to show that $G = G(K/k) \subset Gl(2, C)$ is the intersection of $Gl(2, C)$ and the zero set of a collection of polynomials in four variables with coefficients in C.

Let $Y_{11}, Y_{12}, Y_{21}, Y_{22}$ be new variables and let $\Psi : R = C[x, Y_{11}, Y_{12}, Y_{21}, Y_{22}] \to C[x, y_{11}, y_{12}, y_{21}, y_{22}]$ be the obvious substitution homomorphism. Let P be the kernel of Ψ. For $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in GL(2, C)$, A acts on R by leaving $C[x]$ fixed and sending $(Y_{11}, Y_{12}, Y_{21}, Y_{22})$ to $(aY_{11} + cY_{12}, bY_{11} + dY_{12}, aY_{21} + cY_{22}, bY_{21} + dY_{22})$. A will induce an automorphism of K if and only if A maps P to itself. We may write $P = (p_1, \ldots, p_s)$ where the p_i are linearly independent over C. Let m be the maximum of the degrees of the p_i. A maps $V_m = \{p \in R \mid \deg p \leq m\}$ to itself. Extend $p_1, \ldots, p_s$ to a basis $p_1, \ldots, p_t$ of V_m. There exist polynomials $P_{ij}(a, b, c, d)$ such that the action of A on V_m is given by $A \cdot p_i = \Sigma_j P_{ij}(a, b, c, d) p_j$. We then see that $A \in G$ if and only if $P_{ij}(a, b, c, d) = 0$ for $i = 1, \ldots, s$ and $j = s + 1, \ldots, t$. $\square$

Note that the above proof is not constructive. Given $y' + Ay = 0$, the proof does not show us how to determine the polynomials defining G. In fact it is not known in general how to produce such a set and this is an important open problem in the galois theory.

To show that the closed subgroups of G are precisely the Zariski closed subgroups we need to show the following:

Proposition 3.6. *Let K be a $P-V$ extension of K with galois group G. Any Zariski closed subgroup of G is closed.*

Proof: We shall follow [KAP, p. 37] and only prove this proposition when K is associated with a linear 2×2 system, i.e. $K = k\langle y_{11}, y_{12}, y_{21}, y_{22}\rangle$. It is enough to show that if H is a subgroup of $G \subset GL(2,C)$ then H is Zariski-dense in $H^{\vee\vee}$. Assume not and we will argue to a contradiction. By assumption there is a polynomial $f(a,b,c,d)$ with constant coefficients vanishing on H but not on $H^{\vee\vee}$. Let $\begin{bmatrix} A & B \\ C & D \end{bmatrix} = \begin{bmatrix} y_{11} & y_{12} \\ y_{21} & y_{22} \end{bmatrix}^{-1}$. For new variables, Y_{ij}, define the differential polynomial $F(Y_{11}, Y_{12}, Y_{21}, Y_{22}) = f(AY_{11} + BY_{21}, AY_{12} + BY_{22}, CY_{11} + DY_{21}, CY_{12} + DY_{22})$. Let $\sigma = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ in G. We have $F(\sigma(y_{11}), \sigma(y_{12}), \sigma(y_{21}), \sigma(y_{22})) = f(a,b,c,d) = 0$ for all $\sigma \in H$ but not all $\sigma \in H^{\vee\vee}$. Among all differential polynomials in Y_{ij} with coefficients in K, let $E(Y_{11}, Y_{12}, Y_{21}, Y_{22})$ have the smallest number of terms with this property. We can assume that some coefficient in E is 1. For $\tau \in H$ let E_τ be the polynomial obtained by applying τ to the coefficients of E. $E - E_\tau$ is shorter than E and vanishes for $Y_{ij} = \sigma(y_{ij})$ for all $\sigma \in H$. Therefore it must vanish for $Y_{ij} = \sigma(y_{ij})$ for all σ in $H^{\vee\vee}$. If $E - E_\tau$ is not identically zero, we can find an element $\gamma \in K$ such that $E - \gamma(E - E_\tau)$ is shorter than E and has the same property as E. This contradiction shows that the coefficients of E are left invariant by H. Therefore E has coefficients in $H^\vee$ and these coefficients must be left invariant by $H^{\vee\vee}$. For $\sigma \in H^{\vee\vee}$, $E(\sigma(y_{11}), \sigma(y_{12}), \sigma(y_{21}), \sigma(y_{22})) = \sigma(E(\sigma^{-1}(\sigma(y_{11})), \sigma^{-1}(\sigma(y_{12})), \sigma^{-1}(\sigma(y_{21})), \sigma^{-1}(\sigma(y_{22}))) = \sigma E(y_{11}, y_{12}, y_{21}, y_{22}) = 0$. This contradicts the fact that $E(\sigma(y_{11}), \sigma(y_{12}), \sigma(y_{21}), \sigma(y_{22})) \neq 0$ for some $\sigma \in H^{\vee\vee}$ and finishes the proof of Proposition 3.6. $\square$

The characterization of closed subgroups of G provided by Propositions 3.5 and 3.6 can be useful in the calculation of galois groups, as the following two examples illustrate.

Examples 3.7.

(a) Let $L(y) = y' - ay$ with $a \in k$. The $P - V$ extension of k corresponding to $L(y)$ is of the form $K = k\langle y\rangle = k(y)$, where $y' = ay$ (i.e., $y = \exp(\int a)$). If $\sigma \in G(K/k) = G$ then $\sigma(y'/y) = \sigma(a) = a = y'/y$. From this we can conclude that $(\sigma(y)/y)' = 0$. Therefore $\sigma(y) = c_\sigma \cdot y$ for some $c_\sigma \in C - \{0\} = C^* = Gl(1, C)$. The only Zariski closed subgroups of $Gl(1, C)$ are finite or all of $Gl(1, C)$. If G is finite, then it must be cyclic. This implies that $y^m \in k$ for some m. If G is not finite then $G = Gl(1, C)$.

(b) Let $a \neq 0$ be in k and $L(y) = y'' - (a'/a)y'$. The $P - V$ extension of k corresponding to $L(y) = 0$ is of the form $K = k\langle 1, y\rangle = k(y)$, where $y' = a$. For $\sigma \in G(K/k) = G$, $(\sigma(y) - y)' = 0$, so $\sigma(y) = y + c_\sigma$ for some $c_\sigma \in$ Const$(k) = C$. Since $\sigma(1) = 1$, we can identify σ with the matrix $\begin{bmatrix} 1 & c_\sigma \\ 0 & 1 \end{bmatrix}$. The set of matrices of the form $\begin{bmatrix} 1 & c \\ 0 & 1 \end{bmatrix}$ for $c \in C$ can be identified with the additive subgroup of C. Therefore, G is either the trivial subgroup or isomorphic to C.

We now turn to the problem of characterizing the closed subfields of K. It is a fact that the closed subfields of K are precisely the differential subfields F with $k \subset F \subset K$. The key step in showing this is the following proposition:

Proposition 3.8. *Let K be a $P - V$ extension of k and assume that $C =$ $Const(k)$ is algebraically closed. If $\alpha \in K - k$ then there is a $\sigma \in G(K/k)$ such that $\sigma(\alpha) \neq \alpha$.*

Proof: We only give a rough sketch of the proof of this result, proceeding in two steps. The first step is to show that for any differential fields $k \subset K$ and $\alpha \in K - k$, there is a differential field E containing K and a k-isomorphism $\Psi : K \to E$ such that $\Psi(\alpha) \neq \alpha$. The proof of this fact relies on the fact that one can construct differential ideals not containing certain elements and requires a certain amount of differential ideal theory [KAP, pp. 13-17]. The second step is to use this result when K is a $P-V$ extension of k. In this case the isomorphism Ψ is determined by its effect on $y_1 = (y_{1i}), \ldots, y_n = (y_{ni})$, where $K = k\langle y_{11}, \ldots, y_{nn}\rangle$. Ψ is a matrix (d_{ij}) in $Gl(n, C)$. The fact that Ψ is an isomorphism and $\Psi(\alpha) \neq \alpha$ is equivalent to (d_{ij}) satisfying a system of polynomial equations $f_1(d_{ij}) = 0, \ldots, f_s(d_{ij}) = 0,\ g(d_{ij}) \neq 0$. Since this system is consistent (it has the solution (d_{ij})), and C is algebraically closed, we can find c_{ij} in C satisfying this system. (c_{ij}) defines an element $\sigma \in Gl(n, C)$. This σ will now map K to itself and so define a differential automorphism. $\square$

Proposition 3.8 implies that any differential subfield E of K containing k is closed since K is a $P-V$ extension of E. Combining Propositions 3.3, 3.5, 3.6 and 3.8 we have:

Theorem 3.9. *Let $k \subset K$ be differential fields with K a $P-V$ extension of k and $C = Const(k)$ algebraically closed. Let $G = G(K/k) \subset Gl(n, C)$. Then G is a linear algebraic group and the correspondence described above is a bijective correspondence between Zariski closed subgroups of G and differential subfields E with $k \subset E \subset K$.*

Just as in the galois theory of algebraic equations one is able to identify those subfields corresponding to normal subgroups of the galois group.

Definition 3.10. *Let $k \subset K$ be differential fields. We say that K is* normal *over k if for any $\alpha \in K - k$ there exists a $\sigma \in G(K/k)$ such that $\sigma(\alpha) \neq \alpha$.*

Theorem 3.11. *Let K be a $P-V$ extension of k and assume $Const(k)$ is algebraically closed. A Zariski closed subgroup H of $G = G(K/k)$ is normal in G if and only if $H^\vee$ is normal over k. In this case $G(H^\vee/k)$ is isomorphic to G/H.*

Proof: Assume that H is normal in G. Since K is normal over k, given any

$\alpha \in H^{\vee} - k$ there is a $\sigma \in G$ such that $\sigma(\alpha) \neq \alpha$. Since H is normal in G, any $\sigma \in G$ leaves $H^{\vee}$ invariant and so induces an automorphism of $H^{\vee}$. Therefore $H^{\vee}$ is normal over k.

Now assume that $H^{\vee}$ is normal over k. One first shows that any k-automorphism Ψ of $H^{\vee}$ can be extended to a k-isomorphism $\tilde{\Psi}$ of K into a differential field E containing K. This is done in a way similar to that described in Proposition 3.11. As in Proposition 3.8 the k - automorphism $\tilde{\Psi}$ is determined by a matrix $(d_{ij}) \in Gl(n, C)$, and the fact that $\tilde{\Psi}$ extends Ψ and is an isomorphism is equivalent to a system of equations. Since this system is consistent we can find a solution $(c_{ij}) \in C$ that defines a k-automorphism that extends Ψ. We now let H_1 be the normalizer of H in G. One can show that H_1 is Zariski closed. H_1 contains any automorphism of K that leaves $H^{\vee}$ invariant. Since $H^{\vee}$ is normal over k and all automorphisms of $H^{\vee}$ extend to K, $H_1^{\vee} = k$. Theorem 3.9 now implies that $H_1 = G$.

The final statement comes from the observation that restricting any $\sigma \in G$ to $H^{\vee}$ induces a homomorphism from $G(K/k)$ to $G(H^{\vee}/k)$ whose kernel is H. $\square$

The usual galois theoretic arguments [KAP, pp. 18 - 20] for algebraic extensions and algebraic equations can be generalized to show:

Theorem 3.12. *Let K be a $P - V$ extension of k and assume that $C = Const(k)$ is algebraically closed. Let H be a Zariski closed subgroup of $G = G(K/k)$. Then H has finite index in G if and only if $H^{\vee}$ is algebraic over k. In this case $|G : H| = [H^{\vee} : k]$.*

We close the section with a theorem that relates the galois group of a differential equation on a Riemann surface to that of the equation on a finite cover. We state this in terms of fields. The corresponding statement for connections on bundles over Riemann surfaces can be found in Chapter 4.

Theorem 3.13. *Let $k \subset K$ be differential fields with the same algebraically closed field of constants and assume that $[K : k] < \infty$. Let $y' + Ay = 0$ be a differential equation with coefficients in k. Let k_1 be the $P - V$ extension of k corresponding to this equation and let K_1 be the $P - V$ extension of K corresponding to this equation. Then there is a natural injective homomorphism from the galois group of K_1 over K to the galois group of k_1 over k that is an isomorphism on the connected components of the identity.*

Proof (sketch): It can be shown that there is a large field E having the same constants as k such that k, K, k_1, K_1 are all subfields of E [KOL, 2] and we will assume that this is the case. Since K_1 and $k_1 \cdot K$ are Picard-Vessiot extensions of K corresponding to the same equation, we may identify K_1 with $k_1 \cdot K$ and the galois group $G(K_1/K)$ with $G(k_1 \cdot K/K)$. Each $\sigma \in G(k_1 \cdot K/K)$ will map k_1 to itself and leave $k_1 \cap K$ elementwise fixed. We therefore get a homomorphism from $G(k_1 \cdot K/K)$ to $G(k_1/k_1 \cap K)$. If $\sigma \in G(k_1 \cdot K/K)$ leaves k_1 elementwise fixed, then it leaves a fundamental set of solutions of $y' + Ay = 0$ fixed and so must be the identity in $G(k_1 \cdot K/K)$. Therefore the homomorphism is injective. We can identify $G(k_1/k_1 \cap K)$ with the subgroup of $G(k_1/k)$ of elements that leave $k_1 \cap K$ elementwise fixed. Therefore we get an injective homomorphism of $G(K_1/K) = G(k_1 \cdot K/K)$ with a subgroup $G(k_1/k_1 \cap K)$ of $G(k_1/k)$. The fixed field of this subgroup is an algebraic extension of k, so the subgroup has finite index in the full group. Therefore, topological considerations show that $G(k_1/k_1 \cap K)$ contains the connected component of the identity of $G(k_1/k)$. □

§3.2 Monodromy Groups and Fuchsian Equations

In Section 1.2 we can consider the situation where there is no symplectic structure and Γ is an arbitrary Riemann surface with distinguished nonvanishing vector field X. The monodromy representation, now in $Gl(N_p)$, is defined as before. Here we assume this greater generality, and that the differential bundle (N, Γ, X, D) is *meromorphic* in the following sense: Γ is of the form $\Lambda \backslash \{\text{finite set}\}$ (with $\Gamma = \Lambda$ allowed), where Λ is also a Riemann surface; N extends to a holomorphic bundle N_Λ over Λ of the same rank n; X extends to a meromorphic vector field on Λ; whenever s is a holomorphic section of N which extends to a meromorphic section of N_Λ the same is true of Ds. (The inspiration for this definition is found on page 128 of [V].) Elements of $\Lambda \backslash \Gamma$ are the *singular points* of the differential bundle.

We assume the transition functions of the bundle N_Λ are given by meromorphic functions on Λ. (For Λ noncompact this follows from [F, Theorem 30.4, p. 229]; for Λ compact use Item 3, p. 170, of [G-H] and the fact that Λ is a projective variety.) This easily implies the existence of local frames in which the entries of the matrix $A(t)$ of (1.3) are meromorphic on Λ. We deal exclusively with such frames.

Assume $p \in \Gamma$ (i.e., that p is not a singular point), choose a local frame e over p so as to express the differential bundle as in (1.3), and let $Y = (y_1, \ldots, y_n)$, where $y_j = (y_{1j}, \ldots, y_{nj})^t$ for $j = 1, \ldots, n$, be the germ at p of the corresponding fundamental solution matrix. If k is the field of germs at p of meromorphic functions on Λ then the Picard-Vessiot extension $K = k(Y) = k(y_{11}, \ldots, y_{n1}, y_{12}, \ldots, y_{nn})$ and the associated derivation extensions are frame independent. This follows since, if we are given the analog of (1.3) in some other frame $\hat{e}$, the two sets of germs at p of the fundamental solution matrices Y and Z are related by $A = YP$, where P is some nonsingular $n \times n$ matrix of germs of meromorphic functions on Λ. But then $k(Y) = k(Z)$, and in both cases X defines the derivation.

Given $\sigma \in G(K/k)$ we define the action of σ on a germ at p of a fundamental solution matrix Y of (1.3) to be the result of applying σ to each component of Y. Then in the respective frames e and $\hat{e}$ we have $\sigma(Y) = YA$ and $\sigma(Z) = Z(P^{-1}AP)$ where A is a nonsingular $n \times n$ matrix with entries in k. A canonical representation $\lambda : G(K/k) \to Gl(N_p)$ is then well-defined by the association $\sigma \to A_p \in Gl(n, \mathbb{C})$ given in (3.4). In particular, λ is faithful.

Analytic continuation of elements of K around inverses of loops based at p gives rise to a differential field automorphism of K that keeps k fixed. Calling this map $\varphi : \pi_1(\Gamma, p) \to G(K/k)$ we have the commutative diagram

$$\begin{array}{ccc} \pi_1(\Gamma, p) & \xrightarrow{\rho} & Gl(N_p) \\ & \searrow \varphi & \uparrow \lambda \\ & & G(K/k) \end{array}$$

which shows that the monodromy group $\rho(\pi_1(\Gamma, p))$ can naturally be regarded as a subgroup of the galois group $G(K/k)$ via the embedding λ. Note that the monodromy group can be considerably smaller, e.g., the monodromy group of $y'' + xy = 0$, $' = d/dx$, on $\Gamma = \mathbb{P}^1 \backslash \{\infty\}$ is trivial (as is $\pi_1(\Gamma)$), but the galois group is $Sl(2, \mathbb{C})$ [KAP, pp. 42 - 44].

We now restrict to the case $\Gamma = \mathbb{C} \backslash \{$ finite set $\}$ and $X = d/dz$. Equation (1.3) then becomes

$$Y' + A(z)Y = 0, \quad ' = d/dz \tag{3.14}$$

Definition 3.15. *A point s is a* regular singular point *for (3.14) if in any open angular sector Ω with vertex at s all solutions Y(z) of (3.14) analytic in Ω satisfy $|z - s|^N Y(z) \to 0$ as $z \to s$ for some $N \geq 0$. If $s = \infty$ we replace z by the local uniformizing parameter $t = 1/z$ about ∞ and require $t^N Y(1/t) \to 0$ as $t \to 0$.*

In general it is difficult to determine if a singular point is a regular singular point. One way is to convert the first order system to a single equation and then use the following criterion of Fuchs [HAE]:

Fuchs' Criterion. *The point $z = 0$ is a regular singular point of $L(y) = y^{(n)} + a_{n-1}y^{(n-1)} + \ldots + a_0 y = 0, a_i$ meromorphic at 0, if and only if the order of the pole of a_{n-i} at 0 is at most i.*

Another approach without having to convert to an n^{th} order equation is given in [M] and [HIL]. If all singular points are regular we say the differential equation (3.14) is *fuchsian*. For fuchsian equations we have the following:

Theorem 3.16. *Suppose (3.14) is fuchsian and $M \subset G = G(K/k)$ is the monodromy group, i.e. the image of $\varphi : \pi_1(\Gamma, p) \to G$. Then M is Zariski dense in G.*

Proof: It is enough to show that the fixed field of the monodromy group is the same as the fixed field of the full galois group. If this happens, then the Zariski closure of the monodromy group is a linear algebraic subgroup of the galois group with the same fixed field and so by Theorem 3.9 these groups must be the same. Now suppose w is fixed by the monodromy group. Since the Picard-Vessiot extension is generated by functions satisfying the growth condition in Definition 3.15, any element of this field satisfies these growth conditions. Since w is left fixed by the monodromy and satisfies the above growth conditions it can be extended to a function that is at worst meromorphic at each singular point. Therefore w defines a meromorphic function on Γ and so is in the field of meromorphic functions on Γ. This is just the fixed field of the galois group. $\square$

Theorem 3.17. *Suppose (3.14) is fuchsian with monodromy and galois groups M and G respectively, and $1 \leq r \leq n$. Then the following statements are equivalent:*

$$\begin{array}{lll} (a) & M & is\ (exactly)\ \ r - Ziglin; \\ (b) & G & is\ (exactly)\ \ r - Ziglin; and \\ (c) & G^0 & is\ (exactly)\ \ r - Ziglin. \end{array}$$

Proof: By Theorem 3.16 and Proposition 2.9. $\square$

§3.3 Liouvillian Solutions of Homogeneous Linear Differential Equations

Recall that the notion of "solvability in terms of radicals" for algebraic equations can be formalized in terms of towers of fields and that necessary and sufficient conditions can be given in terms of the galois group. An analogous situation holds for linear differential equations and "solvability in terms of exponentials, integrals and algebraics". We start with the

Definition 3.18. *Let $k \subset K$ be a differential field extension. K is a liouvillian extension of k if there exists a tower $k = K_0 \subset K_1 \subset \ldots \subset K_n = K$ such that, for each i, $K_i = K_{i-1}(t_i)$ where either*

(i) *t_i is algebraic over K_{i-1}, or*

(ii) *$t_i'/t_i \in K_{i-1}$ (i.e. $t_i = e^{\int u_i}$ for some $u_i \in K_{i-1}$), or*

(iii) *$t_i' \in K_{i-1}$ (i.e. $t_i = \int u_i$ for some $u_i \in K_{i-1}$).*

For example $y = e^{\int \sqrt{x}}$ lies in a liouvillian extension of $C(x)$ since $y \in C(x, \sqrt{x}, e^{\int \sqrt{x}})$. The galois theoretic criterion for solvability in terms of these functions is

Theorem 3.19. *Let K be a $P-V$ extension of k with galois group $G = G(K/k)$ and assume that Const(k) is algebraically closed. K is contained in a liouvillian extension of k if and only if the connected component G^0 of the identity is a solvable group.*

Proof: We start by assuming that K is contained in a liouvillian extension E of k with $\text{Const}(E) = \text{Const}(k)$ (in general one can show that if K is contained in a liouvillian extension of k, then it is contained in one with the same constants [KOL, p. 408]). Let E be defined by the tower $k = E_0 \subset$

$\ldots \subset E_m = E$. We proceed by induction on m.

$$\begin{tikzcd} & E_m = E \arrow[d, dash] & \\ & K\langle z\rangle \arrow[dl, dash] \arrow[dr, dash] & \\ E_1 = k\langle z\rangle \arrow[dr, dash] & & K \arrow[dl, dash] \\ & k\langle z\rangle \cap K \arrow[d, dash] & \\ & k & \end{tikzcd}$$

Let $E_1 = k\langle z\rangle$. $K\langle z\rangle$ is a $P-V$ extension of $k\langle z\rangle$, so by the induction hypothesis $G(K\langle z\rangle/k\langle z\rangle)$ has a solvable component of the identity. Restricting any $\sigma \in G(K\langle z\rangle/k\langle z\rangle)$ to K gives a k-automorphism of K. This gives an isomorphism of $G(K\langle z\rangle/k\langle z\rangle)$ onto the subgroup $H = (k\langle z\rangle \cap K)^\vee$ of $G(K/k)$. We now consider the three possibilities for z. If z is algebraic over k, then Theorem 3.12 implies that $|G\colon H| < \infty$. In this case $|G^0\colon H^0| < \infty$, so $G^0 = H^0$, since G^0 is connected. Therefore G^0 is solvable. If $z' \in k$ or $z'/z \in k$, then Examples 3.7 show that $G(k\langle z\rangle/k)$ is abelian. Theorem 3.11 (applied to the $P-V$ extension $k\langle z\rangle$ of k) then implies that $k\langle z\rangle \cap K$ is a normal extension of k with abelian galois group. Therefore H is normal in G and G/H is abelian. We now get the desired conclusion from the following group theoretic fact: If G is a linear algebraic group and H is a Zariski closed subgroup such that H is normal in $G, G/H$ is abelian and H^0 is solvable, then G^0 is solvable. We refer to [KAP, p. 29] for a proof of this fact.

Now let us assume that the connected component G^0 of G is solvable. The proof that K lies in a liouvillian extension of k relies heavily on the following group theoretic result (see [KAP, p. 30] or [ROS] for a proof):

Lie-Kolchin Theorem. *Let C be an algebraically closed field and $G \subset GL(n, C)$ a solvable linear algebraic group that is connected in the Zariski topology. Then there exists an $A \in GL(n, C)$ such that $AGA^{-1} \subset T(n, C)$, i.e. the elements of G can be put in simultaneous triangular form.*

To show that K lies in a liouvillian extension of k we can first replace k by $(G^0)^\vee$. This is a finite algebraic extension of k and allows us to assume that the galois group G is connected and solvable. We now specialize to the case where K is a $P-V$ extension of k associated with a second order linear differential equation $L(y) = 0$ (the general case follows in a similar fashion). G acts on the solution space V of $L(y) = 0$ and, by the Lie-Kolchin Theorem, we may assume that V has a basis y_1, y_2 such that for all $\sigma \in G$ there exists a $c_\sigma \in G$ such that $\sigma(y_1) = c_\sigma(y_1)$. This implies that y_1'/y_1 is left fixed by G and so lies in K. Therefore $k(y_1)$ is a liouvillian extension of k. Let $W = y_1'y_2 - y_2'y_1$. For $\sigma \in G$, $\sigma(W) = (\det\sigma) \cdot W$. Therefore W'/W is left fixed by G and so lies in k. Now $(y_1/y_2)' = W/y_1^2$ so $k(y_1, W, y_2/y_1)$ is a liouvillian extension of k that contains K. $\square$

Theorem 3.19 gives necessary and sufficient conditions for all solutions of a linear differential equation to be liouvillian, but it does not tell us how one can effectively decide this question. For second order linear differential equations Kovacic [KOV] has given a very explicit algorithm to determine this, and the algorithm has been implemented in MAPLE and MACSYMA. For higher order equations an algorithm was given in [S, 1]. All of these algorithms spring from the fact (as can be seen from the proof of Theorem 3.19) that if $L(y) = 0$ has a liouvillian solution, then it has a solution y such that y'/y is algebraic over k.

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§4 Applications

§4.1 A General Class of Examples

Write $\mathbb{C}^{2n} \simeq \mathbb{C}^2 \times \mathbb{C}^{2(n-1)} = \{(x;y) = (x_1, x_2; y_1, \ldots, y_{2(n-1)})\}$ with symplectic form $\omega = \omega_x + \omega_y = dx_1 \wedge dx_2 + \sum_{j=1}^{n-1} dy_{2j-1} \wedge dy_{2j}$. Let $\langle v, w \rangle = \sum_{j=1}^{2(n-1)} v_j w_j$ denote the (non-Hermitian) inner product on $\mathbb{C}^{2(n-1)}$. Consider the meromorphic Hamiltonian H on $\mathbb{C}^{2n}$ given by

$$H(x,y) = h(x) + \frac{1}{2}\langle B(x)y, y\rangle + \mathcal{O}(|y|^3), \tag{4.1}$$

where the $2(n-1)\times 2(n-1)$ matrix $B(x)$ is symmetric. Then the Hamiltonian vectorfield X_H has the coordinate form $(\cdot = d/dt)$:

$$\begin{cases} (a) \quad \dot{x} = J\, h_x(x) + \mathcal{O}(|y|^2) \\ (b) \quad \dot{y} = \tilde{J}\, B(x)\, y + \mathcal{O}(|y|^2), \end{cases} \tag{4.2}$$

where $J = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$, $\tilde{J} = \mathrm{diag}(J, J, \ldots, J)$, and $h_x = \mathrm{grad}_x(h)$. The form of the higher order terms, $\mathcal{O}(|y|^2)$, implies $X_H \mid_{\mathbb{C}^2} = X_h$ where X_h is given w.r.t. $\omega_x = dx_1 \wedge dx_2$. It also provides an uncoupling of the linearized equations so that the NVE of Chapter 1 becomes

$$\dot{y} + A(t)y = 0, \quad A(t) = -\tilde{J}B(x(t)), \tag{4.3}$$

where $x(t)$ is a nonequilibrium solution of X_h with orbit Γ. Here we have identified the totally reduced normal bundle $N \to \Gamma$ with $\Gamma \times \mathbb{C}^{2(n-1)} \to \Gamma$. In particular, the monodromy and differential galois groups of the NVE ∇_Γ are those of (4.3). We recall some previous results.

Theorem 4.4. *Let $U \subset \mathbb{C}^{2n}$ be any neighborhood of the nonequilibrium orbit Γ. Let $1 \leq r \leq n-1$ be an integer and $G \subset Sp(\mathbb{C}^{2(n-1)})$ denote the monodromy group of (4.3). Then we have:*

(a) *G is r-Ziglin when there are r meromorphic integrals of X_H defined on U and functionally independent together with H on this set.*

(b) *When either of G and $\overline{G}$ is not r-Ziglin there can be no such set of r meromorphic integrals as in (a). In particular, for $r = n - 1$ the vectorfield X_H cannot be completely integrable.*

Proof: (a) is a restatement of Theorem 1.29. (b) follows from (a) and Proposition 2.9. $\square$

Example 4.5. As a specific example to be developed in this section take $n = 2$ and let

$$(1) \qquad \begin{aligned} H_\delta(x, y) &= \tfrac{1}{2}(x_2^2 + y_2^2) + \tfrac{1}{4}(x_1^4 + 2\delta x_1^2 y_1^2 + y_1^4) \\ &= h(x) + \tfrac{1}{2}\langle B_\delta(x)y, y\rangle + \mathcal{O}(|y|^3), \end{aligned}$$

where

$$(2) \qquad h(x) = \frac{1}{2}x_2^2 + \frac{1}{4}x_1^4, \quad B_\delta(x) = \begin{pmatrix} \delta x_1^2 & 0 \\ 0 & 1 \end{pmatrix}.$$

Now choose $x_0 \in \mathbb{C}^2$ so that $h_0 = 2h(x_0) \neq 0$. There being no singular points to exclude, Γ is then given by the genus one algebraic curve

$$(3) \qquad x_2^2 + \frac{1}{2}x_1^4 = h_0 \neq 0$$

in $\mathbb{C}^2$ and hence is a punctured torus (the point over ∞ has been removed). The NVE (4.3) along Γ is then

$$(4) \qquad \dot{y} + \begin{pmatrix} 0 & -1 \\ \delta x_1^2(t) & 0 \end{pmatrix} y = 0.$$

§4.2 Symmetry Reduction

To do any reasonable calculation with the NVE in Example 4.5 we must push it down from the totally reduced normal bundle $\Gamma \times \mathbb{C}^2 \to \Gamma$ to the bundle $(\mathbb{P}^1\backslash\{\text{finite set}\}) \times \mathbb{C}^2 \to \mathbb{P}^1\backslash\{\text{finite set}\}$. The general situation is given by a galois (regular), holomorphic, unbranched covering $f : X \to Y$ between Riemann surfaces X and Y. We assume that $N \to X$ and $E \to Y$

are complex vector bundles with respective holomorphic connections $\triangledown_N$ and $\triangledown_E$, and that $N = f^*E$ with $\triangledown_N = f^*\triangledown_E$. Choose $p \in X$ and let $f(p) = \hat{p} \in Y$. We can identify N_p with $E_{\hat{p}}$, and this induces an identification $\hat{f} : Gl(Np) \to Gl(E_{\hat{p}})$. We then have the commutative diagram

$$\begin{array}{ccc} \pi_1(X,p) & \stackrel{\rho(\nabla_N)}{\longrightarrow} & Gl(N_p) \\ f_* \downarrow & & \downarrow \hat{f} \\ \pi_1(Y,\hat{p}) & \stackrel{\rho(\nabla_E)}{\longrightarrow} & Gl(E_{\hat{p}}) \end{array} \tag{4.6}$$

where the $\rho' s$ denote monodromy representations of the respective connections. If $f\colon X \to Y$ is an m-fold (for some $m < \infty$) unbranched galois covering then $f_*\pi_1(X,p)$ is a normal subgroup of $\pi_1(Y,\hat{p})$ with finite index [M, p. 163], and hence, applying the monodromy representation, we have the relation $Mon_p(\triangledown_N) \triangleleft Mon_{\hat{p}}(\triangledown_E)$ between the respective monodromy groups, where the index is bounded by $m = \mid \mathrm{Deck}(X/Y) \mid$.

Proposition 4.7. *With the notation above assume that $f\colon X \to Y$ is an m-fold, unbranched, galois covering for the Riemann surfaces X and Y. Then:*

(a) $(\overline{Mon_p(\triangledown_N)})^0 = (\overline{Mon_{\hat{p}}(\triangledown_E)})^0$*; and*

(b) $Mon_p(\triangledown_N)$ *is (exactly) r-Ziglin iff* $Mon_{\hat{p}}(\triangledown_E)$ *is (exactly) r-Ziglin.*

Proof: (a) $(\overline{Mon_p(\triangledown_N)})^0$ is connected and contains the identity, hence is contained in $(\overline{Mon_{\hat{p}}\triangledown_E)})^0$, but is also of finite index and therefore contains $(\overline{Mon_{\hat{p}}\triangledown_E)})^0$ [H, Proposition 7.3 (b), p. 53].
(b) See Proposition 2.1 (c). $\square$

An important consequence of Proposition 4.7 (a) in the Hamiltonian case is that the group $(\overline{Mon_{\hat{p}}(\triangledown_E)})^0$ is symplectic even though the full group $\overline{Mon_{\hat{p}}(\triangledown_E)}$ may not be. The case $r = 1$ in (b) appeared in [C-R, 2].

In many applications the space Y and the map $f: X \to Y$ arise naturally as the quotient space X/G and quotient map for some finite group G acting freely on X (the action is automatically properly discontinuous since G is finite). That $Y = X/G$ is a Riemann surface follows from [KOD, Theorem 2.2, p. 44]. When this action lifts to a π-equivariant ∇_N–preserving action on the bundle $\pi : N \to X$, linear on fibers, then ∇_E is defined to be the reduced connection ∇_N/G on the quotient bundle $E = N/G$.

In specific examples a finite symmetry group G acting freely on $X = \Gamma\backslash\mathrm{Fix}(G)$ is naturally given, but the question of the existence of a π-equivariant, connection – preserving lift to the bundle $\pi: N \to X$ is more subtle. For rank 2 bundles this extension problem is examined in [B-C-R, Section 4]. However, in Example 4.5 the existence of such a lift can be deduced from elementary calculations (see also [B-C-R, Theorem 4.5]).

Example 4.8. We now apply symmetry reduction to Example 4.5. The orbit Γ in Example 4.5 (3) is invariant with respect to the 8 element group $G = \mathbb{Z}_4 \times \mathbb{Z}_2$ generated by $x_1 \to e^{i\pi/2} \cdot x_1,\ x_2 \to -x_2$. With the energy $h_0 \neq 0$ we see that $X = \Gamma\backslash\mathrm{Fix}(G) = \Gamma\backslash\{6 \text{ points}\}$, where $\mathrm{Fix}(G) = \{p \in \Gamma \mid gp = p \text{ for some } g \in G\backslash\{id\}\}$. This suggests that a suitable candidate for the mapping f of Proposition 4.7 is $f(x_1, x_2) = \frac{1}{4}x_1^4 = z$. The NVE in Example 4.5 (4) becomes, on letting $w = y_1$,

$$\ddot{w} + \delta x_1^2(t) w = 0, \quad \cdot = d/dt \tag{1}$$

where the solution $(x_1(t), x_2(t))$ gives the orbit Γ and satisfies

$$\begin{cases} \dot{x}_1 = x_2 \\ \dot{x}_2 = -x_1^3. \end{cases} \tag{2}$$

From (2) we have $\dot{z} = x_1^3\dot{x}_1 = x_1^3 x_2$, hence

$$\frac{d}{dz} = (x_1^3 x_2)^{-1}\frac{d}{dt}. \tag{3}$$

Setting $h_0 = 2$ in Example 4.5 (3) gives

$$x_2^2 = 2 - \frac{1}{2}x_1^4 = 2(1-z). \tag{4}$$

A direct calculation using (4) gives

$$\frac{d}{dt}(x_1^3x_2)^{-1} = \frac{5z-3}{4z(1-z)}, \tag{5}$$

hence (3) yields

$$\frac{d^2}{dz^2} = \frac{5z-3}{4z(1-z)}\frac{d}{dz} + (x_1^3x_2)^{-2}\frac{d^2}{dt^2}. \tag{6}$$

Applying this to (1) gives a fuchsian reduced equation

$$L_E(w) = w'' + \frac{1}{4}\frac{3-5z}{z(1-z)}w' + \frac{(\delta/8)}{z(1-z)}w = 0, \quad ' = d/dz, \tag{7}$$

which has regular singular points at $\{0, 1, \infty\}$.

This is a concrete instance of the mapping $f\colon X \to Y$ in Proposition 4.7 with $X = \Gamma\backslash\mathrm{Fix}(G)$ and $Y = X/G = \mathbb{C}\backslash\{0,1\}$. The corresponding bundles are $N = X \times \mathbb{C}^2$ and $E = Y \times \mathbb{C}^2$, with $N \simeq f^*E$ and $\nabla_N = f^*\nabla_E$, where the connections ∇_N on N and ∇_E on E are respectively represented by the equations (1) and (7). In the next section we will show that generically in δ the Hamiltonian of Example 4.5 (1) is nonintegrable by computing the differential galois group of (7) above and using the relation in the fuchsian case between this group and the monodromy group of (7) given in Theorem 3.17. Specifically, we will use an algorithm due to Kovacic [KOV] to show that, generically in the parameter δ, this galois group is not 1-Ziglin.

The theory in the above example has the following generalization.

Proposition 4.9. *Let X_H be a meromorphic Hamiltonian vectorfield on $\mathbb{C}^{2n}$ with H as in (4.1), and let $\Gamma \subset \mathbb{C}^2$ be a nonequilibrium orbit. Assume the following:*

(a) *there is a map* $f\colon \Gamma \to \mathbb{C}\backslash\{$ *finite set* $\}$ *so that for* $R = \{$*ramification points* $\{$*of* $f\}$ *and* $X = \Gamma\backslash R$, *the map* $f : X \to Y = \mathbb{C}\backslash\{$ *finite set* $\}$ *is an* m-*fold (unbranched) galois covering projection given in coordinate form by* $(x_1, x_2) \to z$;

(b) *the connection* ∇_N *for the NVE on* $X \times \mathbb{C}^{2(n-1)}$ *can be expressed as a* $2(n-1)$-*nt order homogeneous linear differential equation* $L_N(w) = 0$ *in the time parameter* t; *and*

(c) *there is a substitution* $t = t(z)$ *for the time variable in* $L_N(w) = 0$ *which results in a* $2(n-1)$-*th order fuchsian homogeneous equation* $L_E(w) = 0$ *on* Y.

Let ∇_E *be the connection on* $E = Y \times \mathbb{C}^{2(n-1)}$ *associated to* $L_E(w) = 0$, *choose* $p \in X$, *and set* $\hat{p} = f(p) \in Y$. *Then* $Mon_p(\nabla_N)$ *is (exactly)* r-*Ziglin iff* $Mon_{\hat{p}}(\nabla_E)$ *is (exactly)* r-*Ziglin iff the galois group of* $L_E(w) = 0$ *is (exactly)* r-*Ziglin.*

Proposition 4.9 can be viewed as symmetry reduction by the group $G = \mathrm{Deck}(X/Y)$ with the galois map $f\colon X \to Y = X/G$ being the quotient map [M, pp. 164-165]. One can easily generalize Proposition 4.9 by replacing $\mathbb{C}^{2n}$ with any complex symplectic $(2n)$-manifold and X by any noncompact Riemann surface. Simply recall that a holomorphic vector bundle over a noncompact Riemann surface is holomorphically trivial [F, p. 229]. The trivialization is not required to be symplectic. We only require that the composition of the bundle map associated to f with the trivializing map satisfy the hypotheses of Proposition 4.9.

§4.3 The Kovacic Algorithm

For second order fuchsian equations on $\mathbb{P}^1$ the Ziglin character of the galois group can always be determined by an algorithm due to J. Kovacic [KOV]. The procedure determines if a given (not necessarily fuchsian) second order homogeneous linear differential equation, with rational coefficients, has solutions "expressible in finite terms" (i.e., in terms of liouvillian functions), and provides a basis of solutions if this is the case. We will need only the fuchsian version of the algorithm for our Example 4.5, 4.8. Details for the complete 1-Ziglin case can be found in [C-R, 2].

The theoretical basis for the algorithm is provided by the following two results.

Theorem 4.10. *Let $r(z) \in \mathbb{C}(z)$ and $L(w) = w'' - r(z)w$, $' = d/dz$. Then precisely one of the following four cases occurs for the equation $L(w) = 0$.*

(I) *There is a solution of the form exp $(\int \omega)$ with $\omega \in \mathbb{C}(z)$.*

(II) *Case (I) fails but there is a solution of the form exp $(\int \omega)$ where ω is algebraic of degree 2 over $\mathbb{C}(z)$.*

(III) *Cases (I) and (II) fail but all solutions are algebraic over $\mathbb{C}(z)$.*

(IV) *Cases (I) - (III) fail, in which case $L(w) = 0$ has no liouvillian solutions (as defined in §3.3).*

Proof: See [KOV, p. 5]. □

Theorem 4.11. *The four cases of Theorem 4.10 correspond, in order, with the following four possibilities for the galois group G of $L(w) = 0$.*

(I) *G is reducible (triangulizable).*

(II) *Case (I) fails but G is isomorphic to a subgroup of*
$$DP = \left\{ \begin{pmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{pmatrix} \right\} \cup \left\{ \begin{pmatrix} 0 & \eta \\ -\eta^{-1} & 0 \end{pmatrix} \right\}.$$

(III) *Cases (I) and (II) fail but G is a finite group.*

(IV) $G \simeq Sl(2, \mathbb{C})$.

Proof: See [KOV, p. 7.] $\square$

We need some basic facts concerning second order fuchsian equations. Let $p(z), q(z) \in \mathbb{C}(z)$ and recall that

$$L_E(w) = w'' + p(z)w' + q(z)w = 0, \ ' = d/dz, \tag{4.12}$$

is fuchsian on $Y = \mathbb{P}^1 \backslash \{a_0 = \infty, a_1, \ldots, a_k\}$ iff

$$\begin{cases} p(z) = \sum_{j=1}^{k} \dfrac{A_j}{z - a_j} \\ q(z) = \sum_{j=1}^{k} \dfrac{B_j}{(z - a_j)^2} + \sum_{j=1}^{k} \dfrac{C_j}{z - a_j} \end{cases} \tag{4.13}$$

where the $A_j, B_j, C_j \in \mathbb{C}$ are subject only to the relation $\sum_{j=1}^{k} C_j = 0$ (see [P, p. 77]). Now set

$$\begin{cases} \beta_j = \frac{1}{4}[(1 - A_j)^2 - 4B_j - 1] \\ \delta_j = -C_j + \frac{1}{2}A_j[\sum_{i \neq j}^{k} \dfrac{A_i}{a_j - a_i}] \\ A_\infty = \sum_{j=1}^{k} A_j \\ B_\infty = \sum_{j=1}^{k} (B_j + C_j a_j) \\ \beta_\infty = \sum_{j=1}^{k} (\beta_j + \delta_j a_j) = \frac{1}{4}[(1 - A_\infty)^2 - 4B_\infty - 1], \end{cases} \tag{4.14}$$

and let

$$\begin{aligned} r(z) &= -[q(z) - \tfrac{1}{4}p^2(z) - \tfrac{1}{2}p'(z)] \\ &= \textstyle\sum_{j=1}^{k} \frac{\beta_j}{(z-a_j)^2} + \sum_{j=1}^{k} \frac{\delta_j}{z-a_j}, \quad \sum_{j=1}^{k} \delta_j = 0. \end{aligned} \tag{4.15}$$

The equation

$$L(v) = v'' - r(z)v = 0 \tag{4.16}$$

is called the "normal form" of (4.12).

For $p(z)$ as in (4.13) let

$$u' + \frac{1}{2}p(z)u = 0. \tag{4.17}$$

Then we can represent a solution w of (4.12) as $w = u \cdot v$ where u and v are respective solutions of (4.17) and (4.16).

These ideas can be generalized to the context of the $s = 2(n-1)$-th order operators of Proposition 4.9 on replacing (4.17) by

$$u' + \frac{1}{s}p_{s-1}(z)u = 0 \tag{4.18}$$

where, again, $p_{s-1}(z)$ is as in (4.13). This normalizes the equation

$$L_E(w) = w^{(s)} + p_{s-1}(z)w^{(s-1)} + \cdots + p_0(z)w = 0 \tag{4.19}$$

to

$$L(v) = v^{(s)} + q_{s-2}(z)v^{(s-2)} + \cdots + q_0(z)v = 0, \tag{4.20}$$

which lacks an $(s-1)$-order derivative term. Since fundamental solution matrices of (4.20) have determinant that is constant under analytic continuation, the monodromy representation ρ_2 of (4.20) satisfies $\rho_2 \colon \pi_1(Y, \hat{p}) \to Sl(s, \mathbb{C})$ (see the proof of Lemma 4.24). The monodromy representation ρ_1 of (4.18) for a loop γ_j based at $\hat{p}$, and going around a_j once in the positive direction, with homotopy class $[\gamma_j]$, is defined by

$$\rho_1([\gamma_j]) = \exp\left(-\frac{1}{s}\int_{\gamma_j^{-1}} p_{s-1}(z)dz\right) = \exp\left(\frac{2}{s}\pi i A_j\right). \tag{4.21}$$

The monodromy representation ρ_E of (4.19) then becomes

$$\rho_E = \rho_1 \otimes \rho_2. \tag{4.22}$$

Let G_E be the galois group of (4.19) and G the galois group of the normal form (4.20). We now establish that G_E is (exactly) r-Ziglin iff G is (exactly) r-Ziglin when the connected component of G_E lies in $Sl(s, \mathbb{C})$. For this we first need two lemmas. We will give an algebraic proof, but one can also obtain the result by relating the generators of the respective monodromy groups of (4.19) and (4.20) through (4.22) for $k = \mathbb{C}(z)$.

Lemma 4.23. *Let $k \subset K$ be differential fields with the same constants. Let $u \in K$ satisfy*

(i) $u'/u \in k$

(ii) *u is algebraic over k.*

Then there is an $N \in \mathbb{Z}\backslash\{0\}$ so that $u^N \in k$.

Proof: Let $u'/u = v$, and let

$$\varphi(u) = u^N + a_M u^M + \cdots + a_0 = 0 \tag{1}$$

be the minimal polynomial of u over k, $a_M \neq 0$. Differentiating (1) we have

$$N v u^N + (a_M' + M v a_M) u^M + \cdots + a_0' = 0. \tag{2}$$

By minimality we have

$$a_M' + M v a_M = N v a_M$$

and so

$$\frac{a_M'}{a_M} = (N - M) v.$$

Therefore

$$\frac{(a_M u^{M-N})'}{(a_M u^{M-N})} = 0$$

so $a_M u^{M-N}$ is a constant. This implies $a_M u^{M-N} \in k$, hence $u^{N-M} \in k$. By minimality we have $M = 0$ and $u^N \in k$. □

For Lemma 4.24 and Proposition 4.25 we will assume that the field k_0 of constants of k is algebraically closed.

Lemma 4.24. *Let $L_E(w) = w^{(s)} + p_{s-1}w^{(s-1)} + \cdots + p_0 w$, $p_i \in k$, have galois group G_E whose connected component G_E^0 is in $Sl(s, k_0)$. Let W be an element in the Picard-Vessiot extension K of k corresponding to L_E such that $W'/W = -p_{s-1}$. Then there is an integer $N \neq 0$ so that $W^N \in k$.*

(For the hypothesis that $G_E^0 \subset Sl(s, k_0)$ see the remarks following Proposition 4.7 along with Proposition 4.7 (a) itself.)

Proof: If Wr is the Wronskian of a fundamental system of solutions of $L_E(w) = 0$, then $(Wr)'/Wr = -p_{s-1}$ (see [KAP, Lemma 6.3, p. 41]), so W must be a constant multiple of Wr. If $g \in G_E\, theng(Wr) = (\det g) \cdot Wr$ so $g(W) = W$ for any $g \in G_E^0$. Therefore W belongs to the fixed field F of G_E^0. Since $[F : k] = [G_E : G_E^0] < \infty$, W is algebraic over k. Lemma 4.23 then implies that $W^N \in k$ for some $N \in \mathbb{Z}\backslash\{0\}$. □

Letting $p_i, q_i \in k$ in (4.18) - (4.20) we have $L_E(w) = L_E(uv) = uL(v)$.

Proposition 4.25. *Let L_E and L be as above and let G_E and G be their respective galois groups over k. Assume $G_E^0 \subset Sl(s, k_0)$. Then G_E is (exactly) r-Ziglin iff G is (exactly) r-Ziglin.*

(Note the equivalence stated in Proposition 4.9.)

Proof: First note that given two linear differential operators L_1 and L_2 with coefficients in k, there exists a third operator L_0 with coefficients in k so that $L_0 = \bar{L}_1 \circ L_1 = \bar{L}_2 \circ L_2$ for some linear operators $\bar{L}_1, \bar{L}_2$, with coefficients in k ([P, pp. 35-36] shows how to construct a "least common multiple" of two operators that has this property, and this construction is valid for any differential field).

Let $L_1 = L_E$ and $L_2(u) = (su' + p_{s-1}u)$. Let F be the Picard-Vessiot extension of k corresponding to L_0. Since $L_0 = \bar{L}_1 \circ L_1 = \bar{L}_2 \circ L_2$, we see that F contains a fundamental set of solutions $\{w_1, \ldots, w_s\}$ of $L_E(w) = 0$, and a solution u of $L_2(u) = 0$. Therefore F contains $\{u^{-1}w_1, \ldots, u^{-1}w_s\}$, a fundamental set of solutions to $L(v) = 0$. The fields $K_E = k\langle w_1, \ldots, w_s\rangle$ and $K = k\langle u^{-1}w_1, \ldots, u^{-1}w_s\rangle$ are Picard-Vessiot extensions corresponding to L_E and L. Let G' be the galois group of F over k. G' leaves both K_E and K invariant and restriction to these subfields yields surjective homomorphisms of G' onto G_E and G. Lemma 4.24 implies that since $(u^s)'/(u^s) = -p_{s-1} \in k$, we have $(u^s)^N \in k$ for some integer $N \in \mathbb{Z}\backslash\{0\}$, so $u^m \in k$ for $m = sN$. If $g \in G'$, then $(g \cdot u)^m = g \cdot (u^m) = u^m \in k$, so $g \cdot u = \rho_g \cdot u$ for some ρ_g with $(\rho_g)^m = 1$. Let $w = (w_1, \ldots, w_s)$ and $v = (u^{-1}w_1, \ldots, u^{-1}w_s)$. For $g \in G'$ we have $g \cdot w = A_g\, w$ for some $A_g \in G \subset Gl(s)$ and $g \cdot v = \rho_g^{-1}\, A_g\, w$ for some ρ_g with $\rho_g^m = 1$. Therefore we have shown that with respect to w and v, G_E and G are both subgroups of $G_E \cdot H_m \subset Gl(s, k_0)$, where H_m is the finite group of constant matrices $B = cI, c \in \mathbb{C}\backslash\{0\}$, with $B^m = 1$. In particular, G_E and G are both of finite index in $G_E \cdot H_m$. Therefore, by Theorem 2.10 (e) the group G_E is (exactly) r-Ziglin iff $G \cdot H_m$ is (exactly) r-Ziglin iff G is (exactly) r-Ziglin. □

Corollary 4.26. *Let G be the galois group of (4.16) and assume the hypotheses of Theorem 4.9 for $n = 2$, $r = 1$. If $G \simeq Sl(2, \mathbb{C})$, then the Hamiltonian system X_H does not have an additional meromorphic integral defined and independent of H in any neighborhood of the nonequilibrium curve Γ.*

Proof: G is not 1-Ziglin by Example 2.2 (b). Now use Theorem 1.29 and the equivalences of Proposition 4.25 and Proposition 4.9. □

The Kovacic algorithm involves the characteristic exponents τ_j, τ_∞, of (4.16) at the regular singular points $\{a_0 = \infty, a_1, \ldots, a_k\}$. Using (4.14) these are

$$\begin{cases} \tau_j^{\pm} = \frac{1}{2}[1 \pm \{1 + 4\beta_j\}^{1/2}] = \frac{1}{2}[1 \pm \{(1 - A_j)^2 - 4B_j\}^{1/2}] \text{ at } a_j, \\ \tau_\infty^{\pm} = \frac{1}{2}[-1 \pm \{1 + 4\beta_\infty\}^{1/2}] = \frac{1}{2}[-1 \pm \{(1 - A_\infty)^2 - 4B_\infty\}^{1/2}] \text{ at } a_0 = \infty. \end{cases} \tag{4.27}$$

We now continue with our Example (4.5), (4.8).

Example 4.28. We first put the differential equation (7) of Example 4.8 into the normal form $L(w) = 0$ of (4.16) as required in order to apply the Kovacic algorithm. We calculate (4.13) as

$$(1) \qquad \begin{cases} p(z) = \dfrac{3/4}{z} + \dfrac{1/2,}{z-1}, \\ q(z) = \dfrac{\delta/8}{z} - \dfrac{\delta/8}{z-1}, \end{cases}$$

hence $r(z)$ in (4.15) becomes

$$(2) \qquad r(z) = \frac{1}{16}\left[\frac{(-15/4)}{z^2} - \frac{3}{(z-1)^2} + \frac{(-3-2\delta)}{z} + \frac{(3+2\delta)}{z-1}\right].$$

Thus in (4.14) we have

$$(3) \qquad \begin{cases} \beta_1 = \dfrac{-15}{64}, \beta_2 = \dfrac{-3}{16}, \beta_\infty = \dfrac{1}{64}(-15+8\delta) \\ \delta_1 = \dfrac{1}{16}(-3-2\delta), \ \delta_2 = \dfrac{1}{16}(3+2\delta), \end{cases}$$

and hence the characteristic exponents in (4.27) become

$$(4) \qquad \begin{cases} \tau_1^+ = 5/8 \\ \tau_1^- = 3/8, \end{cases} \quad \begin{cases} \tau_2^+ = 3/4 \\ \tau_2^- = 1/4, \end{cases} \quad \begin{cases} \tau_\infty^+ = \frac{1}{2}[-1 + \frac{1}{4}\{1+8\delta\}^{1/2}] \\ \tau_\infty^- = \frac{1}{2}[-1 - \frac{1}{4}\{1+8\delta\}^{1/2}]. \end{cases}$$

We now apply the Kovacic algorithm to compute the galois group G of (4.16) with $r(z)$ as above. The algorithm consists of three successive cases. Each case is examined in turn, and lack of success in determining a solution in all three cases corresponds to the situation in Corollary 4.26. Our exposition is from [C-R, 2] which gives the algorithm for the fuchsian case.

Case I (The Reducible - Diagonalizable Case):

This portion of the algorithm is phrased in terms of the "modified" characteristic exponents

$$\begin{cases} \alpha_j^\pm = \tau_j^\pm \text{ if } \beta_j \neq 0; \ \alpha_j^\pm = 1 \text{ if } \beta_j = 0 \neq \delta_j; \ \alpha_j^\pm = 0 \text{ if } \beta_j = \delta_j = 0 \\ \alpha_\infty^\pm = \tau_\infty^\pm + 1 \text{ if } \beta_\infty \neq 0; \ \alpha_\infty^+ = 0 \text{ and } \alpha_\infty^- = 1 \text{ if } \beta_\infty = 0. \end{cases} \qquad (4.29)$$

Theorem 4.30. *The following two statements are equivalent:*

(a) *G is reducible; and*

(b) *there is a solution of (4.16) of the form* $\exp\left(\int \theta\right)$, *with* $\theta \in \mathbb{C}(z)$, *which must then be a common eigenvector for* G.

Moreover, (b) will hold iff

(1) *there is a choice* $s(j)$ *and* $s(\infty)$ *of plus and minus signs so that* $d = \left[\alpha_\infty^{s(\infty)} - \sum_{j=1}^{k} \alpha_j^{s(j)}\right]$ *is a nonnegative integer;*

(2) *there is a unique monic degree d polynomial P (which can be determined by the method of undetermined coefficients) satisfying* $P'' + 2\omega P' + (\omega' + \omega^2 - r)P = 0$, *where* $\omega = \omega(z) = \sum_{j=1}^{k} \frac{\alpha_j^{s(j)}}{z - a_j}$; *and*

(3) $\theta = \omega + (P'/P)$.

Proof: See Theorem 4.9, pp. 1796 - 1797 in [C-R, 2] and [KOV]. □

This algorithm will generate two independent solutions θ iff G is diagonalizable. If there are no such solutions, G is irreducible.

Returning to our Example 4.28 we compute the terms in (4.29) as

$$\left\{\begin{array}{l} \alpha_1^+ = 5/8 \\ \alpha_1^- = 3/8 \end{array}\right., \quad \left\{\begin{array}{l} \alpha_2^+ = 3/4 \\ \alpha_2^- = 1/4 \end{array}\right., \quad \left\{\begin{array}{l} \alpha_\infty^+ = \frac{1}{2}[1 + \frac{1}{4}\{1 + 8\delta\}^{1/2}] \\ \alpha_\infty^- = \frac{1}{2}[1 - \frac{1}{4}\{1 + 8\delta\}^{1/2}], \end{array}\right.$$

for $\beta_\infty = \frac{1}{64}(-15 + 8\delta) \neq 0$. We see that for generic $\delta \in \mathbb{C}$ the number d in Theorem 4.30 (1) is not a nonnegative integer. For these values of δ Case I fails, and we pass to the second case.

Case II (The DP - Case):

This stage of the algorithm is phrased in terms of the following sets.

$$\begin{cases} E_j = \{2 + e(1+4\beta_j)^{1/2} \mid e = 0, \pm 2\} \cap \mathbb{Z} \text{ if } \beta_j \neq 0 \\ E_j = \{4\} \text{ if } \beta_j = 0 \neq \delta_j \\ E_j = \{0\} \text{ if } \beta_j = \delta_j = 0 \\ E_\infty = \{2 + e(1+4\beta_\infty)^{1/2} \mid e = 0, \pm 2\} \cap \mathbb{Z} \text{ if } \beta_\infty \neq 0 \\ E_\infty = \{0, 2, 4\} \text{ if } \beta_\infty = 0 \end{cases} \tag{4.31}$$

Theorem 4.32. *The following two statements are equivalent:*

(a) *Case I fails but G is conjugate to a DP-group (see Theorem 4.11 (II)).*

(b) *Case I fails but there is a solution of (4.16) of the form exp $(\int \omega)$ where ω is algebraic of degree 2 over $\mathbb{C}(z)$.*

Moreover, there is a solution as in (b) iff

(1) *there are choices $e_j \in E_j$ and $e_\infty \in E_\infty$, not all even integers, so that $d = \frac{1}{2}[e_\infty - \sum_{j=1}^{k} e_j]$ is a nonnegative integer; and*

(2) *there is a monic, degree d polynomial P (which can be found by the method of undetermined coefficients) satisfying*

$$P''' + 3\theta P'' + (3\theta^2 + 3\theta' - 4r)P' + (\theta'' + 3\theta\theta' + \theta^3 - 4r\theta - 2r')P = 0,$$

where $\theta = \frac{1}{2}\sum_{j=1}^{k}[e_j/(z-a_j)]$.

Finally, if P is as in (2) set $\phi = \theta + (P'/P)$ and choose a solution ω of $\omega^2 + \phi\omega + [\frac{1}{2}\phi' + \frac{1}{2}\phi^2 - r] = 0$; then exp $(\int \omega)$ will be the required solution of (4.16) as in (b) above.

Proof: See Theorem 4.13, pp. 1797 - 1798 in [C-R, 2] and [KOV]. $\square$

We compute from Example 4.28 (3) that

$$\begin{cases} E_1 = \{2\} \\ E_2 = \{1, 2, 3\} \\ E_\infty = \{2\} \quad \text{generically in } \delta \end{cases}$$

Thus generically for $\delta \in \mathbb{C}$ there are no choices of elements in the sets E_j, E_∞, so that d as defined in Theorem 4.32 (1) is a nonnegative integer. Thus we pass to Case III.

Case III (The Finite Case):

For $n = 4, 6$, or 12 set

$$\begin{cases} F_j(n) = \{6 + \frac{12e}{n}(1+4\beta_j)^{1/2} \mid e = 0, \pm 1, \ldots, \pm \frac{n}{2}\} \cap \mathbb{Z} \text{ if } \beta_j \neq 0 \\ F_j(n) = \{12\} \text{ if } \beta_j = 0 \neq \delta_j \\ F_j(n) = \{0\} \text{ if } \beta_j = \delta_j = 0 \\ F_\infty(n) = \{6 + \frac{12e}{n}(1+4\beta_\infty)^{1/2} \mid e = 0, \pm 1, \ldots, \pm \frac{n}{2}\} \cap \mathbb{Z}, \end{cases} \tag{4.33}$$

where the definition of $F_\infty(n)$ is regardless of whether or not $\beta_\infty = 0$.

Theorem 4.34. *When Cases I and II fail adopt the following procedure:*

(1) *Let $n = 4$ and record all choices of $f_j \in F_j(n)$ and $f_\infty \in F_\infty(n)$ for which $d = \left(\frac{n}{12}\right)[f_\infty - \sum_{j=1}^{k} f_j]$ is a nonnegative integer.*

(2) *For each such choice set $\theta = \left(\frac{n}{12}\right)\sum_{j=1}^{k}[f_j/(z-a_j)]$, and let $S = \Pi(z-a_j)$ (where the product is taken over only those a_j which are poles of $r(z)$), and determine (e.g., by the method of undetermined coefficients) if there is a monic, degree d polynomial P such that if we set $P_n = -P$ and recursively define*

$$P_{i-1} = -SP_i' + [(n-i)S' - S\theta]P_i - (n-i)(i+1)S^2 r P_{i+1}$$

for $i = n, n-1, \ldots, 0$, then $P_{-1} = 0$; and

(3) *repeat, if necessary, steps (1) and (2) using $n = 6$ and then $n = 12$.*

G is then finite iff this procedure results in a polynomial P as in (2) - (3). In this case all solutions of (4.16) are algebraic over $\mathbb{C}(z)$. Moreover, given such a P any solution ω of the equation $\sum_{i=0}^{n}[(S^i P_i)/(n-i)!]\omega^i = 0$ will give a solution exp $(\int \omega)$ to (4.16).

Proof: See Theorem 4.18, p. 1798 in [C-R, 2] and [KOV]. □

For our Example 4.28 we compute the sets (4.33) to be

$$\begin{cases} F_1(4) = \{6\} \\ F_2(4) = \{3,6,9\} \end{cases}, \quad \begin{cases} F_1(6) = \{5,6,7\} \\ F_2(6) = \{3,4,5,6,7,8,9\} \end{cases}$$

$$\begin{cases} F_1(12) = \{5,6,7\} \\ F_2(12) = \{3,4,5,6,7,8,9\} \end{cases}$$

$F_\infty(n) = \{6\}$ generically in δ.

Thus generically in δ there are no choices of $f_j \in F_j(n)$, and $f_\infty \in F_\infty(n)$ for $n = 4, 6, 12$, so that d in Theorem 4.34 (1) is a nonnegative integer. Thus Case III of the Kovacic algorithm also fails generically in the parameter δ. We summarize our work in a theorem.

Theorem 4.35. *For $h_0 = 2$ in Example 4.5 (3) the vectorfield X_{H_δ} is nonintegrable for generic $\delta \in \mathbb{C}$. More precisely, there is no meromorphic integral of X_{H_δ}, functionally independent of H_δ, in any neighborhood of the orbit Γ given in (3) of Example 4.5 at $h_0 = 2$.*

Proof: The above calculations with the Kovacic algorithm have shown that the galois groups of (4.16) with $r(z)$ as in Example 4.28 (2) is generically $Sl(2, \mathbb{C})$ (see Theorem 4.11). The conclusion now follows from Corollary 4.26. □

The above result could, of course, have been derived for other values of h_0. For other versions of the Kovacic algorithm see [D], [D, L-R], and [S-U].

For further examples and applications of the above theory see [B-C-R], [C-R, 1] [C-R, 2], [N], [R-C] and [Y]. For the extension to three degrees of freedom see [C-R-S].

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