Group-Theoretic Obstructions to Integrability

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Abstract

Let V be a 4-dimensional complex symplectic vector space. This paper classifies those connected linear algebraic subgroups of the symplectic group $Sp(V)$ that admit two independent rational invariants. As an application we show the non-integrability of a three degree of freedom Hamiltonian system.

Key Words: Rational invariant, transcendence degree, linear algebraic group, symplectic group, monodromy group, galois group, Hamiltonian, non-integrability, fuchsian, linearized equations, Ziglin.

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Contents

0. Introduction

Let V be a $2n$-dimensional complex symplectic vector space, and let $G \subset Gl(V)$ be a linear algebraic group with connected component of the identity G^0 contained in $Sp(V)$, the group of symplectic automorphisms of V. Sections 1-5 of this paper are concerned with the transcendence degree over $\mathbb{C}$ of the field of rational invariants of G, and, in particular, with classifying those G for which this transcendence degree is n. Since G^0 is a normal subgroup of G with finite index this transcendence degree is the same for both groups (Proposition 2.3(c)). For this reason, and because G itself may not be symplectic, the classification scheme is given in terms of connected linear algebraic subgroups of $Sp(V)$.

For rank $(G) = n$ it is shown in [B-C-R-S], and is implicit in [Zi], that this transcendence degree is n if and only if G^0 is an n-torus. Since $G^0 \subset Sp(V)$ implies $0 \leq$ rank $(G) \leq n$ (Theorem 4.6), the classification problem reduces to a study of the cases $0 \leq$ rank $(G) < n$. The case $n = 1$ was solved by A. Baider and is detailed in [C-R]. In this paper we solve the case $n = 2$ (see Summary 5.6). This is done by formulating the ideas in terms of representation theory and giving quite general treatments of the symplectically irreducible and unipotent (i.e., rank=0) cases. We also obtain more general results for solvable G^0.

Sections 6–7 of the paper are concerned with applications. Let X_H be a holomorphic Hamiltonian vector field on some complex analytic symplectic manifold. Then the algebraic group G arises as the Zariski closure of the monodromy representation of a linearization of X_H along some specific trajectory Γ, and in this case we automatically have G and G^0 symplectic. The number of meromorphic integrals of X_H that are independent, together with H, in a neighborhood of Γ is reflected in the transcendence degree associated to G as above. The independent integrals are not required to be in involution. The transcendence degree can thus be viewed as an obstruction to the integrability of X_H.

Now suppose $f : \Gamma \to Y$ is a finite galois covering map of Riemann surfaces and the linearized equations along Γ are the pull-back of some linear differential equation on Y with differential galois group G_Y. Then: G is a normal subgroup of G_Y of finite index (Proposition 7.4); $G^0 = (G_Y)^0$ (Proposition 7.4); and the transcendence degrees over $\mathbb{C}$ of the fixed fields of G, G_Y and $(G_Y)^0$ are identical (Theorem 7.5). For purposes of computing transcendence degrees we can therefore replace G with $(G_Y)^0$. The group G_Y may not be symplectic, but the identity $G^0 = (G_Y)^0$ insures the context of Sections 1–5.

For explicit calculations the desirable choice for Y is $\mathbb{CP}^1\backslash\{\text{finite set}\}$, and the best possible situation occurs when the linear differential equation on Y is of second order. In that case one can use an algorithm of J. Kovacic [KOV] to determine G_Y completely and, when the equation is fuchsian, to calculate the relevant transcendence degree [C-R]. Unfortunately, for higher order equations as occur in this paper the algorithm is far more involved (see [SU1], [SU2] for recent work on second and third order equations and [S] for a review of previous work on higher order equations), and as yet one cannot expect a complete solution. But we are able to offer a computable necessary condition on fourth order equations for the relevant transcendence degree to be less than 2 (Proposition 6.6). The ideas are applied in Section 7 to prove that a specific complex analytic Hamiltonian system of three degrees of freedom is not completely integrable in terms of meromorphic integrals. In that example the mapping $f : \Gamma \to \mathbb{CP}^1\backslash\{\text{finite set}\}$ is viewed as the quotient mapping of Γ by the action of a finite symmetric group. Our example has been chosen for ease of exposition. For a concrete three degree of freedom example from physics see [K-S]. There related methods are used, and a notable simplification is achieved by assuming the integrals are in involution.

1. Generalities on Symplectic Modules

Throughout the section G is a complex linear algebraic group and V is a $2n$-dimensional symplectic G-module, i.e. a $2n$-dimensional complex symplectic vector space together with a rational representation $\rho : G \longrightarrow Sp(V)$. We let ω denote the symplectic form on V, and recall that a basis $e = \{e_j\}_{j=1}^{2n}$ of V is *symplectic* if and only if $\omega(e_i, e_j) = 0 = \omega(e_{n+i}, e_{n+j})$ and $\omega(e_i, e_{n+j}) = \delta_{ij}$ for $1 \leq i, j \leq n$.

A subspace $Y \subset V$ is: *non-degenerate*, or *symplectic*, if $\omega|Y \times Y$ is a symplectic form on Y; *isotropic* if ω vanishes on $Y \times Y$ (any 1-dimensional subspace has this property); *Lagrangian* if maximal isotropic, i.e. if isotropic but not a proper subspace of an isotropic space; *properly isotropic* if isotropic but not trivial and not Lagrangian.

We let $W^\perp$ denote the *symplectic orthocomplement* of any subspace $W \subset V$, i.e. the subspace $\{v \in V : \omega(v,w) = 0 \text{ for all } w \in W\}$. One has: $\dim V = \dim(W) + \dim(W^\perp)$; $W \subset Y \subset V$ if and only if $Y^\perp \subset W^\perp$ when $Y \subset V$ is any subspace; W is isotropic if and only if $W \subset W^\perp$; W is Lagrangian if and only if $W = W^\perp$ [A-M, p. 403].

Proposition 1.1 *When W is a subspace of V and $T \in Sp(V)$ the following assertions hold:*

(a) *W is T-invariant if and only if $W^\perp$ is T-invariant, in which case $W \cap W^\perp$ is also T-invariant.*

(b) *When W is nontrivial precisely one of the following three statements holds:*

  i. *W is non-degenerate;*

  ii. *W is Lagrangian;*

  iii. *$W \cap W^\perp$ is properly isotropic.*

(c) *When W is nontrivial the following statements are equivalent:*

  i. *W is non-degenerate;*

  ii. *$W \cap W^\perp = \{0\}$;*

  iii. *$W^\perp$ is non-degenerate;*

  iv. *$V = W \oplus W^\perp$.*

Proof: This is standard symplectic linear algebra; see [A-M, Chapter 5.3].

Q.E.D.

Proposition 1.2 *Suppose W is a nontrivial subspace of V.*

(a) *W is a (symplectic) G-submodule if and only if $W^\perp$ has the same property.*

(b) *When W is a G-submodule, precisely one of the following three statements holds:*

  i. *W is a symplectic G-submodule, in which case V is the direct sum of the G-submodules W and $W^\perp$.*

  ii. *W is a Lagrangian G-submodule.*

iii. $W \cap W^{\perp}$ *is a properly isotropic* G*-submodule.*

Proof: By Proposition 1.1.

Q.E.D.

V and ρ are *s-(ir)reducible* (i.e. *symplectically (ir)reducible*) if V is (not) the direct sum of symplectic G-modules. To illustrate, take $V = \mathbb{C}^2$ with the usual symplectic form, $G = \{\begin{pmatrix} \lambda & 0 \\ \alpha & \lambda^{-1} \end{pmatrix} \in Sp(1, \mathbb{C})\}$, and let $\rho : G \longrightarrow Sp(V)$ be inclusion: then V is reducible but s-irreducible.

Proposition 1.3 *Suppose* V *is reducible but s-irreducible. Then* V *contains a nontrivial isotropic* G*-submodule.*

Proof: By Proposition 1.2 (b)

Q.E.D.

Any linear algebraic group G admits a semidirect product decomposition

$$G = R_u(G) \rtimes P, \tag{1.4}$$

where $R_u(G)$ is the unipotent radical of G and P is linearly reductive (i.e., for every P-submodule N of a P-module M there is a P submodule N' such that $M = N \oplus N'$). Moreover, P is unique up to conjugation by an element of $R_u(G)$ (e.g. see [Hoch, Theorem 4.3, p. 117]). This reduces the study of G-modules, in some sense, to the cases of unipotent (and therefore solvable) and linearly reductive G. We note that since we are dealing with linear algebraic groups G in characteristic zero, the notions of linearly reductive and reductive (i.e. $R_u(G) = \{e\}$) are the same [F, Exercise 14, p. 216]. We begin with the solvable case, which requires some additional elementary symplectic linear algebra.

Proposition 1.5 *When* W *is a properly isotropic subspace of* V *the following assertions hold.*

(a) *The symplectic form* ω *on* V *passes naturally to* $W^{\perp}/W$*, i.e. there is a unique symplectic form* ω_π *on* $W^{\perp}/W$ *satisfying* $\pi^*\omega_\pi = i^*\omega$*, where* $\pi : W^{\perp} \longrightarrow W^{\perp}/W$ *is the canonical projection and* $i : W^{\perp} \hookrightarrow V$ *is inclusion.*

(b) When $T \in Sp(V)$ and W is T-invariant the induced operator on $W^{\perp}/W$ is in $Sp(W^{\perp}/W)$, i.e. is symplectic relative to ω_{π}.

Proof: See, e.g. [A-M, Chapter 5.3].

Q.E.D.

Theorem 1.6 (The Symplectic Lie-Kolchin Theorem) *When $G \subset Sp(V)$ is connected and solvable the following assertions hold.*

(a) G fixes a full flag $W_1 \subset W_2 \subset \cdots \subset W_n$ in some Lagrangian G-submodule $W = W_n$ of V.

(b) There is a basis $e = \{e_j\}_{j=1}^{2n}$ of V, symplectic when ordered as $(e_1, e_2, \cdots, e_n, e_{2n}, e_{2n-1}, \cdots, e_{n+1})$, relative to which the matrix representations of G are lower triangular with all diagonals of the form $(\lambda_1, \lambda_2, \cdots, \lambda_n, \lambda_n^{-1}, \lambda_{n-1}^{-1}, \cdots, \lambda_1^{-1})$. Moreover, $\{e_j\}_{j=n+1}^{2n}$ spans W.

The result is known (e.g. see [Sp-2, Ex. 6, p. 163]).

Proof: By the standard Lie-Kolchin Theorem [H-2, Theorem 17.6, p. 113] V contains a 1-dimensional G-submodule W_1, which is automatically isotropic. Let e_{2n} span W_1 and (by non-degeneracy and, if necessary, rescaling) choose an $e_1 \in V \backslash W_1^{\perp}$ such that $\omega(e_1, e_{2n}) = 1$. Now pass to $W_1^{\perp}/W_1$ (see Proposition 1.5) and argue by induction.

Q.E.D.

Remark 1.7 *In terms of the basis e of Theorem 1.6(b) the symplectic form ω has matrix representation* $\hat{J} = \begin{pmatrix} 0 & \hat{I} \\ -\hat{I} & 0 \end{pmatrix}$, *where* $\hat{I} = \begin{pmatrix} 0 & \cdots & & 0 & 1 \\ \vdots & \cdot^{\cdot^{\cdot}} & & 1 & 0 \\ & & \cdot^{\cdot^{\cdot}} & & \\ 0 & 1 & & \cdot^{\cdot^{\cdot}} & \vdots \\ 1 & 0 & & \cdots & 0 \end{pmatrix}$, *and a matrix* $M = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \in Gl(2n, \mathbb{C})$ *is therefore symplectic if and only if $M^T \hat{J} M = \hat{J}$, i.e. if and only if* *(a)* $\alpha^T \hat{I}\gamma$ *and* $\beta^T \hat{I}\delta$ *are symmetric and*

(b) $-\gamma^T \hat{I}\beta + \alpha^T \hat{I}\delta = \hat{I}$.

When $\beta = 0$ *these conditions reduce to* $\sigma := \alpha^T \hat{I}\gamma$ *being symmetric and* $\delta = \hat{I}\alpha^{-T}\hat{I}$ *respectively* ($\alpha^{-T} := (\alpha^T)^{-1}$)*, from which we see that all matrices of Theorem 1.6(b) can be written*

(c) $M = \begin{pmatrix} \alpha & 0 \\ \hat{I}\alpha^{-T}\sigma & \hat{I}\alpha^{-T}\hat{I} \end{pmatrix}$,

with α *lower triangular. Since the admissible* α *and* σ *form spaces of dimension* $\frac{n(n+1)}{2}$*, we conclude from (c) that the Borel subgroups of* $Sp(V)$ *have dimension* $n(n+1)$*. For unipotent* α *the dimension reduces to* $\frac{n(n-1)}{2}$*, but all symmetric* σ *are still admissible; hence*

(d) $\dim(R_u(Sp(V))) = n^2$.

To study the case of reductive G we need the following additional facts concerning orthocomplements of subspaces $W, Y \subset V$ (see [A-M, Chapter 5.3]).

$$\text{(1.8)} \qquad \begin{array}{ll} \text{(a)} & W^{\perp\perp}(:= (W^\perp)^\perp) = W; \\ \text{(b)} & W^\perp \cap Y^\perp = (W+Y)^\perp; \\ \text{(c)} & (W \cap Y)^\perp = W^\perp + Y^\perp; \\ \text{(d)} & V = W \oplus Y \text{ if and only if } V = W^\perp \oplus Y^\perp. \end{array}$$

Proposition 1.9 *Suppose* $V = W \oplus Y$*, where* W *is properly isotropic. Then for* $\hat{Y} = W^\perp \cap Y$ *and* $\hat{W} := W \oplus (\hat{Y}^\perp \cap Y)$ *the following assertions hold.*

(a) $W \subset \hat{W}$, $\hat{Y} \neq \{0\}$, *and* $\hat{Y} \subset Y$.

(b) $\hat{W}$ *and* $\hat{Y}$ *are non-degenerate.*

(c) $V = \hat{W} \oplus \hat{Y}$.

(d) $\hat{W}^\perp = \hat{Y}$ *and* $\hat{Y}^\perp = \hat{W}$.

(e) $Y = \hat{Y} \oplus (\hat{Y}^{\perp} \cap Y)$.

(f) $V = W \oplus (\hat{W} \cap Y) \oplus \hat{Y}$.

(g) $\hat{W} = W \oplus (\hat{W} \cap Y)$.

(h) $\hat{W}$ *is a non-degenerate subspace which contains* W *as a Lagrangian subspace.*

(i) *When* W *and* Y *are* G*-submodules the same is true of* $W^{\perp}$, $\hat{Y}$, $\hat{W}$ *and* $\hat{W} \cap Y$. *In particular, there is a* G*-submodule complement to* W *in* $\hat{W}$.

Proof: By (1.8) we have $\hat{Y}^{\perp} = (W^{\perp} \cap Y)^{\perp} = W + Y^{\perp}$. But $W + Y^{\perp} \neq V$ since $\dim(Y^{\perp}) = \dim(W) < n$, and therefore $\hat{Y} \neq \{0\}$. Since the inclusions are obvious this gives (a).

We claim $\hat{Y} \cap \hat{Y}^{\perp} = \{0\}$, which by Proposition 1.1 (c) is equivalent to the non-degeneracy of $\hat{Y}$. First note from $\hat{Y}^{\perp} = W + Y^{\perp}$ (see the previous paragraph) that any $v \in \hat{Y} \cap \hat{Y}^{\perp}$ satisfies $v \in W^{\perp} \cap Y$ as well as $v = w + y^{\perp}$, where $w \in W$ and $y^{\perp} \in Y^{\perp}$. For any w' in the isotropic subspace W this implies $0 = \omega(v, w') = \omega(w, w') + \omega(y^{\perp}, w') = \omega(y^{\perp}, w')$, and therefore $y^{\perp} \in W^{\perp} \cap Y^{\perp}$. But (1.8b) gives $W^{\perp} \cap Y^{\perp} = (W \oplus Y)^{\perp} = V^{\perp} = \{0\}$, forcing $y^{\perp} = 0$, and therefore $v = w \in W \cap Y = \{0\}$.

Before proving the non-degeneracy of $\hat{W}$ we establish (c)-(f), and to this end note from the previous paragraph that $\hat{Y} \cap (\hat{Y}^{\perp} \cap Y) = (\hat{Y} \cap \hat{Y}^{\perp}) \cap Y = \{0\}$. Thus to prove (e) all we need show is that $Y = \hat{Y} + (\hat{Y}^{\perp} \cap Y)$. But $V = \hat{Y} \oplus \hat{Y}^{\perp}$ (Proposition 1.1 (c)), and so any $y \in Y$ may be written $y = \hat{y} + \hat{y}^{\perp}$, where $\hat{y} \in \hat{Y}$ and $\hat{y}^{\perp} \in \hat{Y}^{\perp}$. This gives $\hat{y}^{\perp} = y - \hat{y} \in \hat{Y}^{\perp} \cap Y$, and (e) follows. Assertion (c) is then a consequence of $V = W \oplus Y = W \oplus \hat{Y} \oplus (\hat{Y}^{\perp} \cap Y) = \hat{W} \oplus \hat{Y}$, and (f) follows since $W \oplus (\hat{W} \cap Y) \oplus \hat{Y} = W \oplus (\hat{Y}^{\perp} \cap Y) \oplus \hat{Y} = W \oplus Y = V$.

Now choose $\hat{w} \in \hat{W}$ and write $\hat{w} = w + z$, where $w \in W$ and $z \in \hat{Y}^{\perp} \cap Y$. Then for any $\hat{y} \in \hat{Y} \subset W^{\perp}$ (recall the definition of $\hat{Y}$) we have $\omega(\hat{w}, \hat{y}) = \omega(w, \hat{y}) + \omega(z, \hat{y}) = 0 + 0$, and therefore $\hat{W} \subset \hat{Y}^{\perp}$. But $\hat{Y} \oplus \hat{Y}^{\perp} = V$, hence $\hat{W} = \hat{Y}^{\perp}$ by (c), whence $\hat{W}^{\perp} = \hat{Y}^{\perp\perp} = \hat{Y}$ by (1.8a). This gives (d), and by Proposition 1.1 (c) also completes the proof of (b).

We have now established (a)-(f).

From (a)-(c) we have $\dim(\hat{W} \cap Y) = \dim(V) - \dim(W) - \dim(\hat{Y})$, and (g) then follows from $W \cap (\hat{W} \cap Y) = (W \cap Y) \cap \hat{W} = \{0\}$.

Assertion (h) can be seen as follows: $\hat{W}$ is non-degenerate by (b), and $W^{\perp} \cap (\hat{W} \cap Y) = (W^{\perp} \cap Y) \cap \hat{W} = \hat{Y} \cap \hat{W} = \{0\}$ so $W^{\perp} \cap \hat{W} = W^{\perp} \cap (W \oplus$

$(\hat{W} \cap Y)) = W$.

Assertion (i) is immediate from Proposition 1.1 (a) together with the definitions and constructions of the various spaces.

Q.E.D.

Proposition 1.10 *Suppose V is the direct sum of two G-submodules, one properly isotropic. Then V is s-reducible.*

Proof: Let $V = W \oplus Y$ as in Proposition 1.9. Proposition 1.9 (b), (c), (d), and (i) imply the result.

Q.E.D.

Theorem 1.11 *Assume ρ is semisimple, reducible, and s-irreducible. Then V admits an irreducible "Lagrangian splitting", i.e. V is the direct sum of two irreducible Lagrangian G-submodules.*

Proof: By Proposition 1.3 the space V contains an isotropic G-submodule W, which by semi-simplicity and Proposition 1.10 must be Lagrangian.

Choose a G-submodule complement Y to W, and note that $Y^\perp \cap Y$ must be isotropic. If $Y^\perp \cap Y = \{0\}$ then Y is non-degenerate (Proposition 1.1 (c)), in which case V is s-reducible as $Y \oplus Y^\perp$ (Propositions 1.1 (c) and 1.2 (a)), contrary to our hypotheses. Since $Y \cap Y^\perp$ is a G-submodule, it will have a G-complement. But Proposition 1.10 implies $Y \cap Y^\perp$ cannot be properly isotropic, and it follows that $Y^\perp = Y$, hence that Y is Lagrangian.

The irreducibility of W and Y follows from the semi-simplicity hypothesis together with Proposition 1.10.

Q.E.D.

Theorem 1.12 *Suppose G is linearly reductive and the G-module V is reducible but s-irreducible. Then V admits an irreducible Lagrangian splitting.*

Proof: Immediate from the preceding theorem.

Q.E.D.

We close the section with two results concerning the rank of a linear algebraic subgroup of $Sp(V)$.

The spectrum of any operator $T \in Sp(V)$ has the form $\{\lambda_1, \cdots, \lambda_n, \lambda_1^{-1}, \cdots, \lambda_n^{-1}\}$ [A-M, Theorem 3.1.16, p. 169]: we call T *resonant* if one can find integers $n_j \in \mathbb{Z}$ for $j = 1, \cdots, n$, not all zero, such that $\prod_{j=1}^{n} \lambda_j^{n_j} = 1$. Otherwise we say T is *nonresonant*

Theorem 1.13 *For any linear algebraic group $G \subset Sp(V)$ the following statements hold:*

(a) rank $(G) \leq n$; and

(b) rank $(G) = n$ if and only if G contains a nonresonant element.

Moreover, when (b) holds and $g \in G$ is nonresonant the Zariski closure of the subgroup generated by g is a maximal torus $T \subset G$, and any maximal torus $T \subset G$ contains such an element.

Proof: See [B-C-R-S, Theorem 2.14].

Q.E.D.

Theorem 1.14 *Suppose $G \subset Sp(V)$ is connected, linearly reductive and of rank 1. Then either*

(a) $G \simeq Sl(2, \mathbb{C})$,

(b) $G \simeq PSl(2, \mathbb{C})$, or

(c) $G \simeq G_m$, i.e. G is a 1-dimensional torus.

Proof: As noted, linearly reductive is equivalent to reductive (since we are in characteristic zero), so $R(G)$ is a torus T [H-2, p. 125].

If $\dim(T) = 0$, then G is semisimple so cases (a) or (b) hold by [Sp-2, p. 178]. If $\dim(T) = 1$, then T is a maximal torus in G. Therefore the rank of G/T is 0 [H-2, Ex. 1, p. 136] so G/T is unipotent [H-2, p. 137]. Since $T = R(G)$, G/T is also semisimple and so must be trivial. Therefore (c) follows.

Q.E.D.

Notes:

1. If $G \subset Sp(V)$ is a connected linear algebraic group and V is irreducible then we shall show in Proposition 5.1 that G is semisimple (and so linearly reductive). Therefore if G is, in addition, of rank 1 then cases (a), (b) or (c) of Theorem 1.14 must hold.

2. If $G \subset Sp(V)$ is connected, semisimple and of rank 1, then we can conclude that $G \simeq Sl(2, \mathbb{C})$ or $G \simeq PSl(2, \mathbb{C})$.

2. Generalities on Ziglin Groups and Ziglin Modules

Here G is a group, V is an m-dimensional complex vector space, and $\rho : G \longrightarrow Gl(V)$ is a representation, assumed rational when G is linear algebraic. When V is symplectic we write m as $2n$, denote the symplectic form by ω, and assume $\rho : G \longrightarrow Sp(V)$.

Let $S = S(V)$ denote the $\mathbb{C}$-algebra of polynomial functions on V, and $L = L(V)$ the quotient field. The assignment $(g, f) \mapsto g \cdot f := f \circ \rho(g^{-1})$ defines a left action of G on both S and L, and for $P = S$ or L we let P^G denote the $\mathbb{C}$-algebra of (G)-*invariants*, i.e. those $f \in P$ satisfying $g \cdot f = f$ for all $g \in G$. By identifying the constant functions with $\mathbb{C}$ we can regard L^G as a field extension of $\mathbb{C}$; let $tr.deg._{\mathbb{C}}(L^G)$ denote the associated transcendence degree.

Suppose $1 \leq k \in \mathbb{Z}$. The G-module V, the representation ρ, and the group $\rho(G)$ are: *precisely k-Ziglin* if $tr.deg._{\mathbb{C}}(L^G) = k$; *(at least) k-Ziglin* if $tr.deg._{\mathbb{C}}(L^G) \geq k$; *at most k-Ziglin* if $tr.deg._{\mathbb{C}}(L^G) \leq k$; *not Ziglin* if $tr.deg._{\mathbb{C}}(L^G) = 0$, i.e., if $L^G = \mathbb{C}$. Of course when $G \subset Gl(V)$ and ρ is inclusion these definitions also apply to G. From the isomorphism $L \approx \mathbb{C}(x_1, \cdots, x_m)$ induced by any basis we see that V, ρ and $\rho(G)$ can be at most m-Ziglin; hence $k \leq m$ can be assumed.

Example 2.1 *As is well-known, up to equivalence there is a unique irreducible rational representation $\rho_m : Sl(2, \mathbb{C}) \longrightarrow Gl(V)$ for each $1 \leq m = \dim_{\mathbb{C}}(V)$ (e.g. see [Sp-1, Prop 3.2.1, p. 49]), which for odd m factors through a unique irreducible representation, also denoted ρ_m, of $PSl(2, \mathbb{C})$ in $Gl(V)$.*

If $\rho : Sl(2,\mathbb{C}) \to Gl(V)$ is any representation we denote by χ_ρ the associated multiplicative character, i.e. $\chi_\rho(g) = \det(\rho(g))$. Also well-known is the fact that L^G is the quotient field of S^G when $\chi_\rho(G)$ is trivial [Sp-1, Lemma 2.5.12(b), p. 34], as is the case for $Sl(2,\mathbb{C})$ [H-2, Ex. 2, p. 83]. Classical invariant theory then provides the following results.

- i. *ρ_1 is precisely 1-Ziglin. This is clear from the triviality of $\chi_1(Sl(2,\mathbb{C}))$.*
- ii. *ρ_2 is not Ziglin. This will follow from Theorem 2.2(a) and the observation that $V\backslash\{0\}$ is an orbit.*
- iii. *ρ_3 is precisely 1-Ziglin. [SCHUR, Satz 2.8, p. 45].*
- iv. *ρ_m is precisely $(m-3)$-Ziglin for $m > 3$. [SCHUR, Satz 2.8, p 45] and [Sp-1, Prop. 3.4.9, p. 65].*

Proposition 2.2 *Suppose G is linear algebraic, connected, of rank 1 and V is irreducible as a G-module. Then V is:*

- (a) *precisely 1-Ziglin if $m(= \dim_{\mathbb{C}}(V)) = 1$;*
- (b) *not Ziglin if $m = 2$;*
- (c) *precisely 1-Ziglin if $m = 3$; and*
- (d) *precisely $(m-3)$-Ziglin if $m > 3$.*

Proof: From Theorem 1.14 we see that G must be semisimple of rank 1, hence isomorphic to $Sl(2,\mathbb{C})$ or $PSl(2,\mathbb{C})$. Now apply Example 2.1.

Q.E.D.

Proposition 2.3 *Suppose $G \subset Gl(V)$ and $1 \leq k \leq m$. Then*

- (a) *G is (precisely) k-Ziglin if and only if the Zariski closure $\overline{G}$ is (precisely) k-Ziglin.*
- (b) *When G is a linear algebraic group (as is $\overline{G}$ in (a)), G is (precisely) k-Ziglin if and only if the connected component G^0 of the identity is (precisely) k-Ziglin.*
- (c) *$L^G \subset L^H$ whenever H is a subgroup of G; hence H is at least k-Ziglin whenever G is k-Ziglin. If H is of finite index in G, then H is k-Ziglin if and only if G is k-Ziglin.*

(d) *Finite groups G are k-Ziglin for all $1 \leq k \leq m$.*

Assertions (a) and (b) reduce the study of Ziglin groups to one of connected linear algebraic Ziglin groups.

Proof: See [B-C-R-S, Prop 2.1].

Q.E.D.

Theorem 2.4 *When $G \subset Gl(V)$ is a connected linear algebraic group the following assertions hold.*

(a) *G is k-Ziglin if and only if the maximal dimension of a G-orbit is at most $m - k$.*

(b) *G is precisely k-Ziglin if and only if the points with G-orbits of dimension $m - k$ contain a Zariski dense open subset of V.*

(c) *G is at least k-Ziglin if $\dim(G) \leq m - k$.*

(d) *Suppose some G-orbit has dimension $m - k$ and $\{f_j\}_{j=1}^{k} \subset L^G$ is a functionally independent set. Then this set must be a transcendence basis for $\mathbb{C} \subset L^G$.*

Proof: Assertion (c) is immediate from (a). For the rest see [B-C-R-S, Thoerem 2.10].

Q.E.D.

The proof of the next result is a good illustration of Theorem 2.4(b).

Proposition 2.5 *Suppose $H \subset Gl(n, \mathbb{C})$ is a d-dimensional algebraic group, $\sigma_1, \cdots, \sigma_n$ are fixed inner automorphisms of $Gl(n, \mathbb{C})$, and for each $h \in H$ let $g_h := diag[\sigma_1 h, \cdots, \sigma_n h] \in Gl(n^2, \mathbb{C})$. Then the group $\{g_h\}$ is precisely $(n^2 - d)$-Ziglin.*

Proof: The group $\{g_h\}$ can obviously be conjugated to one with all $\sigma_j = id$, and it therefore suffices to prove only this case. (Any conjugate of a k-Ziglin group is again k-Ziglin.)

Write $x \in \mathbb{C}^{n^2}$ as $x = (x_1, \cdots, x_n)$, $x_j \in \mathbb{C}^n$, and let $U := \{x : x_1 \wedge \cdots \wedge x_n \neq 0\} \subset \mathbb{C}^{n^2}$, which we note is Zariski open and (therefore) dense. If $x \in U$

and $g_h \cdot x = x$ then x provides n linearly independent eigenvectors for h with corresponding eigenvalue 1; hence $h = I$. But this shows x has a trivial stabilizer, and as a consequence the G-orbit of x must be d-dimensional. Now apply Theorem 2.4(b).

Q.E.D.

We can derive an important consequence of Proposition 2.5 with the aid of the following two results.

Lemma 2.6 *Suppose G and H are linear algebraic groups, $\varphi : G \to H$ is an epimorphism and $K := ker(\varphi)$. Let $G \simeq R_u(G) \rtimes P$ by the semi-direct product decomposition of (1.4).*

(a) If rank $(G) =$ rank (H) then rank $(K) = 0$.

(b) G is connected and rank $(G) = 0$ if and only if G is unipotent.

(c) If H is semisimple and if K is unipotent then $G \simeq K \rtimes P$.

Proof:

(a) Let T_K be a maximal torus of K. T_K is contained in a maximal torus T_G of G which by [H-2, p. 136] is carried onto a maximal torus T_H of H. Therefore $T_H \simeq T_G/T_K$ so $\dim T_K = 0$.

(b) This follows from [H-2, Exercise 6, p. 101] and [H-2, Exercise 1, p. 137].

(c) $R_u(G) \subset K$ because H is semisimple, so equality follows from (b). Now use $G \simeq R_u(G) \rtimes P$.

Q.E.D.

Proposition 2.7 *Every automorphism of $Sl(2, \mathbb{C})$ is inner. In particular, any element of $Aut(Sl(2, \mathbb{C}))$ is the restriction of an inner automorphism of $Gl(2, \mathbb{C})$.*

Proof: Use [H-2, Theorem 27.4, p. 166] together with the fact that Γ (in the statement of the theorem) must be trivial for $G = Sl(2, \mathbb{C})$.

Q.E.D.

Proposition 2.8 *Suppose G is a linear algebraic subgroup of $\{g_{A,B,C} = \begin{pmatrix} A & 0 \\ C & B \end{pmatrix} : A, B \in Sl(2,\mathbb{C})\} \subset Gl(4,\mathbb{C})$ which is connected, of rank 1, and is such that the projections $\varphi_1 : g_{A,B,C} \mapsto A$ and $\varphi_2 : g_{A,B,C} \mapsto B$ are epimorphisms. Let $G \simeq R_u(G) \rtimes P$ be the semi-direct product decomposition of (1.4). Then P has the form described for the group $\{g_h\}$ of Proposition 2.5, with $H = Sl(2,\mathbb{C})$, and P is therefore precisely 1-Ziglin.*

Proof: $Sl(2,\mathbb{C})$ has rank 1 [H-2, p. 135] and is simple [H-2, p. 168]. Let $K := ker(\varphi_1)$. By Lemma 2.6(a), rank $(K) = 0$ so K^0, the connected component of the identity of K, is unipotent by Lemma 2.6(b). K^0 is also normal in G and we have a natural morphism

$$G/K^0 \xrightarrow{\psi} G/K \xrightarrow{\simeq} Sl(2,\mathbb{C}).$$

The kernel of ψ is K/K^0 and so is finite. Since $Sl(2,\mathbb{C})$ is simply connected [VAR-1, p. 66] and G/K^0 is a finite connected cover, we must have $K = K^0$. Therefore Lemma 2.6(c) yields $G \simeq K \rtimes P$ so $\varphi_1|_P : P \to Sl(2,\mathbb{C})$ is an isomorphism.

We claim $\varphi_2|_P : P \to Sl(2,\mathbb{C})$ is also an isomorphism. Indeed, it cannot be trivial since $\varphi_2 : K \rtimes P \to Sl(2,\mathbb{C})$ is an epimorphism. Since the only normal nontrivial subgroup of $Sl(2,\mathbb{C})$ is $\{\pm I\}$, we have $\dim Im(\varphi_2|_P(P)) = 3$. Since P and $Sl(2,\mathbb{C})$ are connected we have that $\varphi_2|_P$ is surjective. If $ker(\varphi_2)|_P = \{\pm I\}$, we would again contradict the simply connectedness of $Sl(2,\mathbb{C})$ so $\varphi_2|P$ is an isomorphism on P.

But then $P \simeq \{diag[h, \sigma h] : h \in Sl(2,\mathbb{C})\}$ for some $\sigma \in \mathrm{Aut}(Sl(2,\mathbb{C}))$, and by Proposition 2.7 we may assume σ is an inner automorphism of $Gl(2,\mathbb{C})$. P thus has the asserted form, and since $\dim(Sl(2,\mathbb{C})) = 3$ [H-2, p. 52] Proposition 2.5 now completes the proof.

Q.E.D.

We conclude the section with observations on the passage of the Ziglin property to subspaces and quotient spaces.

Proposition 2.9 *Suppose $G \subset Gl(V)$, S^G contains $l \geq 1$ independent linear functionals v_j^*, $j = 1, \cdots, l$, and $k \geq l$ is any integer. Then:*

(a) The subspace $W := \bigcap_{j=1}^{l} ker(v_j^*) \subset V$ *is a G-submodule.*

(b) If V *is at least* k*-Ziglin then* W *is at least* $(k-l)$*-Ziglin*

(c) If W *is precisely* $(k-l)$*-Ziglin (or not Ziglin, if* $k=l$*) then* V *is at most* k*-Ziglin.*

Proof: (a) and the implication (b)⇒(c) are obvious, and by induction it suffices to prove (b) when $l=1$. Therefore, let $\dim W = n-1$, $\dim V = n$ and assume V is precisely k'-Ziglin, $k' \geq k$. By Theorem 2.4 the maximal orbit dimension in V is $n-k'$. Since W is invariant the maximal orbit dimension in W is at most $n-k'$, so W is at least $(n-1)-(n-k') = k'-1$ Ziglin so it is at least $k-1$ Ziglin.

Q.E.D.

Theorem 2.10 *Suppose* $G \subset Sp(V)$ *is connected and* k*-Ziglin,* S^G *contains a non-trivial linear function* v^**,* $E \subset V$ *is a 1- dimensional (and therefore) isotropic submodule, and* $E^\perp = ker(v^*)$*. Then* $E^\perp/E$ *is at least* $(k-2)$*-Ziglin. In particular,* V *is at most* $(r+2)$*-Ziglin (resp. 2-Ziglin) when* $E^\perp/E$ *is precisely* r*-Ziglin (resp. not Ziglin).*

Proof: We first show that if V is k-Ziglin and E is a 1-dimensional submodule of V then V/E is $(k-1)$-Ziglin. To see this let $\pi : V \to V/E$ and note that any orbit in V/E is the projection of an orbit in V. Let θ' be an orbit of maximal dimension in V/E and let θ be an orbit in V such that $\pi(\theta) = \theta'$. We then have that $\pi(\overline{\theta})$ contains an open subset of $\overline{\theta'}$, so $\overline{\pi(\overline{\theta})} = \overline{\theta'}$. The discussion in [H-2, p. 30] implies that $\dim \overline{\pi(\overline{\theta})} \leq \dim \overline{\theta}$ so we have $\dim \overline{\theta'} \leq \dim \overline{\theta}$. If t is the maximal orbit dimension in V then the maximal orbit dimension in V/E is at most t. Since V is k-Ziglin we conclude that V/E is $(k-1)$-Ziglin. Since $E^\perp/E$ is of codimension 1 in V/E, Proposition 2.9(b) shows that $E^\perp/E$ is at least $(k-2)$-Ziglin.

Q.E.D.

3. Generalities on the Solvable Case

Here W and Y are complex n-dimensional vector spaces, $V = TW$ is the tangent bundle of W, which we identify with the vector space $W \oplus Y$, and $G \subset Gl(V)$ is a connected linear algebraic group. The tangent space $\{w\} \times Y$ to W at a point $w \in W$ will be denoted $T_w W$. We assume G has a closed (and necessarily commutative) subgroup H of elements of the form

$$\begin{pmatrix} I & 0 \\ s & I \end{pmatrix} \tag{3.1}$$

Note that

1. H must be connected since each element is unipotent.
2. $S = \{s | \begin{pmatrix} I & O \\ s & I \end{pmatrix} \in H\}$ is a linear subspace of $L(W, W)$, since clearly S is closed under addition. Furthermore if $s \in S$ then $ns \in S$ for all $n \in \mathbb{Z}$ so $c \cdot s \in S$ for all $c \in \mathbb{C}$ since S must be a closed subset of $L(W, W)$.

Examples 3.2

(a) When G is solvable we can assume (by Lie-Kolchin) that G is a group of lower triangular matrices, from which it is easy to see that elements of the derived series $\mathcal{D}^j G$, for sufficiently large j, will have the form assumed of H.

(b) When V is symplectic and $G \subset Sp(V)$ is solvable we can again assume the situation in (a), by Theorem 1.6, and in addition, that the elements of H have the special structure described in Remark 1.7(c).

(c) When V is symplectic and $G \subset Sp(V)$ admits a Lagrangian submodule $W \subset V$ the kernel of the representation of G on W induced by the inclusion $W \hookrightarrow V$ can be assumed to have the form hypothesized for H. This is because any basis for W can be extended to a symplectic basis of V [A-M, Prop. 3.1.2, pp. 162–163].

Note that the H-orbit of any $v = (w, y) \in TW$ has the form

$$\{(w, sw + y) : s \in S\} \subset T_w W, \tag{3.3}$$

i.e. it is the translate by y of the subspace $\{(w, sw) : s \in S\}$ of $T_w W$.

Proposition 3.4

(a) $\dim(H)$ *coincides with the vector space dimension* $\dim_{\mathbb{C}}(S)$ *of the subspace* $S \subset L(W, W)$.

(b) *The tangent space* $T_w W$*, for each* $w \in W$*, is* H*- invariant.*

(c) *All* H*-orbits in* $T_w W$ *are affine subspaces with dimension equal to that of the subspace* $\{sw : s \in S\}$ *of* W*. In particular,* H*-orbits have dimension at most* n*.*

(d) *The* H*-orbit of* $(w, y) \in V$ *has dimension at most* r*, where* $1 \leq r \leq n$*, if and only if* $s_1 w \wedge \cdots \wedge s_{r+1} w = 0$ *for all choices* $s_1, \cdots, s_{r+1} \in S$*. The* H*-orbit of* $(w, y) \in V$ *has dimension precisely* r*, where* $1 \leq r \leq n$*, if and only if this dimension is at most* r *and* $s'_1 w \wedge \cdots \wedge s'_r w \neq 0$ *for some choice* $s'_1, \cdots, s'_r \in S$*.*

Proof: The assertions are clear from the remarks surrounding (3.1) and (3.3).

Q.E.D.

The Ziglin properties of H are now easily determined.

Proposition 3.5 *H is at least n-Ziglin. Indeed, the coordinate functions* $x \mapsto x_j$ *for* $j = 1, \cdots, n$ *are algebraically independent invariants. Moreover, for any integer* $n \leq k \leq 2n$ *the following assertions hold.*

(a) *H is k-Ziglin if and only if there is an integer* $r \leq 2n-k$ *such that* $s_1 w \wedge \cdots \wedge s_{r+1} w = 0$ *for all* $s_1, \cdots, s_{r+1} \in S$ *and all* $w \in W$*; and*

(b) *H is precisely k-Ziglin if and only if* $s_1 w \wedge \cdots \wedge s_{2n-k+1} w = 0$ *for all* $s_1, \cdots, s_{2n-k+1} \in S$ *and* $w \in W$*, but* $s'_1 w_0 \wedge \cdots \wedge s'_{2n-k} w_0 \neq 0$ *for some choice* $s'_1, \cdots, s'_{2n-k} \in S$ *and* $w_0 \in W$*.*

Moreover, when (b) holds the subset $U := \{w \in W : s'_1 w \wedge \cdots \wedge s'_{2n-k} w \neq 0\}$ *of* W *is Zariski open and (therefore) dense, all* H*-orbits in* $TU := U \times Y$ *will have dimension* $2n - k$*, and all other* H*-orbits will have at most that dimension.*

Proof: The initial assertion is obvious, while (a) is immediate from Proposition 3.4 and Theorem 2.4(a). For (b) and the final assertion simply note that the mapping $w \mapsto s'_1 w \wedge \cdots \wedge s'_{2n-r} w$ is polynomial; then use Theorem 2.4(b).

Q.E.D.

Proposition 3.5 can be useful, in combination with the following result, for proving that G is not k-Ziglin for specific values of k. Note that the hypotheses on V are somewhat more general than assumed at the beginning of the section.

For any subgroup $G' \subset G$ (including $G' = G$) and any $v \in V$ let $G' \cdot v$ denote the G'- orbit of v.

Proposition 3.6 *Assume V is a complex m-dimensional vector space, that $G' \subset G$ are connected linear algebraic subgroups of $Gl(V)$, and that there is a $v \in V$ and a $g \in G \backslash G'$ such that $g \cdot v \notin cl(G' \cdot v)$. Then the following assertions hold.*

(a) $\dim(G \cdot v) > \dim(G' \cdot v)$.

(b) If G' is precisely k-Ziglin, where $1 \leq k \leq m$, and if $\dim(G' \cdot v) = m - k$*, then G is at most $(k-1)$-Ziglin (meaning not Ziglin if $k = 1$).*

Proof:

(a) See, e.g., [H-2, p. 25].

(b) Use (a) together with (a) and (b) of Theorem 2.4.

Q.E.D.

Proposition 3.7 *Suppose H (as in (3.1)) is precisely k-Ziglin for some $1 \leq k \leq 2n$, and G admits an element of the form $\begin{pmatrix} \alpha & 0 \\ \gamma & \delta \end{pmatrix}$ (relative to the decomposition $V = TW$ used in (3.3)) with $\alpha \neq I$. Then G is at most $(k-1)$-Ziglin.*

Proof: By hypothesis there is a $w \in W$ such that $\alpha w \neq w$, and one can assume that w is contained in the set U of Proposition 3.5 (because U is dense in W). But the G- orbit of $v = (w, y)$, for any $y \in Y$, contains the point $(\alpha w, \alpha w + y)$, which by (3.3) is not in $cl(H \cdot v)$. The result now follows from Proposition 3.6.

Q.E.D.

4. The Solvable 2-Ziglin Subgroups of the Symplectic Group

Here V is a complex 4-dimensional symplectic vector space and G_u denotes the unipotent radical $R_u(Sp(V))$ of $Sp(V)$. In view of Theorem 1.6 and Remark 1.7 we can choose a basis $e = \{e_1, e_2, e_4, e_3\}$ which allows an identification

$$G_u \simeq \left\{ g = \begin{pmatrix} 1 & 0 & 0 & 0 \\ a_g & 1 & 0 & 0 \\ b_g & c_g & 1 & 0 \\ d_g & b_g - a_g c_g & -a_g & 1 \end{pmatrix} \right\}, \tag{4.1}$$

and has the additional property that the G_u-submodule W spanned by $\{e_4, e_3\}$ is Lagrangian. Restriction to W defines a rational representation $\varphi : G \to Gl(W)$ with kernel

$$K_u \simeq \left\{ g = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ b_g & c_g & 1 & 0 \\ d_g & b_g & 0 & 1 \end{pmatrix} = \begin{pmatrix} I & 0 \\ s_g & I \end{pmatrix} \right\}, \tag{4.2}$$

where the notation $g = \begin{pmatrix} I & 0 \\ s_g & I \end{pmatrix}$ is in conformity with that used in (3.1).

Call two elements $g_j = \begin{pmatrix} I & 0 \\ s_j & I \end{pmatrix}$ of $H \subset K_u$ *independent*, $j = 1, 2$, if the matrices $s_j = \begin{pmatrix} b_j & c_j \\ d_j & b_j \end{pmatrix}$ are independent, i.e. if the vectors (b_j, c_j, d_j) are linearly independent over $\mathbb{C}$.

Lemma 4.3 *g_1, g_2 as above are independent if and only if there is a vector $w \in \mathbb{C}^2$ such that $s_1 w \wedge s_2 w \neq 0$.*

Proof: We prove the equivalent statement: s_1, s_2 are dependent if and only if $s_1 w \wedge s_2 w = 0$ for all $w \in \mathbb{C}^2$. Moreover, in so doing we regard s_1, s_2 as operators on $\mathbb{C}^2$.

($\Rightarrow$) $s_2 = k s_1$ implies $s_2 w = k s_1 w$ for all $w \in \mathbb{C}^2$, and so $s_1 w \wedge s_2 w = 0$.

($\Leftarrow$) By (c) of Remark 1.7 we have $s_j = \hat{I}\sigma_j$, $\hat{I} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, where σ_j is symmetric. Since $\hat{I}^2 = I$ it follows that $s_1 w = k(w) s_2 w$ if and only if $\hat{I}\sigma_1 w = k(w)\hat{I}\sigma_2 w$ if and only if $\sigma_1 w = k(w)\sigma_2 w$.

By symmetry we can choose a basis $f = \{f_1, f_2\}$ for $\mathbb{C}^2$ consisting of eigenvectors of σ_1 (e.g. see [H-K, Theorem 3, p. 294]), which are easily seen to also be eigenvectors of σ_2. Letting λ_j, η_j denote the eigenvectors of σ_j and writing $w \in \mathbb{C}^2$ as $w_1 f_1 + w_2 f_2$ we then have

$$\sigma_1 w = w_1 \lambda_1 f_1 + w_2 \eta_1 f_2 = k(w)(w_1 \lambda_2 f_1 + w_2 \eta_2 f_2).$$

But this implies $\lambda_1 = k(w)\lambda_2, \eta_1 = k(w)\eta_2$, and the linear dependence of σ_1 and σ_2, hence of s_1 and s_2, follows immediately.

Q.E.D.

In the next result we use coordinates (x_1, x_2, x_3, x_4) with respect to the ordered basis (e_1, e_2, e_4, e_3) of Theorem 1.6 (b).

Theorem 4.4 *The following list of nontrivial 2-Ziglin algebraic subgroups $G \subset G_u$ is exhaustive.*

- *(a) G is any algebraic subgroup of K_u of dimension at least 2. In this case V contains a Zariski open dense set of 2-dimensional orbits, is precisely 2-Ziglin, and $\{x_1, x_2\}$ is a transcendence base for L^G.*
- *(b) G is any one-dimensional algebraic subgroup of K_u, in which case G must have the form $\{\begin{pmatrix} I & 0 \\ ts & I \end{pmatrix} : t \in \mathbb{C}\}$, with $s = \begin{pmatrix} b & c \\ d & b \end{pmatrix}$ a fixed complex matrix. Here G is precisely 3-Ziglin, and $\{x_1, x_2, -dx_1x_3 + cx_2x_4 + b(x_1x_4 - x_2x_3)\}$ is a transcendence base for L^G.*

(c) *G is the Zariski closure of a subgroup of G_u generated by a single element* $g = \begin{pmatrix} 1 & 0 & 0 & 0 \\ a & 1 & 0 & 0 \\ b & c & 1 & 0 \\ d & b-ac & -a & 1 \end{pmatrix}$ *with $a \neq 0$. In this case G is one-dimensional, precisely 3-Ziglin, and* $\{x_1, (2b-ac)x_1x_2 - 2ax_1x_3 + cx_2^2, (\frac{1}{2}a^2c + 3d)x_1^2x_2 - 3ax_1(x_1x_4 + x_2x_3) + (3b - \frac{3}{2}ac)x_1x_2^2 + cx_2^3\}$ *is a transcendence base for L^G.*

(d) *G is the Zariski closure of the (abelian) subgroup of G_u generated by two elements*

$$g_1 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ a_1 & 1 & 0 & 0 \\ b_1 & c_1 & 1 & 0 \\ d_1 & b_1 - a_1c_1 & -a_1 & 1 \end{pmatrix}, g_2 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ d_2 & 0 & 0 & 1 \end{pmatrix}$$

where $a_1d_2 \neq 0$. In this case G is two-dimensional, has a Zariski dense open set of 2-dimensional orbits, is precisely 2-Ziglin, and $\{x_1, (2b_1 - a_1c_1)x_1x_2 - 2a_1x_1x_3 + c_1x_2^2\}$ *is a transcendence base for L^G.*

The groups involved in (a) and (b), which constitute all nontrivial closed subgroups of K_u, can already be seen to be 2-Ziglin from the initial assertion of Proposition 3.5. The formulation given here is necessary for later work.

Proof: That the indicated sets of functions constitute independent invariants can be verified by straightforward calculation. That they are transcendence bases will follow, in each case, from the orbit dimension statements and Theorem 2.4(d). Recall that any unipotent algebraic group is connected.

For any nontrivial algebraic subgroup $G \subset G_u$ let $K(\varphi) := ker(\varphi|G)$ and $R(\varphi) =$ range $(\varphi|G)$ where $\varphi = \varphi_2$ is the projection of Proposition 2.8. For ease of notation write $\varphi|G$ as φ. Recall that (see [H-2, Proposition B, p. 54])

(4.4.1) $$\dim(G) = \dim(K(\varphi)) + \dim(R(\varphi)),$$

and note that $R(\varphi) = \{\begin{pmatrix} 1 & 0 \\ -a_g & 1 \end{pmatrix} : g \in G\}$ has dimension at most 1.

(a) and (b): The assertions are immediate from Lemma 4.3 and Proposition 3.5(b).

Now observe that (a) and (b) cover the case in which φ is trivial, i.e. in which $G = K(\varphi)$.

(c) When φ is an isomorphism $R(\varphi)$ is nontrivial (because G is assumed nontrivial), and we can therefore choose a $g \in G$ as in the statement of (c) (in particular, with $a \neq 0$). But then $R(\varphi)$ must be the Zariski closure of the group generated by $\varphi(g)$, and G therefore has the indicated form.

(d) Suppose, finally, that φ is nontrivial but not an isomorphism. Then we can find a $g_1 \in G\backslash K(\varphi)$, say

$$g_1 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ a_1 & 1 & 0 & 0 \\ b_1 & c_1 & 1 & 0 \\ d_1 & b_1 - a_1c_1 & -a_1 & 1 \end{pmatrix}, a_1 \neq 0, \tag{4.4.2}$$

and a $g_2 \in K(\varphi)$, say

$$g_2 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ b_2 & c_2 & 1 & 0 \\ d_2 & b_2 & 0 & 1 \end{pmatrix}, (b_2, c_2, d_2) \neq (0, 0, 0).$$

Then the commutator $(g_1, g_2,) := g_1g_2g_1^{-1}g_2^{-1}$ is

$$(g_1, g_2) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ -a_1c_2 & 0 & 1 & 0 \\ (-2a_1b_2 + a_1^2c_2) & -a_1c_2 & 0 & 1 \end{pmatrix},$$

which we note is also in $K(\varphi)$,and the condition for the dependence of g_2 and (g_1, g_2) is precisely that $b_2 = c_2 = 0$. If this does not hold, then $K(\varphi)$ is precisely 2-Ziglin by (a), whence G cannot be 2-Ziglin by Proposition 3.6. Thus

$$g_2 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ d_2 & 0 & 0 & 1 \end{pmatrix}, d_2 \neq 0.$$

The Zariski closure of the group generated by g_2 is $\left\{ \begin{pmatrix} I & 0 \\ \begin{pmatrix} 0 & 0 \\ t & 0 \end{pmatrix} & I \end{pmatrix} : t \in \mathbb{C} \right\}$, hence $\dim(K(\varphi)) = 1 = \dim(R(\varphi))$. Since $g_1 \notin K(\varphi)$, and $\dim(G) = 2$ by (i), G is the closure of the group generated by $K(\varphi)$ and g_1, therefore of g_1 and g_2. Since the group generated by g_2 has a Zariski dense set of one-dimensional orbits, Proposition 3.6(a) shows that G has orbits of dimension at least 2. But that must be the maximal dimension because two independent G-invariants have been given (see Theorem 2.4).

Q.E.D.

Theorem 4.5 *The rank 1 2-Ziglin connected algebraic subgroups of*

$$\left\{ \begin{pmatrix} \lambda & 0 & 0 & 0 \\ a & \eta & 0 & 0 \\ b & c & \eta^{-1} & 0 \\ d & -\lambda^{-1}(ac - b\eta) & -(\lambda\eta)^{-1}a & \lambda^{-1} \end{pmatrix} \right\} \tag{4.5.1}$$

are precisely those of dimension ≤ 2.

Any Borel subgroup of $Sp(V)$ can be assumed to have the form (4.5.1), as can be seen from Remark 1.7.

Proof: First recall (Theorem 2.4(c)) that for any connected algebraic subgroup $G \subset Sp(V), \dim(G) \leq 2$ implies G is (at least) 2-Ziglin. It thus suffices to assume G is a connected algebraic subgroup of (4.5.1) of dimension ≥ 3, and prove that G is not 2-Ziglin.

G is solvable, and therefore $G \simeq R_u(G) \rtimes T$, where T is any maximal torus of G [H-2, Theorem 19.3, p. 123]. But note from $\dim(G) \geq 3$ and $\dim(T) = 1$ that $\dim(R_u(G)) > 1$. But then $R_u(G)$ must be as in (a) or (d) of Theorem 4.4, and so all orbits of points in a Zariski dense open set must be 2-dimensional. Since $\{I\} \neq T$, Proposition 3.6 then shows G admits 3-dimensional orbits, and so cannot be 2-Ziglin (Theorem 2.4).

Q.E.D.

For the next statement we drop the $\dim(V) = 4$ hypothesis assumed throughout the section. Theorem 4.6(a) is a restatement of Theorem 1.13(a).

Theorem 4.6 *Suppose V is a $2n$-dimensional complex symplectic space and $G \subset Sp(V)$ is a connected linear algebraic group. Then:*

(a) rank(G) $\leq n$.

(b) If rank(G) $= n$ then G is n-Ziglin if and only if G is an n-torus.

In view of Theorem 1.13 the rank $= n$ hypothesis of (b) can be replaced by the existence of a non-resonant element of G.

Proof: See [B-C-R-S, Theorem 2.15] for part (b).

Q.E.D.

The result reduces the study of n-Ziglin groups to the cases rank$(G) < n$, hence for $n = 2$ to the cases rank$(G) = 1$ and rank$(G) = 0$ (i.e. the unipotent case, by Lemma 2.6(b)).

Summary 4.7 *The results of this section on connected subgroups $G \subset Sp(V)$* ($\dim_{\mathbb{C}}(V) = 4$) *may be summarized as follows:*

(a) If rank (G) = 0 then G is 2-Ziglin if and only if G has one of the forms listed in Theorem 4.4.

(b) If G is solvable and rank(G) = 1 then G is 2-Ziglin if and only if $\dim(G) \leq 2$ *(Theorem 4.5).*

(c) If rank(G) = 2 then G is 2-Ziglin if and only if G is a 2-torus (Theorem 4.6).

As a consequence, for a complete classification of connected algebraic 2-Ziglin subgroups $G \subset Sp(V)$ only the rank 1 non-solvable case remains for examination. This will be handled in Section 5.

5. Classification of the 2-Ziglin Subgroups of the Symplectic Group

The following proposition is well known but is hard to find in print. The Lie algebra version appears in [H-1, Prop. 19.1].

Proposition 5.1 *Let G be a connected linear algebraic group and $\rho : G \to GL(V)$ a faithful irreducible representation. Then*

(1) G is reductive, the radical of G, $R(G)$, is a torus of dimension ≤ 1, and $R(G) \subset Z(G)$.

(2) If $\rho(G) \subset Sl(V)$ then G is semisimple.

Proof:

(1) The unipotent radical $R_u(G)$ is contained in $R(G)$ and we may write $R(G) = R_u(G) \rtimes T$ where T is a torus. Therefore to prove (1) we need to show $R(G) = T$ and $\dim T \leq 1$.

When we identify G with a subgroup of $Gl(V)$, the elements of $R(G)$ will have a common eigenvector in V (since $R(G)$ is solvable and connected [H-2, Theorem 17.6, p. 113]). For each character χ of $R(G)$ let $V_\chi = \{v \in V | g \cdot v = \chi(g) \cdot v \text{ for all } g \in R(G)\}$. $V_\chi = \{0\}$ except for some finite set of characters. Since $R(G)$ is normal in G, G permutes the V_χ. Since G is connected, it must leave the V_χ fixed. Since V is an irreducible G-module and $V_\chi \neq \{0\}$ for some χ, we must have $V = V_\chi$ for some χ. Therefore $R(G)$ is a connected subgroup of the group of constant diagonal matrices so $R(G)$ is a torus and $\dim R(G) \leq 1$.

(2) In $Sl(V)$ the group of constant diagonal matrices is finite, so if $\rho(G) \subset Sl(V)$ then $R(G) = \{e\}$ and G is semisimple.

Q.E.D.

Lemma 5.2 *Suppose H is a connected 3-dimensional subgroup of $Gl(V)$ which acts irreducibly on V. Then $H = Sl(2, \mathbb{C})$.*

Proof: Proposition 5.1 implies that H is reductive with $R(H)$ a central torus of dimension at most 1. If $\dim R(H) = 1$, then $H/R(H)$ would be a semisimple group of dimension 2. Since there are no semisimple groups of dimension 2 [H-2, Exercise 4, p. 137] we have $R(H) = \{e\}$ so H is semisimple. From the classification of simple linear algebraic groups we see that $H \simeq Sl(2, \mathbb{C})$ or $H \simeq PSl(2, \mathbb{C})$. Since the only irreducible representation of $Sl(2, \mathbb{C})$ in $\mathbb{C}^4$ is faithful, $H \not\simeq PSl(2, \mathbb{C})$ so $H \simeq Sl(2, \mathbb{C})$.

Now let V be a 4 dimensional complex symplectic vector space.

Theorem 5.3 *If $G \subset Sp(V)$ is a connected algebraic 2-Ziglin group, then V admits a 1-dimensional (and therefore) isotropic submodule.*

Proof: In view of Summary 4.7 it suffices to prove the result when rank(G) = 1. Note by Proposition 2.2(d) that V must be reducible.

Suppose V is s-reducible with no 1-dimensional submodules. Because all proper, closed, connected subgroups of $Sl(2, \mathbb{C})$ have dimension at most 2, they are solvable [H-2, Exercise 4, p. 137] and so admit a 1-dimensional submodule [H-2, Theorem 17.6, p. 113]. Therefore, G must have the form considered in Proposition 2.8 (with all $C = 0$), and so, by that result and Proposition 2.3(c), cannot be 2-Ziglin.

Alternately, suppose V is reducible but s-irreducible. Proposition 1.3 then guarantees a nontrivial isotropic submodule W, which we may assume is irreducible and Lagrangian (otherwise we are done). Now choose any basis of W and extend to a symplectic basis of V [A-M, Prop. 3.1.2, pp. 162–163] so as to view

$$G \subset \{g = \begin{pmatrix} A & 0 \\ * & A^{-T} \end{pmatrix} : A \in Gl(2, \mathbb{C})\}.$$

For $g \in G$, let $\pi(g) = A$. The irreducibility of W implies $\dim(\pi(G)) \geq 3$ [H-2, Exercise 4, p. 137], hence by Lemma 5.2 we have $\pi(G) = Gl(2, \mathbb{C})$ or $Sl(2, \mathbb{C})$, and by considering only the kernel of the character $\begin{pmatrix} A & 0 \\ * & A^{-T} \end{pmatrix} \mapsto det(A)$, we may in fact assume $\{A\} = Sl(2, \mathbb{C})$. This again places us in the setting of Proposition 2.8, and again G cannot be 2-Ziglin.

Q.E.D.

Lemma 5.4 *Suppose G is an algebraic subgroup of $\{\begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix} : A, B \in Sl(2, \mathbb{C})\}$ with the following two properties:*

(a) the morphism $\varphi_1 : G \to Sl(2, \mathbb{C})$ given by $\begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix} \mapsto A$ is surjective; and

(b) the morphism $\varphi_2 : G \to Sl(2, \mathbb{C})$ given by $\begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix} \mapsto B$ satisfies $\dim(\varphi_2(G)) \leq 2$.

Then $\{\begin{pmatrix} A & 0 \\ 0 & I \end{pmatrix} : A \in Sl(2,\mathbb{C})\}$ *is a closed subgroup of* G.

Proof: Let $K_2 = ker(\varphi_2)$. Note by (a) that $\dim(G) \geq 3$, hence by (b) that $\dim(K_2) > 0$. But $K_2 = \{\begin{pmatrix} A' & 0 \\ 0 & I \end{pmatrix}\}$, where $\{A'\}$ is a normal subgroup of $Sl(2,\mathbb{C})$. Since $Sl(2,\mathbb{C})$ is simple (in the sense of algebraic groups, i.e. all proper normal subgroups are finite), K_2 is a subgroup of the required form.

Q.E.D.

Theorem 5.5 *Suppose* $G \subset Sp(V)$ *is connected, algebraic, non-solvable, of rank 1, and 2-Ziglin. Then* V *is the direct sum of two non-degenerate* G*-submodules, in terms of which* $G \simeq \{\begin{pmatrix} A & 0 \\ 0 & I \end{pmatrix} : A \in Sl(2,\mathbb{C})\}$.

Proof: By Theorem 5.3 the G-module V admits a one-dimensional isotropic submodule E, and by a simple modification of the proof of Theorem 1.6(b) we see that there is a basis $e = \{e_j\}_{j=1}^4$ of V, symplectic when reordered as (e_1, e_2, e_4, e_3), which enables us to view G as a subgroup of

$$\{g = \begin{pmatrix} \lambda & 0 & 0 & 0 \\ * & a & b & 0 \\ * & c & d & 0 \\ * & * & * & \lambda^{-1} \end{pmatrix} : \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in Sl(2,\mathbb{C})\}. \tag{5.5.1}$$

We thus have a morphism $\varphi : G \to Sl(2,\mathbb{C}) \simeq Sp(E^{\perp}/E)$ sending g to $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$, which must be surjective since G is not solvable. Lemma 2.6 then gives $G \simeq K \rtimes P$, hence $\varphi|P : P \to Sl(2,\mathbb{C})$ is an isomorphism.

If we apply the basis construction of the previous paragraph with P replacing G, then by the linear reductivity of P we can assume all asterisks in (5.5.1) vanish, thence by reordering that

$$P \subset \left\{ g = \begin{pmatrix} a & b & 0 & 0 \\ c & d & 0 & 0 \\ 0 & 0 & \lambda & 0 \\ 0 & 0 & 0 & \lambda^{-1} \end{pmatrix} : \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in Sl(2,\mathbb{C}) \right\},$$

wherein all elements of $Sl(2,\mathbb{C})$ occur. Lemma 5.4 then shows $K' := \{\begin{pmatrix} A & 0 \\ 0 & I \end{pmatrix} : A \in Sl(2,\mathbb{C})\}$ to be a closed subgroup of P, hence of G.

We claim K' is precisely 2-Ziglin. Indeed, the last two coordinate functions x_3 and x_4 associated with the newly ordered basis (e_2, e_4, e_1, e_3) are independent invariants, and the orbit of any $v = (v_1, v_2) \in V \simeq \mathbb{C}^2 \times \mathbb{C}^2$ with $v_1 \neq 0$ is obviously 2-dimensional; hence Theorem 2.4(d) applies.

But then $K' = G$; otherwise Proposition 3.6 shows that G is not 2-Ziglin.

Q.E.D.

Summary 5.6 *For a connected algebraic subgroup $G \subset Sp(V)$ (where $\dim_{\mathbb{C}}(V) = 4$) we have the following results (see Summary 4.7).*

- *(a) Suppose rank $(G) = 0$. Then G is 2-Ziglin if and only if G has one of the forms listed in Theorem 4.4.*
- *(b) Suppose rank $(G) = 1$.*
  - *(1) If G is solvable then G is 2-Ziglin if and only if $\dim(G) \leq 2$. (See Theorem 4.5).*
  - *(2) If G is not solvable then G is 2-Ziglin if and only if in some basis we can write $G \simeq \{\begin{pmatrix} A & 0 \\ 0 & I \end{pmatrix} : A \in Sl(2,\mathbb{C})\}$. (See Theorem 5.5).*
- *(c) Suppose rank $(G) = 2$. Then G is 2-Ziglin if and only if G is a 2-torus. (See Theorem 4.6).*

Since rank $(G) \leq 2$ (Theorem 4.6(a)), this covers all possibilities.

As the connected component of the identity is of finite index in a linear algebraic group, Proposition 2.3 (c) together with Summary 5.6 provides a classification of those algebraic subgroups G of $Gl(V)$, with $G^0 \subset Sp(V)$, which admit two independent rational invariants.

6. 2-Ziglin Galois Groups in the Fuchsian Case

In this section we shall derive conditions to insure that the connected component of the identity of the galois group of a fourth order linear differential equation is not 2-Ziglin; by Proposition 2.3(b) the group itself cannot be 2-Ziglin. We begin in complete generality but then restrict to the fuchsian case where the conditions can be stated completely in terms of the exponents at the singular points.

Let k be a differential field of characteristic zero with algebraically closed field of constants (see [KAP], [S], or [B-C-R-S] for the basic concepts of differential fields and their galois theory) and $L(y) = 0$ a homogeneous n-th order linear differential equation with coefficients in k. We define $L^{\circledS 2}(y) = 0$ to be the monic homogeneous linear differential equation whose solution space is spanned by $\{y_1 \cdot y_2 | y_1$ and y_2 are any solutions of $L(y) = 0\}$. $L^{\circledS 2}(y)$ has order at most $\begin{pmatrix} n+1 \\ 2 \end{pmatrix}$ and has coefficients in k [S]. We shall prove the following criterion.

Theorem 6.1 *Let k be a differential field of characteristic 0 with an algebraically closed field of constants C and $L(y) = 0$ a fourth order linear differential equation with coefficients in k. If*

(1) $L(y) = 0$ has no non-zero solutions algebraic over k,

(2) $L^{\circledS 2}(y) = 0$ has order 10 and no non-zero solutions algebraic over k,

(3) $L(y) = 0$ and $L^{\circledS 2}(y) = 0$ each has no solutions y such that $\frac{y'}{y} \in k$,

then G^0, the connected component of the identity of the galois group G of $L(y)$, is not a 2-Ziglin subgroup of $Sp(2, C)$.

Proof: Assume G^0 is a 2-Ziglin subgroup of $Sp(2, C)$. We shall show that this leads to a contradiction.

If $G^0 = \{1\}$, then all solutions of $L(y) = 0$ are algebraic over k, contradicting (1). We therefore assume that $G^0 \neq \{1\}$ and consider the cases of rank (G^0) being 0, 1, and 2 separately.

Case 1. Rank $(G^0) = 0$.

In this case G^0 is unipotent, and as a consequence there is a

solution $y \neq 0$ of $L(y) = 0$ such that $\sigma(y) = y$ for all $\sigma \in G^0$ [H-2, p. 112]. Since y is left fixed by G^0 it must be algebraic over k, contradicting (1).

Case 2. Rank $(G^0) = 1$.

Case 2.1. If G^0 is not solvable, then Theorem 5.5 implies that G^0 leaves a non-zero solution of $L(y) = 0$ fixed. This solution must therefore be algebraic contradicting (1).

Case 2.2. Now assume G^0 is solvable. From Summary 4.7, we see that $\dim(G) \leq 2$. Therefore, we may write $G^0 = U \cdot T$ where U is a 1-dimensional unipotent group and T is a 1-dimensional torus. Note that U is not only normal in G^0 but also in G. Since U must also be 2-Ziglin, we know that it must be of the form described in Theorem 4.4(b) or (c).

Case 2.2a Assume U is as in Theorem 4.4(b), so $U = \left\{ \begin{pmatrix} I & 0 \\ ts & I \end{pmatrix} \middle| t \in \mathbb{C} \right\}$ for some fixed $s = \begin{pmatrix} b & c \\ d & b \end{pmatrix}$. Let $W = \{y \mid \sigma(y) = y \text{ for all } \sigma \in U\}$. The dimension of W is at least 2 and at most 3. If $\dim(W) = 3$, then there exist $z_1, z_2 \in W$ such that for all $\sigma \in T$, $\sigma(z_1) = c_\sigma z_1$, $\sigma(z_2) = c_\sigma^{-1} z_2$ for some $c_\sigma \in \mathbb{C}$ (this is because the eigenvalues of elements of T are always of the form $\{a, a^{-1}, b, b^{-1}\}$). Therefore $z_1 \cdot z_2$ will be left fixed by G^0 and so will be an algebraic solution of $L^{\circledS 2}(y)$, contradicting (2). If $\dim(W) = 2$, then W is the span of y_3 and y_4. We can diagonalize the action of T on W and assume that there exist characters χ_3 and χ_4 on G^0 such that $\sigma(y_i) = \chi_i(\sigma) y_i$ $i = 3, 4$ for all $\sigma \in G^0$. Since U is normal in G, G leaves W invariant. If $\chi_3 \neq \chi_4$, then G must permute the lines spanned by y_3 and y_4. Therefore for any $\sigma \in G$ we have $\sigma(y_3 \cdot y_4) = c_\sigma y_3 \cdot y_4$, and G therefore leaves $\frac{(y_3 \cdot y_4)'}{y_3 \cdot y_4}$ invariant, contradicting (3). If $\chi_3 = \chi_4$, then $z = y_1(-dy_3 + by_4) - y_2(by_3 + cy_4)$ is left invariant by G^0 and so will be an algebraic solution of $L^{\circledS 2}(y) = 0$. If $z = 0$, then the dimension of the span of $\{u \cdot v \mid u \text{ and } v \text{ are solutions of } L(y) = 0\}$ is less than 10, contradicting (2). Therefore z is a non-zero algebraic solution of $L^{\circledS 2}(y) = 0$, again contradicting (2).

Case 2.2b Assume U is as in Theorem 4.4(c) so

$$U \subset \left\{ g = \begin{pmatrix} 1 & 0 & 0 & 0 \\ a_g & 1 & 0 & 0 \\ * & * & 1 & 0 \\ * & * & -a_g & 1 \end{pmatrix} \middle| a_g \in \mathbb{C}, a_g \neq 0 \text{ unless } g = I \right\},$$

$$T \subset \left\{ \begin{pmatrix} \beta^{-1} & 0 & 0 & 0 \\ 0 & \alpha^{-1} & 0 & 0 \\ 0 & 0 & \alpha & 0 \\ 0 & 0 & 0 & \beta \end{pmatrix} \middle| \alpha, \beta \in \mathbb{C}^* \right\}.$$ Let W be the space spanned by y_3 and y_4. W contains a unique 1-dimensional G^0 invariant subspace $\tilde{W}$. For any $g \in G$, gW will be a G^0 invariant subspace with a unique 1-dimensional G^0 invariant subspace. If $gW \cap W \neq \{0\}$ for all $g \in G$, then G will leave $\tilde{W}$ invariant. Because $\tilde{W}$ is spanned by y_4 we have $\sigma(\frac{y_4'}{y_4}) = \frac{y_4'}{y_4}$ for all $\sigma \in G$. This implies $\frac{y_4'}{y_4} \in k$, contradicting (3).

We can thus assume $gW \cap W = \{0\}$ for some $g \in W$. In particular, W has a G^0 invariant complement. We may therefore select a basis $\{y_1, y_2, y_3, y_4,\}$ of the solution space of $L(y) = 0$ so that $U \subset \left\{ \begin{pmatrix} 1 & 0 & 0 & 0 \\ a & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & -a & 1 \end{pmatrix} \middle| a \in \mathbb{C} \right\}$ and T preserves its form. y_2 and y_4 each span G^0 invariant lines, and any other G^0 invariant line lies in the space V spanned by y_2 and y_4 (there may not be any other such lines). Note that V is left invariant by G because G^0 is normal in G. Let χ_2 and χ_4 be characters of G^0 so that $\sigma y_2 = \chi_2(\sigma) y_2$ and $\sigma y_4 = \chi_4(\sigma) y_4$ for any $\sigma \in G^0$. If $\chi_2 \neq \chi_4$, then G must permute the lines spanned by y_2 and y_4. This implies that for $\sigma \in G$, $\sigma(y_2 \cdot y_4) = c_\sigma y_2 \cdot y_4$ so $z = y_2 \cdot y_4$ is a solution of $L^{\circledS 2}(y) = 0$ with $\frac{z'}{z}$ in k, contradicting (2). If $\chi_2 = \chi_4$ then $G^0 \subset \left\{ \begin{pmatrix} \beta^{-1} & 0 & 0 & 0 \\ a & \beta & 0 & 0 \\ 0 & 0 & \beta^{-1} & 0 \\ 0 & 0 & -a & \beta \end{pmatrix} \middle| a \in \mathbb{C}, \beta \in \mathbb{C}^* \right\}$. Therefore $y_2 y_3 + y_1 y_4$ is left fixed by G^0. As before, this must be a non-zero algebraic solution of $L^{\circledS 2}(y) = 0$, contradicting (2).

Case 3. Rank $G^0 = 2$. In this case there is a basis $\{y_1, y_2, y_3, y_4\}$ such that in this basis G^0 is $\left\{ \begin{pmatrix} \lambda_1 & & & 0 \\ & \lambda_2 & & \\ & & \lambda_2^{-1} & \\ 0 & & & \lambda_1^{-1} \end{pmatrix} \right\}$. Therefore $y_2 \cdot y_3$ is left fixed by G^0 and so is a non-zero algebraic solution of $L^{\circledS 2}(y) = 0$, contradicting (2).

Therefore G^0 cannot be a 2-Ziglin subgroup of $Sp(2, \mathbb{C})$.

Q.E.D.

For a $(2n)-th$ order linear differential equation we can offer the following criterion.

Theorem 6.2 *Suppose $L(y)$ is a $(2n)-th$ order linear differential equation, with coefficients in $\mathbb{C}(z)$, having the following two properties:*

(1) The differential galois group G contains a non-resonant element; and

(2) The equation $L^{\circledS 2}(y)$ has no non-zero solutions algebraic over $\mathbb{C}(z)$.

Then G^0 is not an $n-$Ziglin subgroup of $Sp(n, \mathbb{C})$.

Condition (1) is equivalent, by Theorems 1.13 and 4.6, to: G^0 is an $n-$torus.

Proof: Argue as in Case 3 of the previous proof.

Q.E.D.

We shall now use Theorem 6.1 to derive conditions on the exponents of a fuchsian equation to insure that the connected component of the identity of the galois group is not 2-Ziglin.

Let

$$L(y) = a_n(z)y^{(n)} + a_{n-1}(z)y^{n-1} + \cdots + a_0(z)y$$

be a linear differential equation with coefficients in the field $\mathbb{C}((z))$ of formal Laurent series and let $a_i(z) = \sum_{j\geq i_0} a_j z^j$ where $a_{i_0} \neq 0$. If $L(y) = 0$ has a solution of the form $y = z^\rho \sum_{j\geq 0} c_j z^j, c_0 \neq 0$, then formally substituting this expression into $L(y) = 0$ and examining the coefficient of the smallest power of z one sees that ρ will satisfy the equation

$$P(\rho) = \sum a_{k_0}(\rho)_k = 0$$

where $(\rho)_j = \rho(\rho-1)\cdots(\rho-(j-1))$ and the sum is over all k with $k_0 - k = \min_{0\leq i\leq n}\{i_0 - i\}$. $P(\rho)$ is called the *indicial equation of $L(y)$ (at 0)*. We shall refer to the roots of the indicial polynomial as *exponents of $L(y)$ (at 0)*.

If $L(y)$ has coefficients in $\mathbb{C}(z)$, then for any $c \in \mathbb{C}$ we can expand these coefficients in powers of $(z-c)$. Analogously, we can define the indicial equation and exponents at c. Via the transformation $z = \frac{1}{t}$, $\frac{d}{dz} = -t^2\frac{d}{dt}$ we can also define the indicial equation and exponents at infinity by considering the point $t = 0$ of the transformed equation. We say that $z = c$ is an *ordinary point* if, for $0 \leq i \leq n, a_i(z)/a_n(z)$ is analytic at $z = c$ (for $c = \infty$ this property is related to the point $t = 0$ of the transformed equation). The following lemma contains facts that are easily verified by computation or found in [Poole, p. 63].

Lemma 6.3 *Let $L(y) = 0$ have coefficients in $\mathbb{C}(z)$ and let $c \in \mathbb{C}$.*

(1) If $L(y) = 0$ has a formal solution of the form

$$y = (z-c)^\rho \sum_{i\geq 0} c_i (z-c)^{r_i}, r_i \in \mathbb{R}, 0 = r_0 < r_1 < \cdots, c_0 \neq 0, \tag{6.3.1}$$

then ρ is an exponent of $L(y)$ at c.

(2) The degree of $P(\rho)$ is at most n.

(3) If c is an ordinary point then the exponents at c are $\{0, \cdots, n-1\}$.

(4) If ρ is an exponent such that $\rho + k$ is not *an exponent for all $k = 1, 2, \cdots$, then there exists such a formal solution of the form (6.3.1).*

We note that the converse of the above Lemma is not true, that is, there does not necessarily exist a solution of the prescribed form for any exponent. For example, $L(y) = z^2y'' - (3-2z)zy' + (3-6z)y$ has exponents $\{1, 3\}$

at 0 and at this point it has a fundamental set of solutions of the form $\{z^3, -4z^3\ln(z) + z\phi(z)\}$, where $\phi(z)$ is a power series in z with $\phi(0) \neq 0$. No linear combination of these two solutions can be of the form $z\psi(z)$ where $\psi(z)$ is a power series with $\psi(0) \neq 0$.

If a point c is not an ordinary point, it is called a *singular point*. If, in addition, the indicial polynomial at this point has degree n, we say that the point is a *regular point*. An obviously equivalent way of saying this is that the order of the pole of $a_{n-i}(z)/a_n(x)$ at c is at most i. A linear differential equation with only regular singular points (including ∞) is called a *fuchsian linear differential equation.*

We shall need the following Lemma.

Lemma 6.4 *Let $L(y) = 0$ have coefficients in $\mathbb{C}(z)$ and let $c \in \mathbb{C}$.*

(1) c is a regular singular point if and only if in any open angular sector Ω at c all solutions $y = y(z)$ of $L(y) = 0$ analytic in Ω satisfy $\lim_{z\to c}(z-c)^N y(z) = 0$ for some $N \geq 0$.

Now assume $L(y)$ is fuchsian, let $\{z_0 = \infty, z_1, \ldots, z_m\} \subset \mathbb{CP}^1$ be a set containing the singular points, and let $E_j = \{\rho_i(j)\}_{i=1}^m$ be the exponents at z_j. Then the following assertions hold.

(2) If y is a solution of $L(y) = 0$ such that $\frac{y'}{y} \in \mathbb{C}(z)$, then $y(z) = P(z)\prod_{c\in S}(z-c)^{\rho_c}$ where S is the set of finite singular points of $L(y)$, each ρ_c is an exponent at c, $P(z)$ is a polynomial and $-(deg P + \sum_{c\in S}\rho_c)$ is an exponent at infinity.

(3) If $L(y) = 0$ has a non-zero solution algebraic over $\mathbb{C}(z)$ then each E_j must contain a rational number.

(4) Suppose, for some $1 \leq j \leq m$, that the set $E_j - E_j$ contains no positive integers, and that the set $E_j + E_j$ contains exactly $\binom{n+1}{2}$ numbers. Then these numbers must be the exponents of $L^{\circledS 2}(y)$ at z_j. Moreover, if all these numbers are irrational then both $L(y) = 0$ and $L^{\circledS 2}(y) = 0$ have no non-zero solutions algebraic over $\mathbb{C}(z)$.

(5) Suppose the sets $S = \{\sum_{j=1}^m \rho_{i_j}(j)\}$ and $S + S$ contain no positive integers. Then $L(y) = 0$ has no non-zero solution with the property $y'/y \in \mathbb{C}(z)$.

Proof:

(1) This is well known (see [C-L, Sect. 5 of Ch. 4]).

(2) This is less well known, so we include a proof. Let $\frac{y'}{y} = Q(z) + \sum_{i=1}^{s} \sum_{j=1}^{r_i} \frac{a_{ij}}{(z-c_i)^j}$ be the partial fraction decomposition of $\frac{y'}{y}$. Using (1) and the fact that $y = \exp(\int \frac{y'}{y})$, one sees that $Q(z) \in \mathbb{C}$ and each $r_i = 1$. We therefore write $\frac{y'}{y} = \sum_{c_i \in S} \frac{\rho_i}{(z-c_i)} + \sum_{d_i \notin S} \frac{\gamma_i}{(z-d_i)}$ so $y = \prod_{c_i \in S}(z-c_i)^{\rho_i} \prod_{d_i \notin S}(z-d_i)^{\gamma_i}$. This shows that the ρ_i and γ_i must be exponents. Since the d_i are ordinary points we must have $0 \leq \gamma_i \leq (n-1)$ so $P(z) = \prod_{d_i \notin S}(z-d_i)^{\gamma_i}$ is a polynomial. Finally, expanding at infinity shows that $-(degP + \sum \rho_c)$ is an exponent at infinity.

(3) Assume $L(y) = 0$ has a non-zero solution y algebraic over $\mathbb{C}(z)$. For any $z_i, 1 \leq i \leq n$, we may write $y = a_0(z-z_i)^{\rho_0} + a_1(z-z_i)^{\rho_1} + \cdots, \rho_j < \rho_{j+1}, \rho_j \in \mathbb{Q}$, in a Puiseux series at any singular point z_i. ρ_0 will then be an exponent so $E_i \cap \mathbb{Q} \neq \phi$. A similar argument for $z_0 = \infty$ shows $E_0 \cap \mathbb{Q} \neq \phi$.

(4) Since $E_j - E_j$ contains no positive integers, Lemma 6.3(4) implies that there exists for each i a solution of the form $y_i = (z-z_j)^{\rho_i} \sum_{k=0}^{\infty} a_{i_k}(z-z_j)^k$. Since $E_j + E_j$ contains 10 distinct numbers, the series for the 10 functions $y_i y_k$ begin with distinct powers of $z - z_j$. Therefore $E_j + E_j$ is the set of exponents of $L^{\circledS 2}(y)$ at z_j. If none of these is rational, we may conclude from (3) that $L^{\circledS 2}(y) = 0$ has no non-zero algebraic solutions.

(5) If $L(y) = 0$ has a solution $y \neq 0$ such that $\frac{y'}{y} \in \mathbb{C}(z)$ then by (2) there is a choice of exponent $\rho_{i_j}(j)$ at each z_j such that $\sum_{j=0}^{n} \rho_{i_j}(j)$ is a non-positive integer. This contradicts our assumptions, and a similar argument shows that $L^{\circledS 2}(y) = 0$ has no solution $y \neq 0$ with $\frac{y'}{y} \in \mathbb{C}(z)$.

Q.E.D.

Remarks 6.5

(a) *Note that assumption (4) in the above gives local obstructions, whereas (5) gives global obstructions for the solution being of the required format(s).*

(b) *For the effective verification of (4) in the above, consider the resultant* $P(y) = \mathcal{R}(p(y-x), p(x))$ *of the indicial polynomial* $p(x)$ *at some singularity and its "translate"* $p(y-x)$*, both considered as functions of* x*. Then* $P(y)$ *is a polynomial in* y *that vanishes at a point* $y = a$ *where* $p(a-x)$ *and* $p(x)$ *have a common zero* $x = b$*; (i.e.,* $a = (a-b) + b$ *is the sum of two roots of* $p(x)$ *[KIR, pp. 52–53]). Thus, to verify the hypothesis on* $E_j + E_j$*, one need only verify that* $P(y) = 0$ *has* $\binom{n+1}{2}$ *distinct solutions, none of which is rational for the indicial polynomial* $p(x) = p_j(x)$ *at some regular singular point* $z = z_j$*,* $0 \le j \le m$*. This can be done, for example, in MATHEMATICA.*

(c) *One can extend the procedure in (b) to verify the hypothesis on* S *in (5), above. For example, set* $\mathcal{R}(p_i(y-x), p_j(x)) = P_{ij}(y), i \neq j$*, which then vanishes at points* $y = a$ *where* $p_i(a-x)$ *and* $p_j(x)$ *have a common zero* $x = b$*. Here* p_i *and* p_j *are the respective indicial polynomials at the regular singular points* z_i *and* z_j*. Note that as before* $a = (a-b) + b$ *is the sum of two indices from distinct singular points. One then only need concatenate such expressions to form* $\mathcal{R}(P_{ij}(y-x), p_k(x))$*, and so forth, to work up to the sums involved in the definition of the set* S *in hypothesis (5). The implementation of these algorithms on a computer form the basis for practical verification of the conditions (4) and (5) in Lemma 6.4 (or conditions (i) and (ii) in Proposition 6.6 below) as we shall demonstrate in an example of the next section.*

Combining Lemma 6.4 and Theorem 6.1 in the case of fourth order linear differential equations, we have:

Proposition 6.6 *Let* $L(y) = 0$ *be a fourth order linear fuchsian differential equation with coefficients in* $\mathbb{C}(z)$ *and* $Gal^0(L) \subset Sp(2, \mathbb{C})$*. Let* $\{z_0 = \infty, \cdots, z_n\}$ *be a set containing all the singular points and let* $E_j = \{\rho_i(j)\}_{i=1}^4$ *be the exponents at* z_j*. Assume that*

(i) *For some* $0 \leq j \leq n$*, the set* $E_j - E_j$ *contains no positive integers, and the set* $E_j + E_j$ *contains exactly 10 numbers, all irrational.*

(ii) *The sets* $S = \{\sum_{j=0}^{n} \rho_{i_j}(j)\}$ *and* $S + S$ *contain no non-positive integers.*

Then G^0*, the connected component of the identity of the galois group* G *of* $L(y)$*, is not a 2-Ziglin subgroup of* $Sp(2, \mathbb{C})$*.*

7. An Example

In this section we apply the preceding theory to prove the non complete integrability of a specific three degree of freedom holomorphic Hamiltonian system. As far as seems reasonable the treatment will be in general terms.

Let (M, ω) be a complex analytic symplectic manifold with $\dim(M) = (2m + 2)$, and $H : M \to \mathbb{C}$ a holomorphic Hamiltonian with associated vector field X_H. Γ will denote a fixed non equilibrium phase curve of X_H within some regular energy surface $\Sigma := H^{-1}(\{c\})$ of H. Note that Γ is a Riemann surface: a local parameter t is provided by those local integral curves $z = z(t)$ of X_H having images in Γ. The rank $2m$ vector bundle $\pi : N \to \Gamma$, where $N := (T\Sigma|\Gamma)/T\Gamma$, is the *(reduced) normal bundle* of this surface. N is symplectic; the symplectic form ω_N (on fibres) is inherited from ω in the natural way. (For details see Section 1 of [B-C-R-S].)

The Lie derivative $\mathcal{L}_{X_H}$ induces a derivation D on the space of sections of $T\Sigma|\Gamma$ which passes naturally to a second derivation, again denoted D, on the space of sections of N. The associated flat connection $D \otimes dt$ on N is the *normal variational equation* (NVE) of X_H along Γ. For any local frame $\{e_j\}$ on N one has

$$De_j(t) = \sum_{i=1}^{2m} A_{ij}(t) e_i(t) \tag{7.1}$$

where $A(t) = (A_{ij}(t))$ is a holomorphic matrix valued function of the local parameter t, and so in terms of $A(t)$ a section $\sum_{i=1}^{2m} w_i e_i$ is parallel (null) if and only if the coordinate vector $w = (w_1, \cdots, w_{2m})^T$ satisfies the linear differential equation

$$\dot{w} + A(t)w = 0. \tag{7.2}$$

Let G be a finite group acting as automorphisms on Γ and linearly, as well as $\pi-$equivariantly, on $N|X$, where $X := \Gamma\backslash\mathrm{Fix}(G)$. Note the quotient mapping $f : X \to Y := X/G$ is an unbranched covering of Riemann surfaces [KOD, Theorem 2.2, p. 44], and that G can be identified with $Deck(X/Y)$ [MAS, Proposition 8.2, p. 165]. When D is *G-symmetric*, i.e., coincides with the 'averaged' derivation

$$D^* = (\frac{1}{|G|}) \sum_{g \in G} g^* D, \tag{7.3}$$

we can realize the NVE as the pull-back by f of the induced connection D_0 on the quotient bundle $N_0 := (N|X)/G$ over Y. Calculations involving (7.2) can be done in terms of the *reduced connection* D_0, hopefully simplifying matters.

In practice the action of G on Γ can be apparent from symmetries of this surface, but the corresponding lift to a D-symmetric action on $N|X$ is not. The question of when such a lift is possible is examined for rank 2 bundles in [B-C-R, Section 4]. In our example the existence of this lift will be seen from elementary calculations.

Let $u : \tilde{X} \to X$ be the universal cover of X and let $V \subset \tilde{X}$ be an open set with the following properties:

- (a) The restricted pull-back bundle $u^*N|V$ is trivial, i.e., $u^*N|V \approx V \times \mathbb{C}^{2m}$ [FOR, p. 229]; and
- (b) $V = \cup_j V_j$, where the V_j are disjoint, simply connected, and such that $u|V_j : V_j \to u(V)$ is a homeomorphism for j = 1,2,3,....

Property (a) allows us to regard the pull-back derivation $\tilde{D} := u^*D$, when restricted to sections of $u^*N|V$, as a linear differential equation $\tilde{D}\phi = 0$ in the classical sense, and to speak of fundamental systems of solutions $\phi = (\phi_1, \ldots, \phi_{2m})$ of this equation, with the functions ϕ_j being locally defined (on V). Property (b) is to insure that such ϕ_j are in fact globally defined (on V). Now observe that $Deck(\tilde{X}/X)$ acts (on V and) on these fundamental systems of solutions $\phi = (\phi_1, \ldots, \phi_{2m})$ by

$$\sigma \cdot \phi = (\phi_1 \circ \sigma^{-1}, \cdots, \phi_{2m} \circ \sigma^{-1}) = \phi \circ \sigma^{-1}.$$

But then for each $\sigma \in Deck(\tilde{X}/X)$ there is a linear operator $T_\sigma \in Gl(2m, \mathbb{C})$ such that

$$\sigma \cdot \phi = (T_\sigma \cdot \phi_1, \cdots, T_\sigma \cdot \phi_{2m}) = T_\sigma \cdot \phi.$$

The group homomorphism $\sigma \mapsto T_\sigma$ defines the *monodromy representation* of D in $Gl(2m, \mathbb{C})$; the image $Mon(D)$ is the *monodromy group* of D of the NVE. Because the bundle N is symplectic, and because D is induced by a Hamiltonian vector field, we in fact have $Mon(D) \subset Sp(m, \mathbb{C})$. (For details see Section 1 of [B-C-R-S].)

An alternate approach to monodromy groups of linear differential equations is in terms of analytic continuation around inverses of loops based at a distinguished point $p \in X$ and the standard identification $Deck(\tilde{X}/X) \approx \pi_1(X, p)$. This is detailed in Section 1 of [B-C-R-S]. From that perspective all representations are on various fibres of various vector bundles. The advantage of the viewpoint assumed here is that all relevant representations can be realized on the same space. For example, from the fact that $f \circ u : \tilde{X} \to Y$ is the universal cover of Y we see easily that $Mon(D)$ must be a subgroup of $Mon(D_0)$. The latter group, however, may no longer be in $Sp(m, \mathbb{C})$.

In the following result we use the standard notation $\lhd$ to denote the normal subgroup relation, and overlining to denote closures in the Zariski topology.

Proposition 7.4 *Let $\pi : E \to X$ be a holomorphic rank n complex vector bundle over a Riemann surface X and let G be a finite group acting freely as automorphisms on X and $\pi-$equivariantly on E, linearly on fibres. Suppose D is a G-symmetric derivation on this bundle and D_0 is the reduced derivation on the quotient bundle $E/G \to Y := X/G$. Then*

- *(a) $Mon(D) \lhd Mon(D_0)$ with finite index, and*
- *(b) $(\overline{Mon(D)})^0 = (\overline{Mon(D_0)})^0$.*

Proof:

- (a) Since $f : X \to Y$ is Galois, $Deck(\tilde{X}/X) \lhd Deck(\tilde{X}/Y)$ with index $|G| = |Deck(X/Y)| < \infty$. The result follows on applying the monodromy homomorphism to this relation.

(b) Let $M = Mon(D)$ and $M_0 = Mon(D_0)$. $(\overline{M})^0$ is connected and contains the identity, hence is contained in $(\overline{M_0})^0$, but is also of finite index in $\overline{M_0}$ and therefore contains $(\overline{M_0})^0$ (see [H-2, Proposition 7.3(b), p. 53]).

Q.E.D.

As we shall see below, the reason for taking the Zariski closure of the monodromy group is to obtain a linear algebraic group which, in the fuchsian case, is the differential galois group of the linear differential equation of D_0. Note by Proposition 2.3 that any rational invariants of the monodromy group are corresponding invariants for the Zariski closure of this group. As we have seen in the earlier part of this paper, linear algebraic groups are easier to deal with for purposes of classification (see Summary 5.6), and restrictions for a galois group to have rational invariants were given in Theorem 6.1.

Returning to our Hamiltonian context ($n = 2m$), we say a meromorphic function $F : M \to \mathbb{CP}^1$ is an *integral* of X_H if $X_H(F) = 0$.

Theorem 7.5

(a) (Ziglin) Let (M, ω) be a complex analytic symplectic manifold of dimension $(2m+2)$. Let $H : M \to \mathbb{C}$ be a holomorphic Hamiltonian having a non-equilibrium curve Γ in some energy surface $\Sigma = H^{-1}(c)$ on which H is regular. Assume X_H has $1 \le r \le m$ meromorphic integrals $\{F_1, \cdots, F_r\}$, functionally independent together with H on some neighborhood (in M) of Γ (but not necessarily on Γ itself, where the $\{F_j\}$ may be singular, and not necessarily in involution). Then the monodromy group $Mon(D)$ of the NVE is an r-Ziglin subgroup of $Sp(m, \mathbb{C})$.

(b) If, in addition, we have the hypotheses of Proposition 7.4 with $X = \Gamma \backslash Fix(G)$, then

i. $(\overline{Mon(D_0)})^0 \subset Sp(m, \mathbb{C})$, and

ii. $Mon(D)$ is r-Ziglin if and only if $\overline{Mon(D_0)}$ is r-Ziglin.

Proof:

(a) For a complete proof see [Zi] or [B-C-R-S, Section 1], and, in particular, Theorem 1.29 of [B-C-R-S]. Both references also treat the case in which the integrals within some proper subset of $\{F_1, \ldots, F_r\}$ are in involution.

(b) We have already noted that $Mon(D) \subset Sp(m, \mathbb{C})$, and (i) therefore follows from Proposition 7.4(b). Assertion (ii) then follows from Propositions 7.4(a) and 2.3 (a), (c).

Q.E.D.

We now specialize to the case in which the normal bundle N is identified with $\Gamma \times \mathbb{C}^{2m}$ and the induced symplectic form ω_N on fibres is identified with the standard symplectic form on $\mathbb{C}^{2m}$. Henceforth we let $n = 2m$.

Let $Y = \mathbb{C}\backslash\{$ finite set $\}$, and let D_0 be a derivation on the holomorphic sections of the product bundle $Y \times \mathbb{C}^n \to Y$ satisfying

$$D_0(hu) = \frac{dh}{dz} \cdot u + h \cdot D_0 u \tag{7.6}$$

for any holomorphic function $h = h(z)$ on Y and sections $u = u(z)$ of the bundle. In terms of a global frame field $\{f_j(z) | j = 1, \cdots, n\}$ we then have

$$D_0 f_j(z) = \sum_{i=1}^{n} B_{ij}(z) f_i(z), \tag{7.7}$$

where $B(z) = (B_{ij}(z))$ is a holomorphic $n \times n$ matrix valued function on Y. In coordinates $w = (w_1, \cdots, w_n)^T$ with respect to $\{f_j\}$ we then have the analogue of (7.2):

$$w' + B(z)w = 0, \qquad ' = \frac{d}{dz}. \tag{7.8}$$

It is difficult in general to determine if a singular point of a first order system such as (7.8) is a regular singular point. One method consists of converting (7.8) into a single n-th order differential equation

$$L_0(y) = y^{(n)} + a_{(n-1)} y^{n-1} + \cdots + a_0 y = 0, \qquad ' = \frac{d}{dz}, \tag{7.9}$$

where the $a_i = a_i(z)$ are holomorphic on $Y = \mathbb{C}\backslash\{$ finite set $\}$ and meromorphic on $\mathbb{C}$, and using a criterion due to Fuchs (see Lemma 6.3(i)). We say

(7.8) and (7.9) are *equivalent* if there is an $n \times n$ nonsingular matrix $M(z)$ that is holomorphic on Y, meromorphic on $\mathbb{C}$, such that

$$W(z) = M(z)U(z) \tag{7.10}$$

for all z in some open set $V \subset Y$ where $W(z)$ and $U(z)$ are respective (local) fundamental solution matrices of (7.8) and (7.9). In this case one can show that (7.8) and (7.9) have the same differential galois group (see Section 3.2 of [B-C-R-S]). Note that the columns of $U(z)$ have the form $(y(z), y^{(1)}(z), \cdots, y^{(n-1)}(z))^T$ and that in rewriting (7.8) as (7.9) we will often have $w_1 = y$.

For fuchsian equations (see the discussion in Section 6) we have the following result whose proof can be found in [T-T] or [B-C-R-S, Theorem 3.16].

Theorem 7.11 *Assume (7.8) and (7.9) are equivalent fuchsian equations with corresponding derivation D_0 given by (7.6) - (7.7). Then the monodromy group $Mon(D_0)$ is Zariski dense in the galois group of (7.8) (equivalently (7.9)).*

For our example below we now specialize to the case $M = \mathbb{C}^{2m+2} = \{(q_0, q; p_0, p)\}$ where $q = (q_1, \cdots, q_m)$, $p = (p_1, \cdots, p_m)$, and the symplectic $\omega = \sum_{i=0}^{m} dq_i \wedge dp_i$. Let $x = (q_0, p_0)$ and $w = (q, p)$ and consider the holomorphic Hamiltonian $H : \mathbb{C}^{2m+2} \to \mathbb{C}$ given by

$$H(q_0, q; p_0, p) = h(x) + \frac{1}{2}\langle A_0(x)w, w\rangle + \mathcal{O}(|w|^3) \tag{7.12}$$

where the $(2m) \times (2m)$ matrix $A_0(x)$ is symmetric. Then the Hamiltonian vector field X_H will have the coordinate form ($\dot{} = \frac{d}{dt}$)

$$\begin{cases} \text{(a) } \dot{x} = J_0 h_x(x) + \mathcal{O}(|w|^2) \\ \text{(b) } \dot{w} = JA_0(x)w + \mathcal{O}(|w|^2), \end{cases} \tag{7.13}$$

where $J_0 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$, $J = \begin{pmatrix} 0 & I_m \\ -I_m & 0 \end{pmatrix}$, and $h_x = grad(h)$. The form of the higher order terms, $\mathcal{O}(|w|^2)$, implies $X_H|_{\mathbb{C}^2 \times \{0\}} \simeq X_h$ where X_h is given with respect to $\omega_0 = dq_0 \wedge dp_0$. It also provides an uncoupling of the linearized equations so that the NVE (7.2) becomes

$$\dot{w} + A(t)w = 0, \qquad A(t) = -JA_0(x(t)), \tag{7.14}$$

where $x(t)$ is a non-equilibrium solution of X_h with orbit $\Gamma \subset \mathbb{C}^2$. Here we have identified the (reduced) normal bundle $N \to \Gamma$ with $\Gamma \times \mathbb{C}^{2m} \to \Gamma$. We now rephrase Theorem 7.5 for this context, replacing the quotient X/G of X under the action of the group G of Proposition 7.4 by a map $f : X \to Y$, and adding a fuchsian condition (see Proposition 4.9 in [B-C-R-S]).

Theorem 7.15 *Assume the hypotheses of Theorem 7.5(a) with $M = \mathbb{C}^{2m+2}$ and Hamiltonian H as in (7.12) above with $\Gamma \subset \mathbb{C}^2$ being a non-equilibrium orbit of X_h. Assume the following:*

(a) there is a map $f : \Gamma \to \mathbb{C}\backslash\{$ finite set $\}$ so that for $R = \{$ramification points of $f\}$ and $X = \Gamma\backslash R$, the map $f : X \to Y = \mathbb{C}\backslash\{$finite set$\}$ is a k-fold (unbranched) Galois covering projection given in coordinate form by $(q_0, p_0) = x \to z$;

(b) the equation (7.14) for the NVE on $X \times \mathbb{C}^{2m}$ can be expressed as a $(2m)$-th order homogeneous linear differential equation $L(y) = 0$ in the time parameter t; and

(c) there is a substitution $t = t(z)$ for the time parameter in $L(y) = 0$ which results in a $(2m)$-th order fuchsian equation $L_0(y) = 0$ in the parameter z on Y.

Let D be the derivation on $X \times \mathbb{C}^{2m} \to X$ corresponding to (7.15). Then there is a derivation D_0 on $Y \times \mathbb{C}^{2m} \to Y$ such that $L_0(y) = 0$ is the re-expression as a $(2m)$-th order equation of the analogue of (7.15) for D_0. Moreover, letting $Gal(D_0) = Gal(L_0)$ be the differential galois group of $L_0(y) = 0$, we have that

1. *$Mon(D)$ is an r-Ziglin subgroup of $Sp(m, \mathbb{C})$,*

2. *$Gal^0(D_0) \subset Sp(m, \mathbb{C})$, and*

3. *$Mon(D)$ is r-Ziglin if and only if $Gal(D_0)$ is r-Ziglin.*

Proof: The results follow from Theorems 7.5 and 7.11 on noting that the hypotheses (a) - (c) provide a symmetry reduction by the finite group $G = Deck(X/Y)$ of order k with the Galois map $f : X \to Y$ being the quotient map of X to $Y = X/G$ [MAS, pp. 164–165]. The existence of a suitable lift of the G-action to an action on the bundle $X \times \mathbb{C}^{2m} \to X$ under which the derivation D is symmetric is encoded in hypotheses (b) and (c) concerning the existence of a suitable time substitution $t = t(z)$. This then gives the existence of $D_0 = D/G$.

Q.E.D.

Example 7.16 *Now let $m = 2$, hence $M = \mathbb{C}^6$, with Hamiltonian*

$$H(q_0, q; p_0, p) = h(q_0, p_0) + \frac{1}{2}(p_1^2 + p_2^2) + q_0 q_1 q_2 + \mathcal{O}(|w|^3) \tag{7.17}$$

where $w = (q, p) \in \mathbb{C}^4$,

$$h(q_0, p_0) = \frac{1}{2}p_0^2 + a q_0^r (q_0 - 1)^s, \tag{7.18}$$

and $a \in \mathbb{C}\backslash\{0\}$ is a parameter, r and s are non-negative integers. This is of the form (7.12) with $x = (q_0, p_0)$ and $A_0(x) = \begin{pmatrix} 0 & q_0 & 0 & 0 \\ q_0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$. We will later impose restrictions on r and s so that the reduced equation $L_0(y) = 0$ is fuchsian.

Let $\Gamma \subset \mathbb{C}^2$ be a non-equilibrium phase curve of X_h in the analytic set $\mathcal{A} = \{x \mid h(x) = 0\} \subset \mathbb{C}^2$. The involution $(q_0, p_0) \to (q_0, -p_0)$ carries Γ to itself. Letting G be the two element group generated by the involution and S the singularity set of $\mathcal{A}$, we see that $S \cup Fix(G) \subset \{(0, 0), (1, 0)\}$ with equality between these two sets when both r and s are positive. To cover all these cases together we set $X = \mathbb{C}^2\backslash\{(0, 0), (1, 0)\}$, which results in $X/G = Y = \mathbb{C}\backslash\{0, 1\}$ with natural projection $f : X \to Y$ given by $f(q_0, p_0) = q_0 = z$. That the action of G on X has a lift to a π-equivariant action on $\pi : X \times \mathbb{C}^4 \to X$, under which the derivation D corresponding to (7.14) is symmetric, follows from the fact that we can find a suitable time transformation $t = t(z)$ as in Theorem 7.15 (b)-(c). This we do below. Thus we can reduce D to $D/G = D_0$, a derivation on the bundle $Y \times \mathbb{C}^4 \to Y$.

To calculate the 4-th order differential equation $L_0(y) = 0$ associated to D_0 we first convert the equation (7.14) for D to a 4-th order equation $L(y) = 0$ in the time variable t. Using the equation $\dot{q}_0 = p_0$ from X_h, the relation $h(q_0, p_0) = 0$, and the projection $f(q_0, p_0) = q_0 = z$, we can find the substitution $t = t(z)$ required in Theorem 7.15 (c). The result of these calculations is $(' = \frac{d}{dz})$:

$$
\begin{aligned}
L_0(y) &= y^{(4)} + y^{(3)}\left[\frac{3r-2}{z} + \frac{3s}{z-1}\right] \\
&\quad + y^{(2)}\left[(\frac{11}{4})\frac{r(r-1)}{z^2} + (\frac{1}{2})\frac{s(11r-7)}{z(z-1)} + (\frac{11}{4})\frac{s(s-1)}{(z-1)^2} + \frac{2}{z^2}\right] \\
&\quad + y^{(1)}\left[(\frac{3}{4})\frac{r(r-2)^2}{z^3} + (\frac{1}{4})\frac{s(9r^2-17r+4)}{z^2(z-1)}\right. \\
&\qquad \left. + (\frac{1}{4})\frac{s(9rs-7r-s)}{z(z-1)^2} + (\frac{1}{4})\frac{s(s-1)(3s-8)}{(z-1)^3}\right] \\
&\quad + y\left[-(\frac{1}{4})\frac{1}{a^2 z^{2r-2}(z-1)^{2s}}\right] \\
&= 0.
\end{aligned}
\tag{7.19}
$$

Examining the order of the poles at each of the singular points, one sees that in order that the equation $L_0(y) = 0$ be fuchsian the non-negative integers r and s must satisfy

$$
\begin{cases}
0 \leq r \leq 3 \\
0 \leq s \leq 2 \\
3 \leq (r+s) \leq 5.
\end{cases}
\tag{7.20}
$$

Theorem 7.21 *The Hamiltonian vectorfield X_H with Hamiltonian given by (7.17)-(7.18) does not admit two meromorphic integrals that are independent, together with H, in a neighborhood (in $\mathbb{C}^4$) of the non-equilibrium curve $\Gamma \subset \{x | h(x) = 0\} \subset \mathbb{C}^2$ in the case $r = 1$, $s = 2$, $a = \frac{1}{2}$.*

Proof: Assume X_H has two such additional meromorphic integrals as in Theorem 7.5.

The indicial polynomials for (7.19) are calculated to be

$$
\begin{cases}
p_0(\lambda) = \frac{1}{4}[4\lambda^4 - 4\lambda^3 + 6\lambda^2 + 5\lambda - 4] & \text{at } z_0 = \infty \\
p_1(\lambda) = \frac{1}{4}[4\lambda^4 - 20\lambda^3 + 40\lambda^2 - 9\lambda] & \text{at } z_1 = 0 \\
p_2(\lambda) = \frac{1}{2}[2\lambda^4 + 15\lambda^2 - \lambda - 2] & \text{at } z_2 = 1.
\end{cases}
\tag{7.22}
$$

The sets E_j of roots of $p_j(\lambda) = 0$ are given approximately by

$$(7.23) \qquad \begin{cases} E_0 = \{-0.822368, 0.52463, 0.648869 \pm 1.37724i\} \\ E_1 = \{0, 0.256118, 2.37194 \pm 1.77733i\} \\ E_2 = \{-0.331141, 0.395523, -0.0321912 \pm 2.76298i\}. \end{cases}$$

Using these approximations we check that the set $(E_0 - E_0)$ has no positive integers as required in hypothesis (i) of Proposition 6.6.

We also need that the set $(E_0 + E_0)$ contains 10 distinct numbers, none of which is rational (see (i) of Proposition 6.6). We use the resultant trick as discussed in Remark 6.5(b). Set

$$\begin{aligned} P_0(y) &= \mathcal{R}(p_0(y-x), p_0(x)) \\ (7.24) \qquad &= \frac{1}{256}(y^4 - 2y^3 + 6y^2 + 10y - 16)(16y^6 + 48y^5 + 96y^4 - 112y^3 \\ &\quad +128y^2 - 80y - 39)^2. \end{aligned}$$

Using, for example, MATHEMATICA, one verifies that the roots of the two distinct polynomials in this factorization give 10 distinct numbers that form the set $(E_0 + E_0)$. That these polynomials with integer coefficients have no rational roots $y = (\frac{m}{n})$ follows from the fact that m must divide the constant term and n must divide the coefficient of the highest power of y (these conclusions follow directly from a corollary to Gauss' lemma [N-Z-M, Problem 5, p. 414]). One then directly demonstrates that none of these possibilities for m and n gives a rational root for the two polynomials in the factorization (7.24) of $P_0(y)$. Thus hypothesis (i) of Proposition 6.6 is satisfied.

Using the approximate numbers for the sets E_j of (7.23) one directly shows that the sets S and $(S + S)$ of hypothesis (ii) of Proposition 6.6 contain no non-positive integers. The conclusion of Proposition 6.6 then shows that the galois group of (7.19) is not 2-Ziglin. This is a contradiction to Theorems 7.15 and 7.5 with $m = r = 2$. Hence, the alleged pair of additional meromorphic integrals does not exist.

Q.E.D.

We invite the reader to try other parameter values for analogous results.

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