Liouvillian Solutions of Linear Difference-Differential Equations

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Abstract

For a field k with an automorphism σ and a derivation δ, we introduce the notion of liouvillian solutions of linear difference-differential systems $\{\sigma(Y) = AY, \delta(Y) = BY\}$ over k and characterize the existence of liouvillian solutions in terms of the Galois group of the systems. In the forthcoming paper, we will propose an algorithm for deciding if linear difference-differential systems of prime order have liouvillian solutions.

Key words: linear difference-differential equations, Galois theory, liouvillian sequences.

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1. Introduction

One of the initial and key applications of the Galois theory of linear differential equations is to characterize the full solvability of such equations in terms of liouvillian functions, *i.e.*, functions built up iteratively from rational functions using exponentiation, integration and algebraic functions ((Gray, 2000, Appendix 6), (van der Put & Singer, 1997, Chapters 1.5 and 4)). From the differential Galois theory, a linear differential equation

$$L(y) = y^{(n)} + a_{n-1}y^{(n-1)} + \cdots + a_0 y = 0,$$

over $\mathbb{C}(x)$ can be solved in these terms if and only if its Galois group has solvable identity component. This allows us to conclude that if a linear differential equation $L(y) = 0$ has a liouvillian solution then it has a solution of the form $e^{\int f}$ where f is an algebraic function. This characterization is the foundation of many algorithms that allow one to decide if an equation has such solutions and find them if they exist. This theory and these algorithms can be applied for equations of matrix form $Y' = AY$ over $\mathbb{C}(x)$ as well as equations with more general coefficients.

For the case of difference equations, the situation is in many ways not as well developed. A Galois theory of linear difference equations is developed in van der Put & Singer (2001). Later on in Hendriks & Singer (1999), a notion of solving in liouvillian terms is introduced for linear difference equations of the form

$$L(y) = y(x+n) + a_{n-1}y(x+n-1) + \cdots + a_0 y(x) = 0 \quad \text{with} \quad a_i \in \mathbb{C}(x)$$

and for difference equations in matrix form $Y(x+1) = AY(x)$ where A is a matrix over $\mathbb{C}(x)$. In Hendriks & Singer (1999), a characterization of solving in liouvillian terms is presented in terms of Galois groups and an algorithm is given to decide whether a linear difference equation can be solved in liouvillian terms. We note that in Hendriks & Singer (1999), solutions are considered as equivalence classes of sequences of complex numbers $\mathbf{y} = (y(0), y(1), \ldots)$ where two sequences are equivalent if they agree from some point onward. A rational function f is identified with the sequence of the rational values $(f(0), f(1), \ldots)$. The addition and multiplication on sequences are defined elementwise. Liouvillian sequences are built up from rational sequences by successively adjoining solutions of the equations of the form $\mathbf{y}(x+1) = \mathbf{a}(x)\mathbf{y}(x)$ or $\mathbf{y}(x+1) - \mathbf{y}(x) = \mathbf{b}(x)$ and using addition, multiplication and *interlacing* to define new sequences (the interlacing of $\mathbf{u} = (u_0, u_1, \ldots), \mathbf{v} = (v_0, v_1, \ldots)$ is $(u_0, v_0, u_1, v_1, \ldots)$). Similar to the differential case, it is shown in Hendriks & Singer (1999) that a linear difference equation $L(y) = 0$ can be completely solved in terms of liouvillian sequences if and only if its Galois group has solvable identity component. When this is the case, $L(y) = 0$ has a solution that is the interlacing of *hypergeometric* sequences, and Hendriks & Singer (1999) shows how to decide if this is the case. The paper Hendriks & Singer (1999) also gives examples of equations which have no hypergeometric solutions but do have solutions that are interlacings of hypergeometric solutions. Similar results apply to difference equations in matrix form.

By a system of linear difference-differential equations, we mean a system of the form

$$Y(x+1, t) = AY(x, t), \quad \frac{dY(x, t)}{dt} = BY(x, t),$$

where A and B are square matrices over $\mathbb{C}(x, t)$ and A is invertible. The above system is usually written for short as $\{\sigma(Y) = AY, \delta(Y) = BY\}$, with σ and δ an automorphism and a derivation with respect to x and t, respectively. Many higher transcendental functions, such as Hermite polynomial (cf. Bateman & Erdélyi (1953), Ch. 10.13, equations (10), (12) p. 193), Legendre functions (cf. Bateman & Erdélyi (1953), Ch. 10.10, equations (9), (11) p. 179), Bessel functions (cf. Bateman & Erdélyi (1953), Ch. 7.2.1, equation (1), p. 4 and Ch. 7.2.8, equation (56) p. 12), and Tchebychev polynomials (cf. Bateman & Erdélyi (1953), Ch. 10.11, equations (16), (19) p. 185) satisfy such system. For example, the Hermite polynomials

$$H_n(t) = n! \sum_{m=0}^{[n/2]} \frac{(-1)^m (2t)^{n-2m}}{m!(n-2m)!}$$

satisfy a linear differential equation with respect to t and a difference equation with respect to n. In matrix terms, the vector $Y(n,t) = (H_n(t), H_{n+1}(t))^T$ satisfies

$$Y(n+1,t) = \begin{pmatrix} 0 & 1 \\ -2n & 2t \end{pmatrix} Y(n,t), \quad \frac{dY(n,t)}{dt} = \begin{pmatrix} 2t & -1 \\ 2n & 0 \end{pmatrix} Y(n,t).$$

Remark 1. The above assumption that the coefficient matrix A in the difference equation is invertible does not really result in a loss of generality. If A is singular, some linear relations among the unknowns can be obtained, which will reduce the original system to an equivalent system of smaller size (cf. Wu (2005); Li & Wu (2009)). In fact, even if the system $\{\sigma(Y) = AY, \delta(Y) = BY\}$ is not integrable, the non-satisfied integrability conditions will lead to some nontrivial linear relations among the unknowns, thus the original system can be reduced further to an equivalent integrable system (cf. Li & Wu (2009)). So we can always treat integrable systems where the difference coefficient matrix is invertible as a basic problem.

In Bronstein et al. (2005); Labahn & Li (2004); Li et al (2006); Wu (2005), some theories and algorithms have been developed on determining reducibility and existence of *hyperexponential solutions* of such systems. However, as in the pure difference case, there are systems which have no hyperexponential solutions but have solutions that are interlacings of hyperexponential solutions. In this paper, we shall use a Galois theory that appears as a special case of the Galois theory developed in (Hardouin & Singer, 2008, Appendix) to characterize *liouvillian solutions of linear difference-differential systems* as defined in Section 2.2. We will also realize this result by introducing the notion of *liouvillian sequences* as defined in Section 4.

Throughout the paper, we use $(\cdot)^T$ to denote the transpose of a vector or matrix and $\det(\cdot)$ to denote the determinant of a square matrix. The symbols $\mathbb{Z}_{\geq 0}$ and $\mathbb{Z}_{>0}$ represent the set of nonnegative integers and the set of positive integers, respectively. Denote by $\mathbf{1}$ the identity map on the sets in discussion. For a field k, denote by $\mathfrak{gl}_n(k)$ the set of $n \times n$ matrices over k and by $\mathrm{GL}_n(k)$ the set of $n \times n$ invertible matrices over k. In this paper, all difference-differential systems $\{\sigma(Y) = AY,\ \delta(Y) = BY\}$ are assumed to be integrable, which means that $\sigma(B)A = \delta(A) + AB$.

The paper is organized as follows. In Section 2, we will first review some Galois theoretic results in Hardouin & Singer (2008) and then show that the Galois group of a linear difference-differential system has solvable identity component if and only if a certain associated system has a full set of solutions in a tower built up using generalizations of liouvillian extensions. In Section 3, we show that *irreducible* systems over $\mathbb{C}(x,t)$ can be decomposed into some *irreducible* subsystems over $\overline{\mathbb{C}(t)}(x)$ of the same order. In Section 4, we define the notion of liouvillian sequences and characterize the existence of liouvillian sequences solutions by the Galois group of the systems over fields of particular form.

2. Galois Theory

2.1. Picard-Vessiot extensions and Galois groups

In Hardouin & Singer (2008), a general Galois theory is presented for linear integrable systems of difference-differential equations involving parameters. When there exists no parameters this theory yields a Galois theory of difference-differential systems as above. Let us recall some notation and results in Hardouin & Singer (2008).

A *$\sigma\delta$–ring* R is a commutative ring with unit endowed with an automorphism σ and a derivation δ satisfying $\sigma\delta(r) = \delta\sigma(r)$ for any $r \in R$. R is called a *$\sigma\delta$–field* when R is a field. All fields considered in this paper are of characteristic 0.

An element c of R is called a *constant* if $\sigma(c) = c$ and $\delta(c) = 0$, *i.e.*, it is a constant with respect both σ and δ. The set of constants of R, denoted by $R^{\sigma\delta}$, is a subring, and it is a subfield if R is a field.

Throughout this section, unless specified otherwise, we always let k be a $\sigma\delta$–field with an *algebraically closed field* of constants.

Consider a system

$$\sigma(Y) = AY, \quad \delta(Y) = BY, \tag{1}$$

over k where A, B are $n \times n$ matrices over k and Y is a vector of unknowns of size n. By Remark 1, we can always assume in the sequel that A is invertible. The integer n is called the order of the system (1). A $\sigma\delta$-ring R over k is called a *$\sigma\delta$-Picard-Vessiot extension*, or a $\sigma\delta$-PV extension for short, of k for the system (1) if it satisfies the following conditions

(i) R is a simple $\sigma\delta$-ring, *i.e.*, its only $\sigma\delta$–ideals are (0) and R;
(ii) there exists $Z \in \mathrm{GL}_n(R)$ such that $\sigma(Z) = AZ$ and $\delta(Z) = BZ$;
(iii) $R = k[Z, \frac{1}{\det(Z)}]$, that is, R is generated by entries of Z and the inverse of the determinant of Z.

Note that if the system (1) has a $\sigma\delta$-PV extension, the commutativity of σ and δ implies

$$\sigma(B) = \delta(A)A^{-1} + ABA^{-1},$$

which is called the *integrability conditions* for the system (1). Conversely, if the system (1) satisfies the integrability conditions, it is shown in Bronstein et al. (2005) and (Hardouin & Singer, 2008, Appendix) that $\sigma\delta$-PV extensions for (1) exist and are unique up to $\sigma\delta$–k isomorphisms.

The following notations will be used throughout the paper.

Notation 1. Let A be a square matrix over a $\sigma\delta$-ring. For a positive integer m, denote $A_m = \sigma^{m-1}(A)\cdots\sigma(A)A$. For a linear algebraic group G, G^0 represents the identity component of G.

Lemma 2. [Lemma 6.8 in Hardouin & Singer (2008)] *Let k be a $\sigma\delta$-field and R a simple $\sigma\delta$-ring, finitely generated over k as a $\sigma\delta$-ring. Then there are idempotents $e_0, \ldots, e_{s-1}$ in R such that* *(i)* $R = e_0R \oplus \cdots \oplus e_{s-1}R$;

(ii) σ *permutes the set* $\{e_0R, \cdots, e_{s-1}R\}$. *Moreover,* σ^s *leaves each* e_iR *invariant;*

(iii) *each* e_iR *is a domain and a simple* $\sigma^s\delta$*-ring.*

The following lemma is an analogue to Lemma 1.26 in van der Put & Singer (1997).

Lemma 3. *Let* $\{\sigma(Y) = AY,\ \delta(Y) = BY\}$ *be a system over* k, R *a* $\sigma\delta$*–PV extension for the system and* $e_0, e_1, \ldots, e_{s-1}$ *as in Lemma 2. Then each* e_iR *is a* $\sigma^s\delta$*-PV extension of* k *for the system* $\{\sigma^s(Y) = A_sY, \delta(Y) = BY\}$.

Proof. Let R be a $\sigma\delta$-PV extension for $\{\sigma(Y) = AY, \delta(Y) = BY\}$ and F be a fundamental matrix over R for the system. By Lemma 2, each e_iR is a simple $\sigma^s\delta$-ring. Clearly, e_iF are the solutions of $\{\sigma^s(Y) = A_sY, \delta(Y) = BY\}$ since $\sigma^s(e_i) = e_i$ and $\delta(e_i) = 0$ for $i = 0, \ldots, s-1$. Assume that $e_i \det(F) = 0$ for some i. Then

$$\sigma^j(e_i \det(F)) = e_{i+j \mod s} \det(A_j) \det(F) = 0$$

and thus $e_{i+j \mod s} \det(F) = 0$ for $j = 1, \cdots, s-1$. Therefore

$$\det(F) = (e_0 + \cdots + e_{s-1}) \det(F) = 0,$$

a contradiction. So e_iF is a fundamental matrix for $\{\sigma^s(Y) = A_sY, \delta(Y) = BY\}$ for each i. Moreover, $e_iR = k[e_iF, \frac{1}{e_i \det(F)}]$ for each i. This completes the proof. □

Corollary 4. *Let* $d \geq 1$ *be a divisor of* s. *Suppose that there exist idempotents* $e_0, \ldots, e_{s-1}$ *in* R *such that* $R = e_0R \oplus e_1R \oplus \cdots \oplus e_{s-1}R$. *Then for* $i = 0, \ldots, s-1$, *the subring* $\oplus_{j=0}^{\frac{s}{d}-1} e_{i+jd}R$ *of* R *is a* $\sigma^d\delta$*-PV extension of* k *for the system*

$$\sigma^d(Y) = A_dY, \quad \delta(Y) = BY.$$

Here we use a cyclic notation for the indices $\{0, \cdots, s-1\}$.

Proof. The proof is similar to that of Lemma 3. □

Definition 5. Let R be a $\sigma\delta$-PV extension for a system $\{\sigma(Y) = AY,\ \delta(Y) = BY\}$ over k. The group consisting of all $\sigma\delta$-k-automorphisms of R is called the *$\sigma\delta$-Galois group* for the system and denoted $\mathrm{Gal}(R/k)$.

Note that from Hardouin & Singer (2008), $\mathrm{Gal}(R/k)$ is the group of $k^{\sigma\delta}$-points of an algebraic group defined over $k^{\sigma\delta}$. Denote by $\mathrm{Gal}(e_0R/k)$ the $\sigma^s\delta$-Galois group for $\{\sigma^s(Y) = AY, \delta(Y) = BY\}$. Without loss of generality, we assume that $\sigma(e_i) = e_{i+1 \mod s}$. Construct a map Γ from $\mathrm{Gal}(e_0R/k)$ to $\mathrm{Gal}(R/k)$ as follows. Let $\varphi \in \mathrm{Gal}(e_0R/k)$. For any $r = r_0 + r_1 + \cdots + r_{s-1} \in R$ with $r_j \in e_jR$ for $j = 0, \ldots, s-1$, define

$$\Gamma(\varphi)(r) = \sum_{j=0}^{s-1} \sigma^j \varphi \sigma^{-j}(r_j).$$

Let $\phi \in \mathrm{Gal}(R/k)$. Clearly, ϕ permutes the e_i's by the proof of Lemma 6.8 in Hardouin & Singer (2008). Define a map $\Delta : \mathrm{Gal}(R/k) \to \mathbb{Z}/s\mathbb{Z}$ to be $\Delta(\phi) = i$ if $\phi(e_0) = e_i$. We then have the following

Lemma 6. *Let R be a $\sigma\delta$-PV extension for a system $\{\sigma(Y) = AY, \delta(Y) = BY\}$ over k, and Γ and Δ be maps defined above. Then Γ is well-defined, i.e., $\varphi \in \mathrm{Gal}(e_0R/k)$ implies $\Gamma(\varphi) \in \mathrm{Gal}(R/k)$. Moreover, the sequence of group homomorphisms*

$$0 \longrightarrow \mathrm{Gal}(e_0R/k) \xrightarrow{\Gamma} \mathrm{Gal}(R/k) \xrightarrow{\Delta} \mathbb{Z}/s\mathbb{Z} \longrightarrow 0$$

is exact.

Proof. The proof is similar to that of Corollary 1.17 in van der Put & Singer (1997). □

Lemma 7. *Suppose that k has no proper algebraic $\sigma\delta$-field extension and R is a $\sigma\delta$-PV extension for the system $\{\sigma(Y) = AY, \delta(Y) = BY\}$ over k. Then $\mathrm{Gal}(R/k)^0 = \mathrm{Gal}(e_0R/k)$.*

Proof. Let $\hat{k}$ be the algebraic closure of k in the quotient field of e_0R. Then $\mathrm{Gal}(e_0R/k)^0 = \mathrm{Gal}(e_0R/\hat{k})$. Since k has no proper algebraic $\sigma\delta$-field extension, we have $\hat{k} = k$ and therefore $\mathrm{Gal}(e_0R/k) = \mathrm{Gal}(e_0R/k)^0$. From Lemma 6, it follows that $\mathrm{Gal}(e_0R/k)$ is a closed subgroup of $\mathrm{Gal}(R/k)$ of finite index. The proposition in (Humphreys, 1975, p.53) then implies the lemma. □

From Hardouin & Singer (2008), we know that a $\sigma\delta$-PV extension R over k is the coordinate ring of a $\mathrm{Gal}(R/k)$-torsor over k. Let E be an algebraically closed differential field with a derivation δ. Clearly, $E(x)$ becomes a $\sigma\delta$-field endowed with the extended derivation δ such that $\delta(x) = 0$ and with an automorphism σ on $E(x)$ given by $\sigma|_E = \mathbf{1}$ and $\sigma(x) = x + 1$. For such a field $E(x)$, we will get an analogue of Proposition 1.20 in van der Put & Singer (1997). Before stating the result, let us look at the following

Lemma 8. *Let k be a differential field with a derivation δ, S a differential ring extension of k and I a differential radical ideal of S. Suppose that S is Noetherian as an algebraic ring and that I has the minimal prime ideal decomposition $\cap_{i=1}^{t} P_i$ as an algebraic ideal. Then P_i are differential ideals for $i = 1, \ldots, t$.*

Proof. Let $f_1 \in P_1$ and select $f_i \in P_i \setminus P_1$ for $i = 2, \cdots, t$. Then $f = f_1 f_2 \cdots f_t \in I$. By taking a derivation on both sides, we have

$$\delta(f) = \delta(f_1)f_2 \cdots f_t + f_1\delta(f_2)\cdots f_t + \cdots + f_1 f_2 \cdots \delta(f_t) \in I,$$

which implies $\delta(f_1)f_2 \cdots f_t \in P_1$. So $\delta(f_1) \in P_1$ and P_1 is a differential ideal. The proofs for other P_i's are similar. □

Now let S be a finitely generated $\sigma\delta$-ring over k and I a radical $\sigma\delta$-ideal of S. Suppose that S is Noetherian as an algebraic ring and $I = \cap_{i=1}^{s} P_i$ is the minimal prime decomposition of I as an algebraic ideal. Since S is Noetherian, we have $\sigma(I) = I$, which implies that σ permutes the P_i's. From Lemma 8, each P_i is a differential ideal. Therefore if $\{P_i\}_{i \in J}$ with J a subset of $\{1, \ldots, s\}$ is left invariant under the action of σ, then $\cap_{i \in J} P_i$ is a $\sigma\delta$-ideal. We then have the following result. We will use the following notation: if V is a variety defined over a ring k_0 and k_1 is a ring containing k_0, we denote by $V(k_1)$ the points of V with coordinates in k_1.

Proposition 9. *Let $\tilde{k} = E(x)$ be as above, $\{\sigma(Y) = AY, \delta(Y) = BY\}$ a system over $\tilde{k}$, R a $\sigma\delta$-PV extension for the system and $G = \mathrm{Gal}(R/\tilde{k})$. Then the corresponding G-torsor Z has a point which is rational over $\tilde{k}$ and $Z(\tilde{k})$ and $G(\tilde{k})$ are isomorphic. Moreover, G/G^0 is cyclic.*

Proof. The notation and proof will follow that of Proposition 1.20 in van der Put & Singer (1997). Let $Z_0, \cdots, Z_{t-1}$ be the $\tilde{k}$-components of Z. By Lemma 8, the defining ideals P_i of Z_i are differential ideals. As in the proof of Proposition 1.20 in van der Put & Singer (1997), there exists $B \in Z_0(\tilde{k})$ such that $Z_0 = BG^0_{\tilde{k}}$ and $Z = BG_{\tilde{k}}$ where $G_{\tilde{k}}$ denotes the variety G over $\tilde{k}$. Since $Z(\tilde{k})$ is τ-invariant, we have

$$BG_{\tilde{k}} = \tau(BG_{\tilde{k}}) = A^{-1}\sigma(B)G_{\tilde{k}},$$

which implies $B^{-1}A^{-1}\sigma(B) \in G(\tilde{k})$. There exists $N \in G(\tilde{k}^{\sigma\delta})$ such that

$$B^{-1}A^{-1}\sigma(B) \in G^0(\tilde{k})N.$$

Let H be the group generated by G^0 and N. One sees that $\tau(BH_{\tilde{k}}) = BH_{\tilde{k}}$ and therefore the defining ideal $\tilde{I}$ of $BH_{\tilde{k}}$ is σ-invariant. Since the set $BH_{\tilde{k}}$ is the union of some of the Z_i, $\tilde{I}$ is of the form $\cap_{i\in J} P_i$ with J a subset of $\{0, 1, \cdots, t-1\}$. Hence $\tilde{I}$ is a $\sigma\delta$-ideal because each P_i is a differential ideal. Since the defining ideal I of Z is a maximal $\sigma\delta$-ideal, it follows that $\tilde{I} = I$ and so $H = G$. □

From $B^{-1}A^{-1}\sigma(B) \in G^0(\tilde{k})N$ in the proof of Proposition 9, we conclude that N is a generator of the cyclic group G/G^0.

When $\tilde{k} = E(x)$ as above, we will give, in Section 4, a concrete realization of $\sigma\delta$-PV extensions of $\tilde{k}$ in terms of sequences.

We end this subsection with two definitions which will be frequently used.

Definition 10. Two systems $\{\sigma(Y) = AY, \delta(Y) = BY\}$ and $\{\sigma(Y) = \bar{A}Y, \delta(Y) = \bar{B}Y\}$ over k are said to be *equivalent* over k if there is $U \in \mathrm{GL}_n(k)$ such that

$$\sigma(U)A = \bar{A}U, \quad \delta(U) + UB = \bar{B}U.$$

Definition 11. Let R be a $\sigma\delta$-PV extension for a system $\{\sigma(Y) = AY, \delta(Y) = BY\}$ over k and V the solution space of the system in R^n. The system is said to be *irreducible* over k if V has no nontrivial $\mathrm{Gal}(R/k)$-invariant subspaces.

In a manner similar to the purely differential case (van der Put & Singer, 2001, p. 56), one can show that a system $\{\sigma(Y) = AY,\ \delta(Y) = BY\}$ over k is reducible over k if and only if this system is equivalent over k to $\sigma(Z) = \tilde{A}Z, \delta(Z) = \tilde{B}Z$ with

$$\tilde{A} = \begin{pmatrix} A_1 & 0 \\ A_2 & A_3 \end{pmatrix}, \qquad \tilde{B} = \begin{pmatrix} B_1 & 0 \\ B_2 & B_3 \end{pmatrix}.$$

Note that $\tilde{A}$ and $\tilde{B}$ are again $n \times n$ matrices over k.

2.2. Liouvillian solutions

The Galois theory for linear differential equations is stated in terms of differential integral domains and fields (van der Put & Singer, 2001, Chapter 1) and both theory and algorithms for finding liouvillian solutions are well developed (van der Put & Singer, 2001, Chapters 1.5, 4.1 - 4.4). The main result is that the associated Picard-Vessiot extension lies in a tower of fields built up by successively adjoining, exponentials, integrals and algebraics if and only if the associated Galois group has solvable identity component. For linear difference equations, the Galois theory is stated in terms of reduced rings and total rings of fractions. A general theory of liouvillian solutions has not been developed for linear difference equations over arbitrary difference fields k. However, a case has been investigated in Hendriks & Singer (1999) where the coefficient field is of the form $C(x)$ with a shift operator $\sigma : x \mapsto x+1$ and $\sigma|_C = \mathbf{1}$. In this situation, solutions of linear difference equations are identified with sequences whose entries are in C. One says that a linear difference equation is solvable in terms of liouvillian sequences if it has a full set of solutions in a ring of sequences built up by successively adjoining to C sequences representing indefinite sums, indefinite products and interlacings of previously defined sequences. The main result is that a linear difference equation is solvable in terms of liouvillian sequences if and only if its Galois group has solvable identity component.

In this paper we will combine the approaches for differential and difference cases to investigate the solvability of systems of mixed linear difference-differential equations over $E(x)$ where E will always be an algebraically closed differential field unless specified otherwise, $\sigma(x) = x + 1$ and $\sigma|_E = \mathbf{1}$.

In this subsection, we will give a characterization of systems $\{\sigma(Y) = AY, \delta(Y) = BY\}$ whose Galois groups have solvable identity component in terms of liouvillian towers over an arbitrary $\sigma\delta$–field k. Later, in Section 4, we will introduce the notion of $\sigma\delta$-liouvillian sequences and give another characterization of this property for certain $\sigma\delta$–fields in terms of these sequences for $\sigma\delta$-fields of a particular form.

Liouvillian extensions for *$\sigma\delta$-fields* are defined in the usual way.

Definition 12. Let k be a $\sigma\delta$-field. A $\sigma\delta$-field extension K of k is said to be *liouvillian* if there is a chain of $\sigma\delta$-field extensions

$$k = K_0 \subset K_1 \subset \cdots \subset K_m = K$$

such that $k^{\sigma\delta} = K^{\sigma\delta}$, *i.e.*, K shares the same set of constants with k, and $K_{i+1} = K_i(t_i)$ for $i = 0, \ldots, m-1$ where

(1) t_i is algebraic over K_i, or
(2) $\sigma(t_i) = r_1 t_i$ and $\delta(t_i) = r_2 t_i$ with $r_1, r_2 \in K_i$ *i.e.*, t_i is hyperexponential over K_i, or
(3) $\sigma(t_i) - t_i \in K_i$ and $\delta(t_i) \in K_i$.

We now define liouvillian solutions of mixed difference-differential systems. In the sequel, let k be a $\sigma\delta$–field with algebraically closed constants and R be a $\sigma\delta$–PV extension for a system $\{\sigma(Y) = AY,\ \delta(Y) = BY\}$ over k. Suppose that R has a decomposition

$$R = e_0 R \oplus e_1 R \oplus \cdots \oplus e_{s-1} R$$

and $\mathcal{F}_0$ is the quotient field of $e_0 R$. Then $\mathcal{F}_0$ is a $\sigma^s\delta$-field.

Definition 13. Let $v = \sum_{i=0}^{s-1} v_i$ be a solution of $\{\sigma(Y) = AY,\ \delta(Y) = BY\}$ in R^n where $v_i := e_i v \in e_i R^n$. We say that v is *liouvillian* if the entries of v_0 lie in a $\sigma^s\delta$-liouvillian extension of k containing $\mathcal{F}_0$. We say that the original system is *solvable in liouvillian terms* if each solution is liouvillian.

Suppose that v is a solution of $\{\sigma(Y) = AY,\ \delta(Y) = BY\}$ in R^n. From $\sigma(v) = Av$ and $\delta(v) = Bv$, it follows that

$$\sigma(v_{s-1}) \oplus \sigma(v_0) \oplus \cdots \oplus \sigma(v_{s-2}) = Av_0 \oplus Av_1 \oplus \cdots \oplus Av_{s-1}$$
$$\delta(v_0) \oplus \delta(v_1) \oplus \cdots \oplus \delta(v_{s-1}) = Bv_0 \oplus Bv_1 \oplus \cdots \oplus Bv_{s-1},$$

which implies that $\sigma(v_i) = Av_{i+1 \bmod s}$ and $\delta(v_i) = Bv_i$ for $i = 0, \ldots, s-1$. Hence $\sigma^s(v_0) = A_s v_0$ and $\delta(v_0) = Bv_0$, *i.e.*, v_0 is a solution of the system $\{\sigma^s(Y) = A_sY, \delta(Y) = BY\}$. Conversely, assume that v_0 is a solution of $\{\sigma^s(Y) = A_sY, \delta(Y) = BY\}$ in e_0R^n. Let $v_i = A^{-1}\sigma(v_{i-1})$ for $i = 1, \cdots, s-1$ and $v = v_0 \oplus \cdots \oplus v_{s-1}$. We then have $\sigma(v) = Av$. From the fact that $\sigma(B) - \delta(A)A^{-1} = ABA^{-1}$, one can easily check that $\delta(v) = Bv$. Hence v is a solution of $\{\sigma(Y) = AY, \delta(Y) = BY\}$ in R^n. Moreover we have the following

Proposition 14. *The system $\{\sigma(Y) = AY, \delta(Y) = BY\}$ is solvable in liouvillian terms if and only if the system $\{\sigma^s(Y) = A_sY, \delta(Y) = BY\}$ is solvable in liouvillian terms.*

Proof. Let $V_1, \cdots, V_n$ be a basis of the solution space for $\{\sigma(Y) = AY, \delta(Y) = BY\}$. Since V_i is liouvillian, $V_{i0} = e_0V_i$ is liouvillian for each i. It then suffices to show that $V_{10}, \cdots, V_{n0}$ are linearly independent over $k^{\sigma\delta}$. Assume that there exist $c_1, \cdots, c_n \in k^{\sigma\delta}$, not all zero, such that $c_1V_{10} + \cdots + c_nV_{n0} = 0$. Letting $V_{i1} = e_1V_i$ we get $V_{i1} = A^{-1}\sigma(V_{i0})$ for $i = 1, \ldots, n$. Remark that $k^{\sigma\delta} = k^{\sigma^s\delta}$ since $k^{\sigma\delta}$ is algebraically closed. Therefore

$$c_1V_{11} + \cdots + c_nV_{n1} = A^{-1}\sigma(c_1V_{10} + \cdots + c_nV_{n0}) = 0.$$

Similarly, $c_1e_iV_1 + \cdots + c_ne_iV_n = 0$ for each i. Hence $c_1V_1 + \cdots + c_nV_n = 0$, a contradiction.

Conversely, suppose that $V_{10}, \cdots, V_{n0}$ is a basis of the solution space for the system $\{\sigma^s(Y) = A_sY, \delta(Y) = BY\}$, and that all the V_{i0}'s are liouvillian. For $i = 1, \cdots, n$ and $k = 1, \cdots, s-1$, let

$$V_{ik} = A^{-1}\sigma(V_{ik-1}) \quad \text{and} \quad V_i = V_{i0} \oplus V_{i1} \oplus \cdots \oplus V_{i,s-1}.$$

Then each V_i is a solution of $\{\sigma(Y) = AY, \delta(Y) = BY\}$. Clearly, $V_1, \cdots, V_n$ are linearly independent over $k^{\sigma\delta}$. This concludes the proposition. □

Theorem 15. *The system $\{\sigma(Y) = AY, \delta(Y) = BY\}$ is solvable in liouvillian terms if and only if* $\mathrm{Gal}(R/k)^0$ *is solvable.*

Proof. By Proposition 14, $\{\sigma(Y) = AY, \delta(Y) = BY\}$ is solvable in liouvillian terms if and only if the associated system $\{\sigma^s(Y) = A_sY, \delta(Y) = BY\}$ is solvable in liouvillian terms. By Lemma 6, $\mathrm{Gal}(e_0R/k)$ is a subgroup of $\mathrm{Gal}(R/k)$ of finite index. Then $\mathrm{Gal}(R/k)^0 = \mathrm{Gal}(e_0R/k)^0$. Hence it suffices to show that the associated system is solvable in liouvillian terms if and only if $\mathrm{Gal}(e_0R/k)^0$ is solvable. Note that the PV extension of k for the associated system is a domain. Thus the proof is similar to that in differential case. □

3. Decomposition of the System

In this section, let k_0 be the $\sigma\delta$-field $\mathbb{C}(t, x)$ with an automorphism $\sigma : x \mapsto x + 1$ and a derivation $\delta = \frac{d}{dt}$, and k be its extension field $\overline{\mathbb{C}(t)}(x)$. Consider a system of difference-differential equations $\{\sigma(Y) = AY, \delta(Y) = BY\}$ over k_0. We shall analyze this system by focusing on its difference part $\sigma(Y) = AY$ and use techniques from the theory of difference equations. In this latter theory, one assumes that the fixed field of σ, that is, the σ-constants, are algebraically closed. For this reason we will need to consider properties of $\sigma(Y) = AY$ over k as well as over k_0. We shall first show that the $\sigma\delta$-Galois group of this system over k can be identified with a normal subgroup of the Galois group of the same system over k_0, and then conclude some results on orders of the factors. For example, if the above system is irreducible over k_0, it is possible that the system is reducible over k. In this case, we will prove that the factors of the above system over k have the same order. A similar result is well known for differential equations: if one makes a normal algebraic extension of the base field then the differential Galois group over this new field is a normal subgroup of the differential Galois group over the original field and an irreducible equation factors into factors of equal order. In the mixed difference-differential case or even the difference case, the fact that Picard-Vessiot extensions may contain zero divisors introduces some small complication.

We start with some lemmas. Let R and R_0 be the $\sigma\delta$-PV extensions of k and k_0 for the system $\{\sigma(Y) = AY, \delta(Y) = BY\}$ respectively.

Lemma 16. *(i) There is a k_0-monomorphism of $\sigma\delta$-rings from R_0 to R.*
(ii) Identify R_0 with a subring of R as in the first assertion. Suppose that

$$R_0 = f_0 R_0 \oplus \cdots \oplus f_{d-1} R_0$$

where $f_i R_0$ is a domain, $f_i^2 = f_i$ and $\sigma(f_i) = f_{i+1 \bmod d}$, and that

$$R = e_0 R \oplus \cdots \oplus e_{s-1} R$$

is a similar decomposition of R. Then $s = md$ for some $m \in \mathbb{Z}_{>0}$. Moreover, after a possible renumbering of the f_i, we have

$$f_i = e_i + e_{i+d} + \cdots + e_{i+(m-1)d} \quad \text{for } i = 0, \cdots, d-1.$$

PROOF. (i) Clearly, the ring $R_0 \otimes_{k_0} k$ becomes a $\sigma\delta$-ring endowed with the actions

$$\delta(r \otimes h) = \delta(r) \otimes h + r \otimes \delta(h) \quad \text{and} \quad \sigma(r \otimes h) = \sigma(r) \otimes \sigma(h)$$

for any $r \in R_0$ and $h \in k$. Since k_0 is a field, the two canonical embeddings

$$R_0 \to R_0 \otimes_{k_0} 1 \subset R_0 \otimes_{k_0} k \quad \text{and} \quad k \to 1 \otimes_{k_0} k \subset R_0 \otimes_{k_0} k$$

are both injective, and clearly are homomorphisms of $\sigma\delta$-rings. Let M be a maximal $\sigma\delta$-ideal of $R_0 \otimes_{k_0} k$ and consider the ring $(R_0 \otimes_{k_0} k)/M$. Since R_0 and k are both simple $\sigma\delta$-rings, the above embeddings factor through to $(R_0 \otimes_{k_0} k)/M$ and are still injective. Note that $(R_0 \otimes_{k_0} k)/M$ is a $\sigma\delta$–PV extension of k for $\{\sigma(Y) = AY, \delta(Y) = BY\}$. So by uniqueness, we may write $R = (R_0 \otimes_{k_0} k)/M$. Assume that $R_0 = k_0[Z, \frac{1}{\det(Z)}]$ where Z is a fundamental matrix of the system. Let $\bar{Z} = Z \mod M$. One sees that $\bar{Z}$ is still a fundamental matrix of the system and that $\det(\bar{Z}) \neq 0$. Hence

$$R = \left(k_0 \left[Z, \frac{1}{\det(Z)} \right] \otimes_{k_0} k \right) / M = k \left[\bar{Z}, \frac{1}{\det(\bar{Z})} \right]$$

is a $\sigma\delta$-PV ring for the given system over k and R_0 can be embedded into R.

(ii) Write $f_0 = \sum_{j=0}^{s-1} a_j e_j$ with $a_j \in e_j R$. Squaring both sides yields $f_0 = \sum_{j=0}^{s-1} a_j^2 e_j$, thus $a_j^2 e_j = a_j e_j$. Since $e_j R$ is a domain, a_j is either e_j or 0 for each j. The same holds for other f_i's. Then for any $i = 0, \cdots, d-1$, there is a subset $T_i \subset \{0, \cdots, s-1\}$ such that $f_i = \sum_{j\in T_i} e_j$. Assume that $T_{i_0} \cap T_{i_1}$ is not empty for two different i_0 and i_1. Let $l \in T_{i_0} \cap T_{i_1}$. Since $\sum_{i=0}^{d-1} f_i = \sum_{j=0}^{s-1} e_j = 1$,

$$0 = \sum_{i=0}^{d-1} f_i - \sum_{j=0}^{s-1} e_j = \sum_{i=0}^{d-1} \sum_{j\in T_i} e_j - \sum_{j=0}^{s-1} e_j = pe_l + H$$

where $p > 0$ and H is the sum of all the e_q's with $q \neq l$. Multiplying both sides of the above equality by e_l, we get $pe_l = 0$, a contradiction. Hence the T_i's form a partition of $\{0, \cdots, s-1\}$. Since $\sigma(f_i) = f_{i+1 \bmod d}$ and $\sigma(e_j) = e_{j+1 \bmod s}$, one sees that the sets T_i have the same size and then a renumbering yields the conclusion. □

According to Lemma 16, we can view R_0 as a subring of R and assume $R = k[\bar{Z}, \frac{1}{\det(\bar{Z})}]$ in the sequel. In particular, we can view $R_0 = k_0[\bar{Z}, \frac{1}{\det(\bar{Z})}]$.

Lemma 17. *Let* $\gamma : \mathrm{Gal}(R/k) \to \mathrm{Gal}(R_0/k_0)$ *be a map given by* $\gamma(\phi)=\phi|_{R_0}$ *for any* $\phi \in \mathrm{Gal}(R/k)$. *Then* γ *is a monomorphism. Moreover, we can view the identity component of* $\mathrm{Gal}(R/k)$ *as a subgroup of that of* $\mathrm{Gal}(R_0/k_0)$.

Proof. Assume $R = k[\bar{Z}, \frac{1}{\det(\bar{Z})}]$. Let $\phi \in \mathrm{Gal}(R/k)$. If $\phi(\bar{Z}) = \bar{Z}[\phi]_{\bar{Z}}$ for some $[\phi]_{\bar{Z}} \in \mathrm{GL}_n(\mathbb{C})$, then $\gamma(\phi)(\bar{Z}) = \bar{Z}[\phi]_{\bar{Z}}$. Hence $\gamma(\phi)$ is an automorphism of R_0 over k_0, that is, $\gamma(\phi) \in \mathrm{Gal}(R_0/k_0)$. Note that $\det(\bar{Z}) \neq 0$. If $\gamma(\phi) = \mathbf{1}$, then $[\phi]_{\bar{Z}} = I_n$, which implies that $\phi = \mathbf{1}$. So γ is an injective homomorphism. Therefore, we can view $\mathrm{Gal}(R/k)$ as a subgroup of $\mathrm{Gal}(R_0/k_0)$. Since γ is continuous in the Zariski topology, $\gamma(\mathrm{Gal}(R/k)^0)$ is in $\mathrm{Gal}(R_0/k_0)^0$. So the lemma holds. □

Lemma 18. $\mathrm{Gal}(R/k)^0 = \mathrm{Gal}(R_0/k_0)^0$.

Proof. Let $G = \mathrm{Gal}(R/k)$ and $G_0 = \mathrm{Gal}(R_0/k_0)$. From Hardouin & Singer (2008), the $\sigma\delta$-PV ring R (resp. R_0) is the coordinate ring of a G-torsor (resp. G_0-torsor). From (Kunz, 1985, p.40, (3)), the Krull dimension of R (resp. R_0) equals the Krull dimension of G (resp. G_0). Since all the components of a linear algebraic group are isomorphic as varieties, one sees that the Krull dimension of G (resp. G_0) equals the Krull dimension of G^0 (resp. G_0^0). Since R is generated over R_0 by the elements of k, by Proposition 2.2 and Corollary 2.3 in (Kunz, 1985, p. 44), R is an integral ring extension of R_0. By (Kunz, 1985, Corollary 2.13), the Krull dimension of R equals that of R_0. Hence the Krull dimension of G^0 equals that of G_0^0. Since both G^0 and G_0^0 are connected and $G^0 \subset G_0^0$, by the proposition in (Humphreys, 1975, p. 25) we have $G^0 = G_0^0$. □

From Lemma 17, $\mathrm{Gal}(R/k)$ can be viewed as a subgroup of $\mathrm{Gal}(R_0/k_0)$. In the following, we prove that $\mathrm{Gal}(R/k)$ is a normal subgroup of $\mathrm{Gal}(R_0/k_0)$. Let $\mathcal{F}_0$ and $\mathcal{F}$ be the total ring of fractions of R_0 and R respectively. Note that $\mathrm{Gal}(R/k) = \mathrm{Gal}(\mathcal{F}/k)$ and $\mathrm{Gal}(R_0/k_0) = \mathrm{Gal}(\mathcal{F}_0/k_0)$. The following Corollary allows us to assume that $\mathcal{F}_0 \subset \mathcal{F}$.

Lemma 19. *Let* k *be a* $\sigma\delta$*–field and* S *a* $\sigma\delta$*–PV extension of* k. *A nonzero element* $r \in S$ *is a zero divisor in* S *if and only if there exists* $j \in \mathbb{Z}_{>0}$ *such that* $\prod_{i=0}^{j} \sigma^i(r) = 0$.

Proof. Suppose that $\prod_{i=0}^{j} \sigma^i(r) = 0$ for some $j \in \mathbb{Z}_{>0}$ and let j be minimal with respect to this assumption. If $0 = \prod_{i=1}^{j} \sigma^i(r) = \sigma(\prod_{i=0}^{j-1} \sigma^i(r))$, then we would also have $\prod_{i=0}^{j-1} \sigma^i(r) = 0$ contradicting the minimality of j. Therefore r is a zero divisor. Now suppose that r is a zero divisor. We write $S = \oplus_{i=0}^{j-1} S_i$ where the S_i are domains and $\sigma(S_i) = S_{i+1 \bmod j}$. Then $r = \sum_{i=0}^{j-1} r_i$ with $r_i \in S_i$. Writing $r = (r_0, \ldots, r_{j-1})$, one sees that r is a zero divisor in S if and only if some r_i is zero. Assume $r_0 = 0$. Then

$$\sigma(r) = (\sigma(r_{j-1}), 0, \ldots), \quad \sigma^2(r) = (\sigma^2(r_{j-2}), \sigma^2(r_{j-1}), 0, \ldots), \quad \ldots$$

so $0 = \prod_{i=0}^{j-1} \sigma^i(r)$. □
Consequently, we have

Corollary 20. *Let k and S be as above and $\bar{S}$ a σ-subring of S. An element $r \in \bar{S}$ is a zero divisor in $\bar{S}$ if and only if it is a zero divisor in S. Therefore the total ring of fractions of $\bar{S}$ embeds in the total ring of fractions of S.*

Lemma 21. *Let $u \in k$ be of degree m over k_0 and let $u = u_1, u_2, \ldots, u_m$ be its conjugates. Then there exist $M \in \mathrm{GL}_m(k_0)$ and $N \in \mathfrak{gl}_m(k_0)$ such that*

$$Z = \begin{pmatrix} 1 & 1 & \ldots & 1 \\ u_1 & u_2 & \ldots & u_m \\ \vdots & \vdots & \vdots & \vdots \\ u_1^{m-1} & u_2^{m-1} & \ldots & u_m^{m-1} \end{pmatrix}$$

satisfies

$$\sigma(Z) = MZ \quad \textit{and} \quad \delta(Z) = NZ.$$

Proof. We claim that any field automorphism τ of k over k_0 is a $\sigma\delta$–field automorphism. Indeed, since k is an algebraic extension of k_0 so any automorphism τ is automatically a δ-field automorphism. One sees that τ is a σ-field automorphism by noting that for any $f \in \overline{\mathbb{C}(t)}(x)$, τ acts on the coefficients of powers of x while σ acts only on x. For any g in the automorphism group of k over k_0, we have $g(Z) = Z[g]$ where $[g]$ is a permutation matrix. Since g is also an automorphism of $\sigma\delta$–fields, both $M = \sigma(Z)Z^{-1}$ and $N = \delta(Z)Z^{-1}$ are left invariant by g and therefore must have entries in k_0. □

We now proceed to prove the main result of this section.

Proposition 22. $\mathrm{Gal}(R/k)$ *is a normal subgroup of* $\mathrm{Gal}(R_0/k_0)$.

Proof. Consider the following diagram

$$\begin{array}{ccc} & \mathcal{F} & \\ / & & \backslash \\ \mathcal{F}_0 & & k \\ \backslash & & / \\ & \mathcal{F}_0 \cap k & \\ & | & \\ & k_0 & \end{array}$$

First, we claim that the map $\mathrm{Gal}(\mathcal{F}/k) \to \mathrm{Gal}(\mathcal{F}_0/\mathcal{F}_0 \cap k)$ that sends $h \in \mathrm{Gal}(\mathcal{F}/k)$ to its restriction $h|_{\mathcal{F}_0}$ on $\mathcal{F}_0$ is an isomorphism. Any automorphism of $\mathcal{F}$ over k is determined by its action on a fundamental matrix of $\{\sigma(Y) = AY, \delta(Y) = BY\}$ and its restriction on $\mathcal{F}_0$ is determined in the same way. This implies that the restricted map is injective.

To see that it is surjective, note that its image is closed and has $\mathcal{F}_0 \cap k$ as a fixed field. Therefore, the Galois theory implies that the restricted map must be $\mathrm{Gal}(\mathcal{F}_0/\mathcal{F}_0 \cap k)$.

We now show that ghg^{-1} is in $\mathrm{Gal}(\mathcal{F}_0/\mathcal{F}_0 \cap k)$ for any $h \in \mathrm{Gal}(\mathcal{F}_0/\mathcal{F}_0 \cap k)$ and $g \in \mathrm{Gal}(\mathcal{F}_0/k_0)$. It suffices to show that g leaves $\mathcal{F}_0 \cap k$ invariant, which will imply that $ghg^{-1}(u) = u$ for any $u \in \mathcal{F}_0 \cap k$ and so $ghg^{-1} \in \mathrm{Gal}(\mathcal{F}_0/\mathcal{F}_0 \cap k)$. Now let $u \in \mathcal{F}_0 \cap k$ be of degree m over k_0. From Lemma 21, $U = (1, u, u^2, \ldots, u^{m-1})^T$ satisfies some difference-differential system over k_0. Therefore the vector $g(U)$ satisfies the same system and so must be a $\mathbb{C}$-linear combination of the columns of Z. In particular, we have $g(u) \in k$. Therefore g leaves $\mathcal{F}_0 \cap k$ invariant. This completes the proof. □

Theorem 23. *If $\{\sigma(Y) = AY, \delta(Y) = BY\}$ is irreducible over k_0, then there is some positive integer d such that the system is equivalent over $\hat{k}_0 := \mathcal{F}_0 \cap k$ to the system*

$$\sigma(Y) = \mathrm{diag}(A_1, A_2, \cdots, A_d)Y, \quad \delta(Y) = \mathrm{diag}(B_1, B_2, \cdots, B_d)Y$$

where $A_i \in \mathrm{GL}_\ell(\hat{k}_0), B_i \in \mathfrak{gl}_\ell(\hat{k}_0)$, $\ell = \frac{n}{d}$, and the system $\{\sigma(Y) = A_iY, \delta(Y) = B_iY\}$ is irreducible over k for $i = 1, \ldots, d$. Moreover, there exists $g_i \in \mathrm{Gal}(R_0/k_0)$ such that $g_i(A_1) = A_i$ and $g_i(B_1) = B_i$.

Proof. By Proposition 22, $\mathrm{Gal}(R/k)$ is isomorphic to $\mathrm{Gal}(\mathcal{F}_0/\hat{k}_0)$ and then $\mathrm{Gal}(\mathcal{F}_0/\hat{k}_0)$ is a normal subgroup of $\mathrm{Gal}(R_0/k_0)$. Let V be the solution space in R_0^n of $\{\sigma(Y) = AY, \delta(Y) = BY\}$. Then Clifford's Theorem (Dixon, 1971, p.25, Theorem 2.2) tells us that V can be decomposed into $V = V_1 \oplus V_2 \oplus \cdots \oplus V_d$ where the V_i are minimal $\mathrm{Gal}(\mathcal{F}_0/\hat{k}_0)$-invariant subspaces of V and, for each i, there exists $g_i \in \mathrm{Gal}(R_0/k_0)$ such that $V_i = g_i(V_1)$. Furthermore, $g \in \mathrm{Gal}(R_0/k_0)$ permutes the V_i. Let Z_1 be an $n \times \ell$ matrix whose columns are the solutions in V_1 of the original system. Then Z_1 has the full rank. Then for each i, the columns of $Z_i = g_i(Z_1)$ are the solutions in V_i of the original system and Z_i has the full rank too. Let $Z = (Z_1, \cdots, Z_d)$. Then Z is a fundamental matrix of the original system. By Lemma 1 in Grigoriev (1990), there exists $\ell \times n$ matrix P_1 of the rank ℓ with entries from $\hat{k}_0$ such that $P_1V_i = 0$ for $i = 2, \cdots, d$. Since $g \in \mathrm{Gal}(R_0/k_0)$ permutes the V_i,

$$\{g_i(V_2), \cdots, g_i(V_d)\} = \{V_1, \cdots, V_{i-1}, V_{i+1}, \cdots, V_d\}.$$

Let $P_i = g_i(P_1)$. Then $P_iV_j = 0$ for $j \neq i$. Therefore, $P := (P_1^T, \cdots, P_d^T)^T \in \mathfrak{gl}_n(\hat{k}_0)$ satisfies that

$$PZ = \mathrm{diag}(U_1, U_2, \cdots, U_d) \tag{2}$$

with $U_i \in \mathfrak{gl}_\ell(\mathcal{F}_0)$. Moreover we have $U_i = g_i(U_1)$ for each i. We now prove that $\det(P) \neq 0$. Assume the contrary that $\det(P) = 0$. Then $w^TP = 0$ for some nonzero $w \in \hat{k}_0^n$. Therefore there exists $w_i \in \hat{k}_0^\ell$ for $1 \leq i \leq d$ such that $w_1^TP_1 + \cdots + w_d^TP_d = 0$. Since the P_i have full rank, there exists at least one i such that $w_i^TP_i \neq 0$. Without loss of generality, assume that $w_1^TP_1 \neq 0$. From $w_1^TP_1 = -(w_2^TP_2 + \cdots + w_d^TP_d)$ and $P_iZ_1 = 0$ for $i = 2, \cdots, d$, we have $w_1^TP_1Z = 0$. Since $\det(Z) \neq 0$, $w_1^TP_1 = 0$, a contradiction. Therefore $\det(P) \neq 0$. Let $\ell = \frac{n}{d}$. From (2), $\{\sigma(Y) = AY, \delta(Y) = BY\}$ is equivalent over $\hat{k}_0$ to

$$\sigma(Y) = \mathrm{diag}(A_1, \cdots, A_d)Y, \quad \delta(Y) = \mathrm{diag}(B_1, \cdots, B_d)Y$$

where $A_i \in \mathrm{GL}_\ell(\hat{k}_0)$ and $B_i \in \mathfrak{gl}_\ell(\hat{k}_0)$. Furthermore, U_i is a fundamental matrix of the system $\{\sigma(Z) = A_iZ, \delta(Z) = B_iZ\}$. Since $U_i = g_i(U_1)$, we have that $A_i = g_i(A_1)$

and $B_i = g_i(B_1)$ for each i. From the minimality of V_i, the system $\{\sigma(Z) = A_iZ, \delta(Z) = B_iZ\}$ is irreducible over k. □

Corollary 24. *Let A_i and B_i be as in Theorem 23 for $i = 1, \ldots, d$. Then for each i, the Galois group of $\{\sigma(Z) = A_iZ, \delta(Z) = B_iZ\}$ over k has solvable identity component if and only if so does the Galois group of $\{\sigma(Y) = AY, \delta(Y) = BY\}$ over k_0.*

Remark 25. General factorization of systems of difference-differential equations has been considered in the context of Ore algebras by several authors (Wu (2005), Wu & Li (2007), Cluzeau & Quadrat (2008) and further references in these papers). In Wu (2005) and Wu & Li (2007), the authors consider systems with coefficients in a field whose solution spaces are finite dimensional and gives complete factorization algorithms. In Cluzeau & Quadrat (2008), the authors look at systems that do not necessarily have finite dimensional solutions spaces and also work with systems having coefficients in certain rings. They do not give complete factorization algorithms but rather focus on powerful homological techniques which yield factorizations and first integrals in many physically relevant systems. The factorization results of this section are more specialized and aimed at what is needed for the question at hand - finding liouvillian solutions. On the other hand, we are also concerned with characterizing factorizations in (*a priori* unspecified) extension fields, a situation not considered in the above mentioned papers.

4. Rings of Sequences

In this section we will introduce a $\sigma\delta$–ring of sequences S_K and show that in many cases, a $\sigma\delta$–PV ring can be embedded in this ring. We will then define a notion of liouvillian sequences and show that a system having a full set of solutions of this type will have a Galois group with solvable identity component (Proposition 31). In a later result (Proposition 40), we will show the converse is true as well, at least when $E = \overline{\mathbb{C}(t)}$.

4.1. The Ring of Sequences S_K

Let K be a differential field with a derivation δ. Denote by S_K the set of all sequences of the form $\mathbf{a} = (a_0, a_1, \ldots)$ with $\mathbf{a}(i) = a_i \in K$. Define an equivalence relation on S_K as follows: any two sequences $\mathbf{a}$ and $\mathbf{b}$ are *equivalent* if there exists $N \in \mathbb{Z}_{>0}$ such that $\mathbf{a}(n) = \mathbf{b}(n)$ for all $n > N$. Denote by $\mathcal{S}_K$ the set of equivalence classes of S_K modulo the equivalence relation. One sees that $\mathcal{S}_K$ forms a differential ring with the addition, multiplication and a derivation δ defined on $\mathcal{S}_K$ coordinatewise. Clearly, the map σ given by $\sigma((a_0, a_1, \ldots)) = (a_1, a_2, \ldots)$ is an automorphism of $\mathcal{S}_K$ that commutes with the derivation δ. In addition, any element $e \in K$ is identified with $(e, e, \ldots)$. So we can regard K as a (differential) subfield of $\mathcal{S}_K$.

Let E be a differential field with a derivation δ and suppose that E has algebraically closed constants. Construct an automorphism σ on $E(x)$ given by $\sigma|_E = \mathbf{1}$ and $\sigma(x) = x + 1$ and extend δ to be a derivation δ on $E(x)$ such that $\delta(x) = 0$. Assume that K is a differential field extension of E with an extended derivation δ. The map $E(x) \to S_K$ given by $f \mapsto (0, \ldots, 0, f(N), f(N+1), \ldots)$, where N is a non-negative integer such that f has no poles at integers $\geq N$, induces a $\sigma\delta$-embedding of $E(x)$ into $\mathcal{S}_K$. Consequently, we may identify any matrix M over $E(x)$ with a sequence of matrices $(0, \ldots, 0, M(N), M(N+1), \cdots)$ where $M(i)$ means the evaluation of the entries of M at $x = i$.

We now turn to realize $\sigma\delta$-Picard-Vessiot extensions for linear difference-differential systems over fields of a particular form. We proceed in a manner similar to that of (van der Put & Singer, 2001, Chapter 3). In the following, unless specified otherwise, we always let E be an algebraically closed differential field.

Proposition 26. *Let E be an algebraically closed differential field, $K \supset E$, $E(x)$, $\mathcal{S}_K$ be as above and $\tilde{k} = E(x)$. Assume that E and K have the same algebraically closed field of constants as differential fields. Let R be a $\sigma\delta$-PV extension for a system $\{\sigma(Y) = AY, \delta(Y) = BY\}$ over $\tilde{k}$. Let $N \in \mathbb{Z}_{>0}$ be such that $A(m)$ and $B(m)$ are defined and $\det(A(m)) \neq 0$ for all $m \geq N$, and assume that $\delta(Y) = B(N)Y$ has a fundamental matrix $\overline{Z} \in \mathrm{GL}_n(K)$. Then there exists a $\sigma\delta$-$\tilde{k}$-monomorphism of R into $\mathcal{S}_K$. Moreover, the entries of any solution of $\{\sigma(Y) = AY, \delta(Y) = BY\}$ in $\mathcal{S}_K^n$ lies in the image of R in $\mathcal{S}_K$.*

Proof. Let $R = \tilde{k}[Y, \frac{1}{\det Y}]/I$ be the $\sigma\delta$-PV ring extension for $\{\sigma(Y){=}AY, \delta(Y){=}BY\}$ and let G be its Galois group. From Proposition 9, the corresponding torsor has a point P with coordinates in $\tilde{k}$. This implies that for a new matrix of variables X with $Y = PX$, we have $R = \tilde{k}[X, \frac{1}{\det X}]/J$ where J is the defining ideal of G. Furthermore, $\sigma(X) = \tilde{A}X$ and $\delta(X) = \tilde{B}X$ with $\tilde{A} = \sigma(P)^{-1}AP \in G(\tilde{k})$ and $\tilde{B} = P^{-1}BP - P^{-1}\delta(P) \in \mathfrak{g}(\tilde{k})$, where $\mathfrak{g}$ is the lie algebra of G.

Define recursively a sequence of matrices $Z_m \in \mathrm{GL}_n(K)$ for $m \geq N$:

$$Z_N = \overline{Z} \quad \text{and} \quad Z_{m+1} = A(m)Z_m \quad \text{for any } m \geq N.$$

The integrability condition on A and B implies that Z_m satisfies $\delta(Y) = B(m)Y$ for any $m \geq N$ and so $Z = (\ldots, Z_N, Z_{N+1}, \ldots)$ is a fundamental matrix in $\mathrm{GL}_n(\mathcal{S}_K)$ of $\{\sigma(Y) = AY, \delta(Y) = BY\}$.

Remark that the key to proving the proposition is to show that Z generates a $\sigma\delta$-Picard-Vessiot extension. Unfortunately, we do not see a direct way to show this and our proof is a little circuitous. Clearly, $U := P^{-1}Z$ satisfies that $\sigma(U) = \tilde{A}U$ and $\delta(U) = \tilde{B}U$. Then $\delta(U(N')) = \tilde{B}(N')U(N')$ for a sufficiently large N' and therefore K contains a (differential) Picard-Vessiot extension of E for the equation $\delta(U) = \tilde{B}U$. Since $\tilde{B}(N') \in \mathfrak{g}(\tilde{k})$, Proposition 1.31 (and its proof) in van der Put & Singer (2001) together with the uniqueness of Picard-Vessiot extensions imply that there exists $\overline{V} \in G(K)$ such that $\delta(\overline{V}) = \tilde{B}(N')\overline{V}$. Define $V \in \mathcal{S}_K$ by $V(N') = \overline{V}$ and $V(m+1) = \tilde{A}(m)V(m)$ for $m \geq N'$. Then $\sigma(V) = \tilde{A}V$ and $\delta(V) = \tilde{B}V$ and $V \in G(\mathcal{S}_K)$. This implies that the map from $R = \tilde{k}[X, \frac{1}{\det X}]/J$ to $\mathcal{S}_K$ given by $X \mapsto V$ is a $\sigma\delta$-$\tilde{k}$-homomorphism. Since I is a maximal $\sigma\delta$-ideal, this map must be injective, and so is the desired embedding from R into $\mathcal{S}_K$.

Let $W \in \mathcal{S}_K^n$ be a solution of $\{\sigma(Y) = AY, \delta(Y) = BY\}$. For a sufficiently large M, $W(M)$ is defined and is a solution of $\delta(Y) = B(M)Y$ and $W(m+1) = A(m)W(m)$ for $m \geq M$. Therefore $W(M) = V(M)D$ for some constant vector D and thus $W = VD \in R^n$. It follows that $Z = PU$ also generates a $\sigma\delta$-Picard-Vessiot extension, as claimed above. □

Remark 27. If L is a maximal Picard-Vessiot extension of E with the same constants, then the hypothesis on the existence of $\overline{Z}$ in Proposition 26 is always satisfied. Therefore for such a field L, $\mathcal{S}_L$ contains a $\sigma\delta$-Picard-Vessiot ring for any system $\{\sigma(Y) = AY, \delta(Y) = BY\}$.

4.2. The Ring of Liouvillian Sequences

Let L be a field containing E which satisfies the following conditions:

(i) L is a differential field extension of E having the same field of constants as E;

(ii) every element in L is liouvillian over E (in the standard differential sense);

(iii) L is maximal with respect to (i) and (ii).

Zorn's Lemma guarantees that such a field exists. We refer to L as a *maximal liouvillian extension of the differential field E*. One can show that any differential liouvillian extension of E having the same field of constants as E can be embedded into a maximal liouvillian extension and that any two maximal liouvillian extensions of E are isomorphic over E as differential fields.

Recall that for two sequences $\mathbf{a}$ and $\mathbf{b}$, and for a nonnegative integer m, $\mathbf{b}$ is called the *ith m-interlacing* (Hendriks & Singer, 1999, Definition 3.2) of $\mathbf{a}$ with zeroes if

$$\mathbf{b}(mn+i) = \mathbf{a}(n) \quad \text{and} \quad \mathbf{b}(r) = 0 \quad \text{for any } r \not\equiv i \mod m.$$

A sequence $\mathbf{a}$ is called the *ith m-section* of $\mathbf{b}$ if $\mathbf{a}(mn+i) = \mathbf{b}(mn+i)$ and $\mathbf{a}(r) = 0$ for $r \not\equiv i \mod m$.

Definition 28. Let E be a differential field with algebraically closed constants, and σ an automorphism on $E(x)$ satisfying $\sigma(x) = x+1$ and $\sigma|_E = \mathbf{1}$. Let L be a maximal liouvillian extension of E. The *ring of liouvillian sequences* over $E(x)$ is the smallest subring $\mathcal{L}$ of $\mathcal{S}_L$ such that

(1) $L(x) \subset \mathcal{L}$;

(2) For $\mathbf{a} \in \mathcal{S}_L$, $\mathbf{a} \in \mathcal{L}$ if and only if $\sigma(\mathbf{a}) \in \mathcal{L}$;

(3) Supposing $\sigma(\mathbf{b}) = \mathbf{ab}$ with $\mathbf{a}, \mathbf{b} \in \mathcal{S}_L$, then $\mathbf{a} \in E(x)$ implies $\mathbf{b} \in \mathcal{L}$. $\mathbf{b}$ is called a hypergeometric sequence over $E(x)$;

(4) Supposing $\sigma(\mathbf{b}) = \mathbf{a} + \mathbf{b}$ with $\mathbf{a}, \mathbf{b} \in \mathcal{S}_L$, then $\mathbf{a} \in \mathcal{L}$ implies $\mathbf{b} \in \mathcal{L}$;

(5) For $\mathbf{a} \in \mathcal{S}_L$, $\mathbf{a} \in \mathcal{L}$ implies $\mathbf{b} \in \mathcal{L}$, where $\mathbf{b}$ is the ith m-interlacing of $\mathbf{a}$ with zeroes for some $m \in \mathbb{Z}_{>0}$ and $0 \leq i \leq m-1$.

Set $\tilde{k} = E(x)$ and let $\mathcal{L}$ be the ring of liouvillian sequences over $\tilde{k}$. From the remarks in (Hendriks & Singer, 1999, p. 243), if $\mathbf{b} \in \mathcal{S}_L$ belongs to $\mathcal{L}$ then the ith m-section of $\mathbf{b}$ also belongs to $\mathcal{L}$ for any i and m. We claim that $\mathcal{L}$ is a $\sigma\delta$-ring. Since $\mathcal{L}$ can be constructed inductively using (1) - (5) above, it is enough to show the following statements:

(1') If $f \in L(x)$ then $\delta(f) \in L(x)$;

(2') If $\mathbf{a} \in \mathcal{S}_L$, $\mathbf{a}, \delta(\mathbf{a}) \in \mathcal{L}$ then $\delta(\sigma(\mathbf{a})) \in \mathcal{L}$;

(3') Supposing $\sigma(\mathbf{b}) = \mathbf{ab}$ with $\mathbf{a}, \mathbf{b} \in \mathcal{S}_L$, then $\mathbf{a}, \delta(\mathbf{a}) \in E(x)$ implies $\delta(\mathbf{b}) \in \mathcal{L}$;

(4') Supposing $\sigma(\mathbf{b}) = \mathbf{a} + \mathbf{b}$ with $\mathbf{a}, \mathbf{b} \in \mathcal{S}_L$, then $\mathbf{a}, \delta(\mathbf{a}) \in \mathcal{L}$ implies $\delta(\mathbf{b}) \in \mathcal{L}$;

(5') For $\mathbf{a} \in \mathcal{S}_L$, $\mathbf{a}, \delta(\mathbf{a}) \in \mathcal{L}$ implies $\delta(\mathbf{b}) \in \mathcal{L}$, where $\mathbf{b}$ is the ith m-interlacing of $\mathbf{a}$ with zeroes for some $m \in \mathbb{Z}_{>0}$ and $0 \leq i \leq m-1$.

Verifying $(1')$, $(2')$ an $(5')$ is straightforward. To verify $(3')$, suppose that $\mathbf{b} \in \mathcal{L}$ with $\sigma(\mathbf{b}) = \mathbf{ab}$. Set $y = \delta(\mathbf{b})$. Then $\sigma(y) - \mathbf{a}y = \delta(\mathbf{a})\mathbf{b}$. Since $\mathbf{a} \in \tilde{k}$, $\mathbf{b}$ is invertible. Then $\mathbf{u} := \delta(\mathbf{b})/\mathbf{b}$ satisfies $\sigma(\mathbf{u}) - \mathbf{u} = \delta(\mathbf{a})/\mathbf{a}$. Therefore $\mathbf{u} \in \mathcal{L}$ and so $\delta(\mathbf{b}) = \mathbf{ub} \in \mathcal{L}$. To verify $(4')$, suppose that $\mathbf{b} \in \mathcal{L}$ with $\sigma(\mathbf{b}) = \mathbf{a} + \mathbf{b}$. Then $\sigma(\delta(\mathbf{b})) = \delta(\mathbf{b}) + \delta(\mathbf{a})$ which implies that $\delta(\mathbf{b}) \in \mathcal{L}$.

A vector is said to be *hypergeometric* over $\tilde{k}$ if it can be written as Wh where $W \in \tilde{k}^n$ and h is a hypergeometric sequence over $\tilde{k}$.

Let $V \in \mathcal{L}^n$ be a nonzero solution of $\{\sigma^d(Y) = A_d Y, \delta(Y) = BY\}$, where $d \in \mathbb{Z}_{>0}$ and $N \in \mathbb{Z}_{>0}$ be such that $V(N) \neq 0$, $A(j)$ and $B(j)$ are well defined and $\det(A(j)) \neq 0$ for $j \geq N$. We define a vector W in the following way:

$$W(N) = V(N) \quad \text{and} \quad W(j+1) = A(j)W(j) \quad \text{for } j \geq N.$$

Since $\delta(V(N)) = B(N)V(N)$, the integrability condition on σ and δ implies that W is a nonzero solution of $\{\sigma(Y) = AY, \delta(Y) = BY\}$. The proposition below says that W also belongs to $\mathcal{L}^n$.

Proposition 29. *Let W be as above. Then $W \in \mathcal{L}^n$.*

Proof. Let $N = d\ell + m$ where $\ell, m \in \mathbb{Z}_{>0}, 0 < m < \ell$ and V_0 be the mth d-section of V. Then V_0 is a solution of $\{\sigma^d(Y) = A_d Y, \delta(Y) = BY\}$ and $V_0 \in \mathcal{L}^n$. Let

$$V_i(j) = A(j)^{-1} V_{i-1}(j+1) \quad \text{for } j \geq N \text{ and } i = 1, \cdots, d-1,$$

and $U = V_0 + V_1 + \cdots + V_{d-1}$. Then $U \in \mathcal{L}^n$. We shall prove that $W = U$. Note that for $j > N$,

$$V_i(j) = A(j)^{-1} A(j+1)^{-1} \cdots A(j+i-1)^{-1} V_0(j+i).$$

In particular, $V_{d-1}(j+1) = A(j)V_0(j)$. Then $V_i(N) = 0$ for $i = 1, \cdots, d-1$. Therefore $W(N) = V_0(N) + V_1(N) + \cdots + V_{d-1}(N) = U(N)$ and

$$\begin{aligned} U(j+1) &= V_0(j+1) + V_1(j+1) + \cdots + V_{d-1}(j+1) \\ &= A(j)V_1(j) + A(j)V_2(j) + \cdots + A(j)V_0(j) = A(j)U(j) \end{aligned}$$

for $j \geq N$. Hence $W = U \in \mathcal{L}^n$. □

Definition 30. Let E be an algebraically closed differential field with a derivation δ and $E(x)$ be a $\sigma\delta$–field constructed as above. A system $\{\sigma(Y) = AY, \delta(Y) = BY\}$ over $E(x)$ is said to be *solvable in terms of liouvillian sequences* if the $\sigma\delta$-PV extension of this system embeds, over $E(x)$, into $\mathcal{L}$, the ring of liouvillian sequences over $E(x)$.

Clearly, Definition 30 generalizes the notion of solvability in liouvillian terms for linear differential equations and that for linear difference equations. In addition, we shall show in Proposition 31 that this property is also equivalent to the Galois group having solvable identity component.

4.3. Systems with Liouvillian Sequences as Solutions

In this subsection, we first prove that the Galois group of a system, which is solvable in terms of liouvillian sequences, has solvable identity component. Then we prove that if the Galois group of an irreducible system over $\mathbb{C}(x,t)$ has solvable identity component, then it will be equivalent over $\overline{\mathbb{C}(t)}(x)$ to a system in diagonal form. Based on this, we will show that the converse of Proposition 31 is true as well, in the case $E = \overline{\mathbb{C}(t)}$.

Proposition 31. *Let $E(x)$ be as in Definition 30. If a system $\{\sigma(Y) = AY, \delta(Y) = BY\}$ over $E(x)$ is solvable in terms of liouvillian sequences, then the identity component of its Galois group is solvable.*

Proof. Let R, L and $\mathcal{L}$ be the $\sigma\delta$-PV ring, a maximal liouvillian extension of the differential field E and the ring of liouvillian sequences for the given system, respectively. We may assume $R \subset \mathcal{L}$. Consider the following diagram

$$\begin{array}{ccc} & RL(x) & \\ / & & \backslash \\ R & & L(x) \\ \backslash & & / \\ & R \cap L(x) & \\ & | & \\ & E(x) & \end{array}$$

where $RL(x)$ is the field generated by R and $L(x)$. We will show that

(i) $RL(x) \subset \mathcal{L}$ is a $\sigma\delta$-PV extension of $L(x)$ and its Galois group has solvable identity component.

(ii) The $\sigma\delta$-Galois group of $RL(x)$ over $L(x)$ is isomorphic to the subgroup H of the Galois group G of R over $E(x)$ that leaves the quotient field of $R \cap L(x)$ fixed. Moreover, H is a closed normal subgroup of G.

(iii) G/H is the Galois group of the quotient field of $R \cap L(x)$ over $E(x)$ and has solvable identity component.

Once the above claims are proven, the group G has a solvable identity component since both H and G/H have solvable identity component.

To prove (i), we consider $\{\sigma(Y) = AY, \delta(Y) = BY\}$ as a system over $L(x)$. Since $R \subset \mathcal{L}$, then for a sufficiently large N there is a fundamental matrix of $\delta(Y) = B(N)Y$ with entries in L. Applying Proposition 26, we conclude that $\mathcal{S}_L$ contains a $\sigma\delta$-PV extension T of $L(x)$ for the system $\{\sigma(Y) = AY, \delta(Y) = BY\}$ and that $R \subset T$. This implies that $RL(x)$ is a $\sigma\delta$-PV extension of $L(x)$. Proposition 4.1 of van der Put & Singer (1997) implies that $RL(x)$ is also a difference Picard-Vessiot extension for $\sigma(Y) = AY$ and the results of Hendriks & Singer (1999) then imply that the *difference* Galois group has solvable identity component. The $\sigma\delta$-Galois group is a subgroup of this latter group and its identity component is a subgroup of the identity component of the larger group. Therefore it is also solvable.

To prove (ii), let F_L be the total ring of fractions of $RL(x)$ and F_E the total ring of fractions of R. Corollary 20 implies that we can regard F_E as a subset of F_L. The elements of the $\sigma\delta$-Galois group $\bar{G}$ of F_L over $L(x)$ restrict to automorphisms of F_E over $E(x)$, and this gives a homomorphism of this group into H. Clearly the image is closed and the set of elements left fixed by this group is $L(x) \cap F_E$, the quotient field of $L(x) \cap R$. Therefore this image is H.

Before proving that H is normal in G, we first show that

$$F_E \cap L(x) = (F_E \cap L)(x).$$

Since $x \in F_E$ we have $F_E \cap L(x) \supset (F_E \cap L)(x)$. To get the reverse inclusion, let $f \in F_E \cap L(x)$ and write

$$f = \frac{a_r x^r + \ldots + a_0}{b_s x^s + \ldots + b_0}, \ a_i, b_i \in L$$

where the numerator and denominator are relatively prime. We then have that

$$\{x^s f, x^{s-1} f, \ldots, f, x^r, \ldots, 1\}$$

are linearly dependent over L. Since L is the set of σ-invariant elements of $L(x)$, the cassoratian of these elements must vanish (Cohn (1967), p.271). This implies further that these elements are linearly dependent over the σ-invariant elements of the field $F_E \cap L(x)$.

Therefore there exist σ-invariant elements $\tilde{a}_i$, $\tilde{b}_j \in F_E \cap L$ for $i = 0, 1, \ldots, r$ and $j = 0, 1, \ldots, s$, not all zero, such that

$$\tilde{a}_r x^r + \cdots + \tilde{a}_0 - (\tilde{b}_s x^s f + \cdots + \tilde{b}_0 f) = 0.$$

Since x is transcendental over σ-invariant elements, there exists at least one $\tilde{b}_i$ which is not zero. Hence

$$f = \frac{\tilde{a}_r x^r + \ldots \tilde{a}_0}{\tilde{b}_s x^s + \ldots + \tilde{b}_0} \in (F_E \cap L)(x).$$

To show that H is normal, it now suffices to prove that any $\sigma\delta$-automorphism of F_E over k leaves the field $F_E \cap L(x) = (F_E \cap L)(x)$ invariant. Note that L is the set of σ-invariant elements of F_L and so $F_E \cap L$ is the set of σ-invariant elements of F_E. This set is clearly preserved by any $\sigma\delta$–automorphism.

To prove (iii), note that since H is normal, the *field* $(F_E \cap L)(x)$ is a $\sigma\delta$–PV extension of $E(x)$. Furthermore, $(F_E \cap L)(x)$ lies in the liouvillian extension $L(x)$ of $E(x)$. Theorem 15 implies that its Galois group is solvable. □

In what follows, let k_0 and k be the same as in Section 3, that is, k_0 denotes the field $\mathbb{C}(t, x)$ with an automorphism $\sigma : x \mapsto x + 1$ and a derivation $\delta = \frac{d}{dt}$ and k denotes the extension field $\overline{\mathbb{C}(t)}(x)$. We will first prove that if a system $\{\sigma(Y) = AY, \delta(Y) = BY\}$ over k_0 is irreducible over k_0 and its Galois group over k_0 has solvable identity component then there exists $\ell \in \mathbb{Z}_{>0}$ with $\ell | n$ such that the solution space of $\{\sigma^\ell(Y) = A_\ell Y, \delta(Y) = BY\}$ has a basis each of whose members is the interlacing of hypergeometric solutions over k. We will further show that if the Galois group for the system $\{\sigma(Y) = AY, \delta(Y) = BY\}$ has solvable identity component, then the system will be solvable in terms of liouvillian sequences.

By Theorem 23, if $\{\sigma(Y) = AY, \delta(Y) = BY\}$ is an irreducible system over k_0, then it can be decomposed into factors that are irreducible over k and if the Galois group of the system over k_0 has solvable identity component then so do the Galois groups of these factors over k. Hence it is enough to consider factors of the original system over k.

Proposition 32. *Suppose that $\{\sigma(Y) = \mathcal{A}Y, \delta(Y) = \mathcal{B}Y\}$ is an irreducible system over k and that its Galois group over k has solvable identity component. Then the system is equivalent over k to*

$$\sigma(Y) = \bar{\mathcal{A}}Y, \quad \delta(Y) = \bar{\mathcal{B}}Y$$

where $\bar{\mathcal{B}} \in \mathfrak{gl}_\ell(k)$ and

$$\bar{\mathcal{A}} = \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \\ a & 0 & 0 & \cdots & 0 \end{pmatrix} \in \mathrm{GL}_\ell(k)$$

with $a \in k$.

Proof. The proof is similar to those of Lemma 4.1 and Theorem 5.1 in Hendriks & Singer (1999). □

Remark 33. From the proof of Lemma 4.1 in Hendriks & Singer (1999), we know that ℓ divides $|\mathrm{Gal}(\mathcal{R}/k)/\mathrm{Gal}(\mathcal{R}/k)^0|$ because $\{\sigma(Y) = \mathcal{A}Y, \delta(Y) = \mathcal{B}Y\}$ is irreducible over k. From the proof of Theorem 5.1 in Hendriks & Singer (1999), $\mathrm{Gal}(\mathcal{R}/k)^0$ is diagonalizable.

As a consequence of Proposition 32, we have the following

Corollary 34. *If $\{\sigma(Y) = \mathcal{A}Y, \delta(Y) = \mathcal{B}Y\}$ is an irreducible system over k and its Galois group over k has solvable identity component, then $\{\sigma^\ell(Y) = \mathcal{A}_\ell Y, \delta(Y) = \mathcal{B}Y\}$ is equivalent over k to*

$$\sigma^\ell(Y) = \mathcal{D}Y, \quad \delta(Y) = \bar{\mathcal{B}}Y$$

where $\bar{\mathcal{B}} \in \mathfrak{gl}_\ell(k)$ and $\mathcal{D} = \mathrm{diag}(a, \sigma(a), \cdots, \sigma^{\ell-1}(a))$ with a as indicated in Proposition 32.

Next, we shall prove further that $\{\sigma^\ell(Y) = \mathcal{A}_\ell Y, \delta(Y) = \mathcal{B}Y\}$ is equivalent over k to a system in diagonal form. Note that equivalent systems have the same Picard-Vessiot extension. We start with two lemmas.

Lemma 35. *Assume that $w \in k$ satisfies*

$$\sigma^s(w) = \sigma^{s-1}(b) \cdots \sigma(b)bw$$

where $s \in \mathbb{Z}_{>0}$, $b \in k \setminus \{0\}$ and $b = x^\nu + b_1 x^{\nu-1} + \cdots$ with $\nu \in \mathbb{Z}$ and $b_i \in E$. Then $\sigma(w) = bw$.

Proof. We have

$$\sigma^s\left(\frac{\sigma(w)}{b}\right) = \frac{1}{\sigma^s(b)}\sigma\left(\sigma^{s-1}(b)\cdots\sigma(b)bw\right) = \sigma^{s-1}(b)\cdots\sigma(b)b\frac{\sigma(w)}{b}.$$

Since $\frac{\sigma(w)}{b} \in k$, $\frac{\sigma(w)}{b} = cw$ for some $c \in k^{\sigma^s} = k^\sigma$. Note that w and b are rational functions in x. Expanding w and b as Laurent series at $x = \infty$. By comparing the coefficients, we get $c = 1$, so $\sigma(w) = bw$. □

Lemma 36. *Let $A = \mathrm{diag}(a_1, \cdots, a_n)$ where $a_i \in k \setminus \{0\}$ and $a_i = cx^{\nu_i} + a_{i1}x^{\nu_i - 1} + \cdots$ with $\nu_i \in \mathbb{Z}$ and $c, a_{ij} \in \overline{\mathbb{C}(t)}$. Assume that the system $\{\sigma^d(Y) = A_d Y, \delta(Y) = BY\}$ is equivalent over k to*

$$\sigma^d(Y) = A_d Y, \quad \delta(Y) = \mathrm{diag}(b_1, \cdots, b_n)Y$$

where $b_i \in k$ for $i = 1, \ldots, n$. Then $\{\sigma(Y) = AY, \delta(Y) = BY\}$ is equivalent over k to

$$\sigma(Y) = AY, \quad \delta(Y) = \mathrm{diag}(b_1, \cdots, b_n)Y.$$

Proof. From the assumption, there exists $G \in \mathrm{GL}_n(k)$ such that

$$\sigma^d(G)A_d = A_d G, \quad G^{-1}BG - G^{-1}\delta(G) = \mathrm{diag}(b_1, \cdots, b_n).$$

It then suffices to prove that $\sigma(G)A = AG$. Let $G = (g_{ij})_{n\times n}$. Then

$$\begin{cases} \sigma^d(g_{ii}) - g_{ii} = 0, & i = 1, \cdots, n; \\ \sigma^d(g_{ij}) = \sigma^{d-1}\left(\frac{a_i}{a_j}\right)\cdots\sigma\left(\frac{a_i}{a_j}\right)\frac{a_i}{a_j}g_{ij}, & 1 \le i \neq j \le n. \end{cases}$$

By Lemma 35, $\sigma(g_{ij}) = \frac{a_i}{a_j}g_{ij}$ for all $i, j = 1, \cdots, n$. This implies that $\sigma(G)A = AG$. □

Proposition 37. *If $\{\sigma(Y) = \mathcal{A}Y, \delta(Y) = \mathcal{B}Y\}$ is an irreducible system over k and its Galois group over k has solvable identity component, then $\{\sigma^\ell(Y) = \mathcal{A}_\ell Y, \delta(Y) = \mathcal{B}Y\}$ is equivalent over k to*

$$\sigma^\ell(Y) = \mathcal{D}Y, \quad \delta(Y) = \mathrm{diag}(b_1, \cdots, b_\ell)Y$$

with $\mathcal{D}$ as in Corollary 34 and $b_i \in k$ for $i = 1, \ldots, \ell$.

Proof. By Corollary 34, $\{\sigma^\ell(Y) = \mathcal{A}_\ell Y, \delta(Y) = \mathcal{B}Y\}$ is equivalent over k to $\{\sigma^\ell(Y) = \mathcal{D}Y, \delta(Y) = \bar{\mathcal{B}}Y\}$ where $\bar{\mathcal{B}} \in \mathfrak{gl}_\ell(k)$ and

$$\mathcal{D} = \mathrm{diag}(a, \sigma(a), \cdots, \sigma^{\ell-1}(a))$$

with a as in Proposition 32. Let $\mathcal{R} = \bar{e}_0\mathcal{R} \oplus \bar{e}_1\mathcal{R} \oplus \cdots \oplus \bar{e}_v\mathcal{R}$ be the decomposition of $\mathcal{R}$. Then $\mathrm{Gal}(\mathcal{R}/k)^0 = \mathrm{Gal}(\bar{e}_0\mathcal{R}/k)$ by Lemma 7 and $\mathrm{Gal}(\bar{e}_0\mathcal{R}/k)$ is diagonalizable by Remark 33. From Lemma 3, it follows that $\bar{e}_0\mathcal{R}$ is a $\sigma^v\delta$-PV extension of k for the system

$$\sigma^v(Y) = \sigma^{\frac{v}{\ell}-1}(\mathcal{D}) \cdots \sigma(\mathcal{D})\mathcal{D}Y, \quad \delta(Y) = \bar{\mathcal{B}}Y. \tag{3}$$

Let $\tilde{\mathcal{D}} = \sigma^{\frac{v}{\ell}-1}(\mathcal{D}) \cdots \sigma(\mathcal{D})\mathcal{D} = \mathrm{diag}(\bar{d}_1, \cdots, \bar{d}_\ell)$. By Lemma 2, $\bar{e}_0\mathcal{R}$ is a domain. As in the differential case, we can show that (3) is equivalent over k to the system

$$\sigma^v(Y) = \mathrm{diag}(\bar{a}_1, \cdots, \bar{a}_\ell)Y, \quad \delta(Y) = \mathrm{diag}(\bar{b}_1, \cdots, \bar{b}_\ell)Y \tag{4}$$

where $\bar{a}_i, \bar{b}_i \in k$. Then there exists $G = (g_{ij}) \in \mathrm{GL}_\ell(k)$ such that

$$\sigma^v(G)\mathrm{diag}(\bar{a}_1, \cdots, \bar{a}_\ell) = \tilde{\mathcal{D}}G,$$

which implies $\sigma^\ell(g_{ij})\bar{a}_j = g_{ij}\bar{d}_j$. Since $\det(G) \neq 0$, there is a permutation $i_1, \cdots, i_\ell$ of $\{1, 2, \cdots, \ell\}$ such that $g_{1i_1}g_{2i_2} \cdots g_{\ell i_\ell} \neq 0$. Hence

$$\bar{d}_j = \frac{\sigma^\ell(g_{ji_j})}{g_{ji_j}}\bar{a}_{i_j} \quad \text{for } j = 1, \cdots, \ell.$$

Let P be a multiplication of some permutation matrices such that

$$P^{-1}\mathrm{diag}(\bar{a}_1, \cdots, \bar{a}_\ell)P = \mathrm{diag}(\bar{a}_{i_1}, \cdots, \bar{a}_{i_\ell}),$$

and let $T = P\mathrm{diag}(1/g_{1i_1}, \cdots, 1/g_{\ell i_\ell})$. Under the transformation $Y \to TY$, the system (4), and therefore (3), is equivalent over k to

$$\sigma^v(Y) = \tilde{\mathcal{D}}Y, \quad \delta(Y) = \mathrm{diag}(b_1, \cdots, b_\ell)Y$$

where $b_i \in k$ for $i = 1, \ldots, \ell$. Lemma 36 implies that $\{\sigma^\ell(Y) = \mathcal{D}Y, \delta(Y) = \bar{\mathcal{B}}Y\}$ is equivalent over k to

$$\sigma^\ell(Y) = \mathcal{D}Y, \quad \delta(Y) = \mathrm{diag}(b_1, \cdots, b_\ell)Y.$$

This concludes the proposition. □
Theorem 23 together with Proposition 37 leads to the following

Proposition 38. *If $\{\sigma(Y) = AY, \delta(Y) = BY\}$ is an irreducible system over k_0 and its Galois group over k_0 has solvable identity component, then there exists $\ell \in \mathbb{Z}_{>0}$ with $\ell | n$ such that the solution space of $\{\sigma^\ell(Y) = A_\ell Y, \delta(Y) = BY\}$ has a basis consisting of the interlacing of hypergeometric solutions over k.*

Proof. By Theorem 23 and Proposition 37, there is $\ell \in \mathbb{Z}_{>0}$ with $\ell | n$ such that $\{\sigma^\ell(Y) = A_\ell Y, \delta(Y) = BY\}$ is equivalent over k to a system in diagonal form. Since the solution space of the latter system has a basis consisting of the interlacing of hypergeometric solutions over k, so does the solution space of $\{\sigma^\ell(Y) = A_\ell Y, \delta(Y) = BY\}$. □

Corollary 39. *Let $\mathcal{L}$ be the ring of liouvillian sequences over k. Assume that $\{\sigma(Y) = AY, \delta(Y) = BY\}$ is an irreducible system over k_0 and its Galois group over k_0 has solvable identity component. Then the solution space of the system has a basis with entries in $\mathcal{L}$.*

Proof. Proposition 38 implies that there is $\ell \in \mathbb{Z}_{>0}$ such that the solution space of $\{\sigma^\ell(Y) = A_\ell Y, \delta(Y) = BY\}$ has a basis with entries in $\mathcal{L}$. The corollary then follows from Proposition 29. □

Let us turn to a general case where a difference-differential system may be reducible over the base field. If the Galois group over the base field of the given system has solvable identity component, then the Galois group over the base field of each factor is of the same type. The method in Hendriks & Singer (1999) together with the results in Bronstein et al. (2005) implies the following

Proposition 40. *If the Galois group for a system $\{\sigma(Y) = AY, \delta(Y) = BY\}$ over k_0 has solvable identity component, then the solution space of the system has a basis with entries in $\mathcal{L}$.*

Proof. By induction, we only need to prove the proposition for the case where the given system has two irreducible factors over k_0. In this case, the given system is equivalent over k_0 to

$$\sigma(Y) = \begin{pmatrix} A_1 & 0 \\ A_3 & A_2 \end{pmatrix} Y, \qquad \delta(Y) = \begin{pmatrix} B_1 & 0 \\ B_3 & B_2 \end{pmatrix} Y$$

where the systems $\{\sigma(Y) = A_i Y, \delta(Y) = B_i Y\}$ for $i = 1, 2$ are both irreducible over k_0. Let d_i be the order of A_i for $i = 1, 2$. By Corollary 39, each system $\{\sigma(Y) = A_i Y, \delta(Y) = B_i Y\}$ has a fundamental matrix $U_i \in \mathrm{GL}_{d_i}(\mathcal{L})$. From the proof of Theorem 3 in Bronstein et al. (2005), Proposition 26 and Remark 27, it follows that the original system has a $\sigma\delta$-PV extension $\mathcal{R}$ of k_0 which contains entries of the U_i's and, moreover, has a fundamental matrix over $\mathcal{S}_K$ of the form

$$\begin{pmatrix} U_1 & 0 \\ V & U_2 \end{pmatrix}.$$

So $\sigma(V) = A_1 U_1 + A_2 V$. Let $V = U_2 W$. Then $\sigma(W) = W + \sigma(U_2)^{-1} A_1 U_1$. Since $U_i \in \mathrm{GL}_{d_i}(\mathcal{L})$, the entries of W are in $\mathcal{L}$ and so are the entries of V. □

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