LIOUVILLIAN SOLUTIONS OF n-th ORDER HOMOGENEOUS LINEAR DIFFERENTIAL EQUATIONS

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1. Introduction. Let

$$L = \frac{d^n}{dx^n} + p_{n-1}(x)\frac{d^{n-1}}{dx^{n-1}} + \cdots + p_0(x) \tag{1.1}$$

be a linear differential operator with coefficients in F, a finite algebraic extension of $\mathbf{Q}(x)$. We shall show that one can find, in a finite number of steps, a basis for the vector space of liouvillian solutions of $L(Y) = 0$ (i.e., those solutions which can be built up from the rational functions by algebraic operations, taking exponentials and by integration; see Section 2 for a precise definition). In particular, we show how to decide if all solution of $L(Y) = 0$ are liouvillian or if there are any solutions which are liouvillian. Our algorithm in conjunction with the algorithm in [8], also allows one to determine the algebraic relationships among the liouvillian solutions of $L(Y) = 0$, and, in particular, determine if all solutions are algebraic (c.f. [11]).

For second order linear homogeneous equations over $\mathbf{Q}(x)$, the problem of determining if all solutions are algebraic functions was considered by Fuchs, Klein and Schwarz, but none of these mathematicians seems to have presented a complete decision procedure. Building on the work of Klein, Baldassarri and Dwork [2] have given such a procedure. Baldassarri [1] has extended this to consider linear homogeneous equations whose coefficients are algebraic functions. For third order equations, Painlevé and Boulanger gave a procedure which, in effect, reduced the problem of finding algebraic solutions of $L(Y) = 0$ to the problem of effectively bounding the torsion of the jacobian variety of a given curve. A complete procedure for deciding if all solutions of an n-th order homo-

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geneous linear differential equation are algebraic, is presented in [13], where additional references can also be found.

An algorithm for finding all liouvillian solutions of a second order linear homogeneous equation over $\mathbf{C}(x)$, was presented by J. Kovacic [7]. Although the algorithm we present here for equations of arbitrary order is not nearly as efficient as Kovacic's for second order equations, we hope the ideas presented here will lay the groundwork for practical algorithms for solving differential equations of higher order.

Our algorithm relies on the following fact (Proposition 2.2): There is a computable integer valued function $I(n)$ such that if G is a subgroup of $GL(n, \mathbf{C})$ and H is a subgroup of G of finite index which leaves a one dimensional subspace of $\mathbf{C}^n$ invariant, then there exists a subgroup H' of G of index at most $I(n)$ which leaves a one dimensional subspace of $\mathbf{C}^n$ invariant. The galois theory of linear differential equations then allows us to conclude that if $L(Y) = 0$ has a liouvillian solution, then $L(Y) = 0$ has a solution u such that u'/u is algebraic over F of degree at most $I(n)$. Letting u be such a solution and $P(u)$ be the minimal polynomial of u over $\mathbf{C}(x)$, we then show in Sections 3 and 4, how to get a priori bounds, in terms of L, for the degrees of the numerators and the denominators of the coefficients of P. This allows us to reduce the question of the existence of a liouvillian solution of $L(Y) = 0$ to a question in elimination theory. We then show how to inductively find a basis for the space of liouvillian solutions of $L(Y) = 0$.

$\mathbf{Q}$ will denote the rational numbers and $\mathbf{C}$ will denote the complex numbers. All fields mentioned in this paper will be assumed to be of characteristic zero.

2. Liouvillian Solutions of Linear Differential Equations. In this section we shall give a precise definition of a liouvillian function and develop the group theory which, when combined with the galois theory of differential equations, gives us one of the basic facts (Theorem 2.4) on which our decision procedure is based.

The group theory we need is based on the following theorem of Jordan:

THEOREM 2.1. *Let C be an algebraically closed field of characteristic zero. There exists an integer-valued function $J(n)$, depending only on n, such that every finite subgroup of $GL(n, C)$ contains an abelian normal subgroup of index at most $J(n)$.*

Various authors have given bounds for $J(n)$. For example, Schur showed ([3], p. 258) that we can take $J(n) \leq (\sqrt{8n} + 1)^{2n^2} - (\sqrt{8n} -$

$1)^{2n^2}$. For small values of n, this can be greatly improved, for example, $J(2) = 12$, $J(3) = 120$.

Our main group theoretic tool is the following:

PROPOSITION 2.2. *There exists an integer-valued function $I(n)$, depending on n, such that: if G is a subgroup of $GL(n, C)$ and H is a subgroup of G of finite index which leaves a one dimensional subspace of C^n invariant, then there exists a subgroup H' of G of index $\leq I(n)$ which leaves a one dimensional subspace of C^n invariant.*

In order to prove this proposition, we will need a technical lemma. Let S be a subset of $GL(n, C)$. A subspace $V \subset C^n$ is called a *maximal eigenspace* for S if, for each $\sigma \in S$ there is a $c_\sigma \in C$ such that $\sigma v = c_\sigma v$ for all v in V and V is maximal with respect to this property.

LEMMA 2.3. *Let $S \subset GL(n, C)$. There exists at most n maximal eigenspaces for S.*

Proof. It is enough to show that if $V_1, \ldots, V_k$ are distinct maximal eigenspaces for S then the sum $V_1 + \cdots + V_k$ is direct. We do this by induction on k. First notice that if V and W are maximal eigenspaces then either $V \cap W = \{0\}$ or $V = W$. Therefore our assertion is true for $k = 2$. Now assume $k > 2$ and $V_2 + \cdots + V_k = V_2 \oplus \cdots \oplus V_k$. We wish to show $V_1 \cap V_2 \oplus \cdots \oplus V_k = \{0\}$. If not, let $v \neq 0$ be in $V_1 \cap V_2 \oplus \cdots \oplus V_k$. Then $v = v_2 + \cdots + v_k$ with $v_i \in V_i$. We can assume $v_2 \neq 0$. Since V_1 and V_2 are maximal eigenspaces for S and $V_1 \neq V_2$ we can find a $\sigma \in S$ such that $\sigma v = cv$, $\sigma v_2 = dv_2$ and $c \neq d$. On the other hand, $\sigma v = c(v_2 + \cdots) = dv_2 + \cdots$. Since $V_2 \oplus \cdots \oplus V_n$ is a direct sum, we must have $c = d$, a contradiction. Therefore, $V_1 + \cdots + V_k = V_1 \oplus \cdots \oplus V_k$. □

Proof of Proposition 2.2. We first show that we can assume that H is normal in G. Let G act, by left multiplication, on the set of left cosets of H. This gives a homomorphism of G into some finite symmetric group. The kernel $\overline{H}$ of this homomorphism is a subgroup of H, normal in G and of finite index in G. Since $\overline{H} \subset H$, $\overline{H}$ leaves a one dimensional subspace of C^n invariant. Therefore we can replace H by $\overline{H}$, if necessary, and assume H is normal in G.

Let $\{V_1, \ldots, V_k\}$ be the maximal eigenspaces of H. By hypothesis, this set is not empty and by Lemma 2.3, this set contains at most n elements. Let $m = \max_i \{\dim V_i\}$.

Case 1. $m = n$. In this case $k = 1$, $V_1 = C^n$, and H is a subgroup C_n, the group of scalar matrices. Let $\mathrm{PGL}(n-1, C) = \mathrm{GL}(n, C)/C_n$. Note $\mathrm{PGL}(n-1, C) = \mathrm{SL}(n, C)/(C_n \cap \mathrm{SL}(n, C))$. Let $\Phi: \mathrm{GL}(n, C) \to \mathrm{PGL}(n-1, C)$ and $\Psi: \mathrm{SL}(n, C) \to \mathrm{PGL}(n-1, C)$ be the canonical homomorphisms. If we restrict Φ to G, its kernel contains H. Therefore, $\Phi(G)$ is finite. Furthermore, $\Psi^{-1}(\Phi(G))$ will be a finite subgroup of $\mathrm{SL}(n, C)$. Theorem 2.3 implies that there exists an abelian subgroup of $\tilde{H}$ of $\Psi^{-1}(\Phi(G))$ of index $\leq J(n)$. Since C_n is the center of $\mathrm{GL}(n, C)$, $H' = \Phi^{-1}(\Psi(\tilde{H}))$ will be an abelian subgroup of G of index $\leq J(n)$. Since H' is abelian, we can simultaneously put the elements of H' in Jordan normal form. H' therefore leaves a one dimensional subspace of C^n invariant.

Case 2. $m < n$. Since H is normal in G, G permutes the elements $\{V_1, \ldots, V_k\}$ and we get a homomorphism of G into the symmetric group on k elements. Let K be the kernel of this homomorphism. K leaves V_1 invariant and contains H. Since $\dim V_1 < n$, induction on n allows us to assume K contains a subgroup H' of index at most $I(n-1)$ in K, such that H' leaves a one dimensional subspace of C^n invariant. Since the index of K in G is at most $k! \leq n!$, we have that the index of H' in G is at most $n!I(n-1)$. □

Remark. By examining the proof of Proposition 2.2, we can give an inductive bound on $I(n)$:

$$I(n) \leq \max\{J(n), n!I(n-1)\}.$$

It would be of some interest to get a better estimate on $I(n)$. Towards this aim, we note that we do not need to use the full strength of Jordan's theorem in our proof. It would suffice to know that a finite subgroup of $\mathrm{GL}(n, C)$ contains a subgroup H' of bounded index $J'(n)$ such that H' leaves a one dimensional subspace of C^n invariant. A good estimate for $J'(n)$ would yield a good estimate for $I(n)$.

Let F be a differential field, that is a field together with an operation $': F \to F$ (called a derivation) satisfying $(a+b)' = a' + b'$ and $(ab)' = (a')b + a(b')$ for all a, b in F. For example we can take $F = \mathbf{C}(x)$ with derivation d/dx or take F be a field of functions, meromorphic in some domain in $\mathbf{C}$, such that F is closed under the operation of taking derivatives. The set $\{c \mid c \in F \text{ and } c' = 0\}$ forms of subfield of F called the *field of constants of F* and is denoted $\mathcal{C}_F$. Let E be a differential extension of F, that is, E is a field extension of F and the derivation of E extends the derivation of F. We say E is a *liouvillian extension* of F if there exist intermediate fields

$$F = F_0 \subset F_1 \subset \cdots \subset F_n = E$$

such that $F_i = F_{i-1}(\eta_i)$, $i = 1, \ldots, n$ where either:

1. η_i is algebraic over F_{i-1}; or
2. η_i' is in F_{i-1} (in which case η_i is said to be an integral of an element in F_{i-1}); or
3. $\eta_i \neq 0$ and η_i'/η_i is in F_{i-1} (in which case η_i is said to be an exponential of an integral of an element in F_{i-1}).

An element of a liouvillian extension of F is said to be *liouvillian over F*. The elements of liouvillian extensions of $\mathbf{C}(x)$ are called *liouvillian functions.*

Let L be a linear differential operator with coefficients in F. A differential extension field E or F is called a *Picard-Vessiot extension of F for L* if:

1. $E = F\langle u_1, \ldots, u_n \rangle$, where $u_1, \ldots, u_n$ are n solutions of $L(Y) = 0$, linearly independent over $\mathcal{C}_E$ ($E = F\langle u_1, \ldots, u_n\rangle$ means that E is the smallest differential field containing F and $u_1, \ldots, u_n$).
2. E and F have the same field of constants.

When the field of constants of F is algebraically closed, then a Picard-Vessiot extension exists for any linear differential operator defined over F([6] p. 412). There is a well developed galois theory of Picard-Vessiot extensions, for which we refer the reader to [5] or [6].

We now combine Proposition 2.2 with the galois theory for differential equations to show:

THEOREM 2.4. *Let F be a differential field with algebraically closed field of constants, and let L be a linear differential operator with coefficient in F. If $L(Y) = 0$ has a solution liouvillian over F, then $L(Y) = 0$ has a solution z such that z'/z is algebraic over F of degree $\leq I(n)$.*

Proof. Let E be the Picard-Vessiot extension of F associated with L. One can show [6], p. 408, 412) that if $L(Y) = 0$ has a solution liouvillian over F, then $L(Y) = 0$ has a solution y in E such that y is also liouvillian over F.

Let us first assume that all solutions in E of $L(Y) = 0$ are liouvillian over F. In this case, Theorem 5.12 of ([5], p. 39) implies that the galois group G has a solvable connected component H of the identity. The Lie-Kolchin Theorem ([5], p. 38 and [6], p. 367) says that H is conjugate to a group of upper triangular matrices and, in particular, H leaves a one dimensional space invariant. Since H is of finite index in G([5], p. 28), Proposition 2.2 allows us to conclude that there exists a subgroup H' of index $\leq I(n)$ in G which also leaves a one dimensional subspace invariant. We can further assume that H' is closed. Let z be an element of E which is an eigenvector of H'. In this case, $\sigma(z'/z) = z'/z$ for all σ in H'. Therefore z'/z lies in the fixed field $E_{H'}$ of H'. Since $[E_{H'}:F] = |G:H'| \leq I(n)$, ([5], p. 18–19), we see that the degree of z'/z over F is at most $I(n)$.

If it is not the case that all solutions of $L(Y) = 0$ are liouvillian over F, we proceed as follows. The set of solutions of $L(Y) = 0$ in E which are liouvillian over F forms a vector space V. Let $u_1, \ldots, u_k$ be a basis for this vector space. For σ in G and v in V, σv is again in V. Form

$$L_1(Y) = \frac{Wr(Y, u_1, \ldots, u_k)}{Wr(u_1, \ldots, u_k)}$$

where Wr is the usual wronskian determinant. We wish to show that the coefficients of L_1 are in F. Let σ be an element of G and $L_1{}^\sigma$ be the operator obtained by applying σ to the coefficients of L_1.

$$L_1{}^\sigma(Y) = \frac{Wr(Y, \sigma u_1, \ldots, \sigma u_k)}{Wr(\sigma u_1, \ldots, \sigma u_k)} = \frac{(\det \sigma)\, Wr(Y, u_1, \ldots, u_k)}{(\det \sigma)\, Wr(u_1, \ldots, u_k)}$$

$$= L_1$$

Therefore the coefficients of L_1 are fixed by all elements of G and so lie in F. L_1 has only liouvillian solutions, so the preceding discussion applies to L_1 and guarantees the existence of a solution u of $L_1(Y) = 0$ such that u'/u is algebraic over F of degree at most $I(k) \leq I(n)$. Since u is also a solution of $L(Y) = 0$, we are done. □

Remarks. Proposition 2.2 has the following corollary:

COROLLARY 2.5. *There exists an integer valued function $\tilde{I}(n)$, depending only on n, such that if $H \subset G$ are subgroups of $GL(n, C)$ with H triangulizable and of finite index in G, then there exists a subgroup $\tilde{H}$ of G such that $\tilde{H}$ is triangulizable and of index $\leq \tilde{I}(n)$ in G.*

Proposition 2.2 and Corollary 2.5 are also immediate consequences of Theorem 3.6 ([15], p. 45) and Corollary 10.11 ([15], p. 142), although the bound on $I(n)$ achieved by applying these latter results in not as good as our bound. It appears to have already been known that techniques similar to ours will also yield our results ([15], p. 142, Exercise 10.2) although no explicit bound using these techniques is mentioned.

Corollary 2.5 and techniques from ([5], p. 39) can be combined to yield the following result:

COROLLARY 2.6. *Let E be a Picard-Vessiot extension of F (a differential field with algebraically closed field of constants) corresponding to a linear operator of order n. Suppose that the galois group of E over F has a solvable component of the identity. Then there exist intermediate fields*

$$F = E_0 \subset E_1 \subset \cdots \subset E_n = E$$

such that E_1 is algebraic over $E_0 = F$ of degree $\leq \tilde{I}(n)$ and $E_{i+1} = E_i(\eta_i)$ for $i = 1, \ldots, n-1$, where either $\eta_i' \in E_i$ or $\eta_i'/\eta_i \in E_i$.

3. Ancillary Decision Procedures. a. *The field of constants in an algebraic extension of* $\mathbf{Q}(x)$. In the proof of Proposition 3.6 we will need to effectively present the field of constants $\mathcal{C}_F$ of an algebraic extension F of $\mathbf{Q}(x)$. To do this, we will need some facts about rational solutions of linear differential equations. The following is a generalization of a result of Risch [9] and the proof is a modification of his proof.

LEMMA 3.1. *Let $q(x)$, $q_0(x)$, $\ldots$, $q_n(x)$ be in $\mathbf{Q}(x)$. One can decide in a finite number of steps if*

$$L(Y) = q_n(x)\frac{d^n Y}{dx^n} + \cdots + q_0(x) Y = q(x) \tag{3.1}$$

has a solution in $\mathbf{Q}(x)$ and if so, find a basis for the space of all such solutions.

Proof. Let $y \in \mathbf{Q}(x)$ be a solution of $L(Y) = q(x)$. If $p(x)$ is a monic irreducible factor of the denominator of y, then $p(x)$ must divide the denominator of some q_i or q. We wish to bound the power to which p appears in the denominator of y, that is, if

$$y = \frac{A}{p^\alpha} + \cdots$$

is the p-adic expansion of y, we wish to bound α. Let the p-adic expansions of the q_i and q begin with:

$$q_i = \frac{B_i}{p^{\beta_i}} + \cdots \qquad i = 0, \ldots, n$$

$$q = \frac{C}{p^{\gamma}} + \cdots$$

Substituting into (3.1), we get

$$\frac{(-1)^n \alpha(\alpha+1)\cdots(\alpha+n-1)AB_n(p')^{n-1}}{p^{\alpha+n+\beta_n}}$$

$$+ \cdots + \frac{(-1)^{n-1}\alpha(\alpha+1)\cdots(\alpha+n-2)AB_{n-1}(p')^{n-2}}{p^{\alpha+n-1+\beta_{n-1}}} \tag{3.2}$$

$$+ \cdots + \cdots + \frac{AB_0}{p^{\alpha+\beta_0}} = \frac{C}{p^{\gamma}} + \cdots$$

Note that $p \nmid AB_{n-i}(p')^{n-(i+1)}$.

We now compare the highest powers of p^{-1}. If $\gamma \geq \alpha + i + \beta_i$ for some i, we get a bound on α in terms of the β_i and γ. Therefore we can assume that $\alpha + i + \beta_i > \gamma$ for $i = 0, \ldots, n$. We now collect the terms on the left hand side of (3.2) involving the highest power, say m, of p^{-1}. Let $\{i_1, \ldots, i_k\} \subset \{0, \ldots, n\}$ be the set of numbers such that $\alpha + i + \beta_i = m$, that is, those i such that $i + \beta_i = \max\{j + \beta_j \mid 0 \leq j \leq n\}$. Comparing coefficients of p^{-m}, we must have

$$(-1)^{i_1}\alpha(\alpha+1)\cdots(\alpha+i_1-1)AB_{i_1}(p')^{i_1-1} + \cdots$$
$$+ (-1)^{i_k}\alpha(\alpha+1)\cdots(\alpha+i_k-1)AB_{i_k}(p')^{i_k-1} \equiv 0 \bmod(p)$$

and in particular

$$(-1)^{i_1}\alpha(\alpha+1)\cdots(\alpha+i_1-1)B_{i_1}(p')^{i_1-1} + \cdots$$
$$+ (-1)^{i_k}\alpha(\alpha+1)\cdots(\alpha+i_k-1)B_{i_k}(p')^{i_k-1} \equiv 0 \bmod(p) \tag{3.3}$$

The residue of the left-hand side of (3.3) mod p gives us a polynomial in x with coefficients involving α. Since these coefficients must vanish, we have that α must be the root of some explicitly constructible polynomials. We can therefore give an a priori bound on the size of α.

We therefore know that if $y \in \mathbf{Q}(x)$ is a solution of $L(Y) = 0$, then

$$y(x) = \frac{R(x)}{p_1^{\alpha_1} \cdots p_r^{\alpha_r}} \tag{3.4}$$

where each p_i is a monic irreducible divisor of a denominator of some q_i or q and each α_i can be determined. We now wish to bound the degree of $R(x)$. Substituting (3.4) in $L(Y) = 0$ and clearing denominators, we can find polynomials $Q_n(x), \ldots, Q_0(x), Q(x)$ such that

$$Q_n(x)R^{(n)}(x) + \cdots + Q_0(x)R(x) = Q(x) \tag{3.5}$$

Let $R(x) = ax^m +$ lower terms, $Q_i(x) = b_i x^{m_i} +$ lower degree terms and $Q(x) = cx^l +$ lower degree terms. Equating coefficients as before will give us a bound on m. If we then write $R(x) = a_m x^m + \cdots + a_0$ and substitute into (3.5), we will get a system of linear equations which determines the a_i. Let $(a_{im}, \ldots, a_{i0})$ $i = 1, \ldots, k$ be a basis for the set of solutions of this latter collection of linear equations. Let $P_i(x) = a_{im}x^m + \cdots + a_{i0}$ and

$$y_i(x) = \frac{P_i(x)}{p_1^{\alpha_1} \cdots p_r^{\alpha_r}} \qquad \text{for } i = 1, \ldots, k.$$

We then have that $y_1, \ldots, y_k$ is a basis for the space of solutions in $\mathbf{Q}(x)$ of $L(Y) = 0$. □

Let $\mathbf{Q}(x)[d/dx]$ be the ring of linear operators in d/dx, with coefficients in $\mathbf{Q}(x)$. We then have:

PROPOSITION 3.2. *Let A be an $n \times n$ matrix with entries in $\mathbf{Q}(x)[d/dx]$ and B on $n \times 1$ matrix with entries in $\mathbf{Q}(x)$. One can effectively decide if there exist $(y_1, \ldots, y_n)$ in $(\mathbf{Q}(x))^n$ such that*

$$A \begin{pmatrix} y_1 \\ \vdots \\ y_n \end{pmatrix} = B$$

and, if so, find a basis for the set of such solutions.

Proof. It is known that the ring $\mathbf{Q}(x)[d/dx]$ has both a right and left division algorithm ([8], p. 31). Using this, one can row and column reduce the matrix A ([8], p. 39), that is, one can find invertible matrices U and V with entries in $\mathbf{Q}(x)[d/dx]$ such that $UAV = D$, D a diagonal matrix. $Y = (y_1, \ldots, y_n)^T$ is a solution of $AY = B$ if and only if $W = V^{-1}Y$ is a solution of $DW = UB$. Therefore finding a basis for the solutions of $AY = B$ is equivalent to finding a basis for the solutions of $DW = UB$. Since D is diagonal, we can apply Lemma 3.1. □

PROPOSITION 3.3. *Let $F = \mathbf{Q}(x, \alpha)$ be an algebraic extension of $\mathbf{Q}(x)$. Given the minimum polynomial of α, one can effectively find a β such that $\mathcal{C}_F = \mathbf{Q}(\beta)$.*

Proof. $\mathcal{C}_F$ is a vector space over $\mathbf{Q}$. We will first show how to find a $\mathbf{Q}$ basis of $\mathcal{C}_F$. Letting $[F:\mathbf{Q}(x)] = m$, we shall find a basis for the set of $(c_{m-1}, \ldots, c_0) \in (\mathbf{Q}(x))^m$ such that $(c_{m-1}\alpha^{m-1} + \cdots + c_0)' = 0$. Carrying out this differentiation, we get

$$\begin{aligned} 0 &= (c_{m-1}\alpha^{m-1} + \cdots + c_0)' \\ &= c'_{m-1}\alpha^{m-1} + \cdots + c_0' + ((m-1)c_{m-1}\alpha^{m-2} + \cdots + c_1)\alpha' \end{aligned} \tag{3.6}$$

Given the minimum polynomial q of α over $\mathbf{Q}(x)$, one can effectively find a polynomial $p \in \mathbf{Q}(x)[y]$ such that $\alpha' = p(\alpha)$. Substituting in (3.6) and using $q(\alpha) = 0$ to replace α^i, $i \geq m$, with polynomials of degree $< m$ in α, we get:

$$\begin{aligned} 0 &= (c_{m-1}\alpha^{m-1} + \cdots + c_0)' \\ &= (c'_{m-1} + L_{m-1}(c_{m-1}, \ldots, c_0))\alpha^{m-1} + \cdots \\ &\qquad + (c_0' + L_0(c_{m-1}, \ldots, c_0)) \end{aligned}$$

where $L_{m-1}, \ldots, L_0$ are linear (algebraic) polynomials with coefficients in $\mathbf{Q}(x)$. Therefore, there exists an $m \times m$ matrix A with coefficients in $\mathbf{Q}(x)[d/dx]$ and an $m \times 1$ matrix B, with coefficients in $\mathbf{Q}(x)$, such that $(c_{m-1}\alpha^{m-1} + \cdots + c_0)' = 0$ if and only if

$$A \begin{pmatrix} c_{m-1} \\ \vdots \\ c_0 \end{pmatrix} = B. \tag{3.7}$$

Using Proposition 3.2, we can find a basis $\{(c_{i,m-1}, \ldots, c_{i,0})\}_{i=1}^{k}$ for the space of solutions of (3.7). Letting $u_i = c_{i,m-1}\alpha^{m-1} + \cdots + c_{i,0}$, then $\mathcal{C}_F = \mathbf{Q}(u_1, \ldots, u_k)$. Each u_i is algebraic over $\mathbf{Q}$. To see this let $y^l + b_{l-1}y^{l-1} + \cdots + b_0$ be the minimum polynomial of some u_i over $\mathbf{Q}(x)$. Since

$$0 = (u_i^l + b_{l-1}u_i^{l-1} + \cdots + b_0)'$$
$$= b'_{l-1}u^{l-1} + \cdots + b_0'$$

we have $b'_{l-1} = \cdots = b_0' = 0$ or $b_i \in \mathbf{Q}$ for $i = 0, \ldots, l-1$. Therefore, the primitive element theorem lets us find a β such that $\mathcal{C}_F = Q(\beta)$. □

b. *Solutions of linear differential equations whose logarithmic derivatives are rational functions.* In this section we will discuss the problem of deciding if a linear differential operator, with coefficients in a finite algebraic extension of $\mathbf{Q}(x)$, has a solution y such that $y'/y \in \mathbf{C}(x)$. In what follows, we shall take K to be a finite algebraic extension of $\mathbf{Q}$.

The following lemma states one of the fundamental facts from the theory of normal solutions of linear differential equations (see ([3], p. 424 and [11] V.I, Section 94-95, VII.1, Section 177) for a proof).

LEMMA 3.4. *Let*

$$L = \frac{d^n y}{dx^n} + p_{n-1}(x)\frac{d^{n-1}}{dx^{n-1}} + \cdots + p_0(x)$$

be a linear differential operator with coefficients in $K(x)$. There exist finite sets $S_1 \subset \mathbf{C}$ and $S_2 \subset \mathbf{C}[x]$, which depend on L and can be effectively determined, such that: if $L(Y) = 0$ has a solution of the form

$$y = Ax^{\rho}e^{P(x)}\Phi(x)$$

where $\rho, A \in \mathbf{C}$, $P(x) \in \mathbf{C}[x]$ and $\Phi(x)$ is of the form

$$\Phi(x) = c_0 + c_1x^{-1} + c_2x^{-2} + \ldots, c_i \in \mathbf{C}, c_0 \neq 0,$$

then $\rho \in S_1$ and $P(x) \in S_2$.

PROPOSITION 3.5. *Let L be as in Lemma 3.4. One can decide in a finite number of steps if $L(Y) = 0$ has a solution z with $z'/z \in \mathbf{C}(x)$ and if* *so, find such a solution. In particular, one can effectively find integers M and N, depending on L, such that if $L(z) = 0$ and $z'/z \in \mathbf{C}(x)$ then*

1. *the degrees of the numerator and denominator of z'/z are bounded by M,*
2. *the residue of z'/z at any point has absolute value $\leq N$.*

Proof. Let z be a solution of $L(Y) = 0$ and assume $z'/z = R(x) \in \mathbf{C}(x)$. Let a be a pole of $R(x)$. If a is a regular point of L, then $R(x)$ can have at most a pole of order 1 at a with residue a positive integer. Therefore we can write

$$R(x) = p(x) + \frac{c_{s+1,1}}{(x - a_{s+1})} + \frac{c_{s+2,1}}{(x - a_{s+2})}$$

$$+ \cdots + \frac{c_{m,1}}{(x - a_m)} + \sum_{i=1}^{s} \sum_{j=1}^{n_i} \frac{c_{i,j}}{(x - a_i)^j}$$

where $a_1, \ldots, a_s$ are the finite singular points of L. We shall show how to determine the $c_{i,j}$, n, n_j, m and the coefficients and degree of $p(x)$ up to some finite set of possibilities.

Let a_j be a finite singular point of L. Expanding $z(x)$ at a_j we have

$$z(x) = (x - a_j)^{c_{j,1}} \exp\left(- \frac{c_{j,2}}{x - a_j} - \frac{c_{j,3}}{2(x - a_j)^2} \right.$$

$$\left. - \cdots - \frac{c_{j,n_j}}{(n_j - 1)(x - a_j)^{n_j - 1}} \right) \Phi_j(z)$$

where $\Phi_j(z)$ is analytic at a_i and $\Phi_j(0) \neq 0$. Via the transformation $t = 1/(x - a_j)$ and Lemma 3.4, we can determine n_j and $c_{j,1}, \ldots, c_{j,n_j}$ up to a finite set of possibilities.

If we consider $z(x)$ at infinity, we get

$$z(x) = x^{\rho} \exp\left(\int p(x) \right) \Phi_{\infty}(x)$$

where Φ_{∞} is analytic at infinity, $\Phi_{\infty}(\infty) \neq 0$ and $\rho = c_{1,1} + \cdots + c_{s,1} + \cdots + c_{m,1}$. Using Lemma 3.4, we can determine the degree and coefficients of $p(x)$ up to a finite set of possibilities. Since the set $\{c_{11}, \ldots, c_{s1}\}$ is determined up to a finite set of possibilities, we can bound the $c_{s+1,1}, \ldots, c_{m,1}$ and in particular bound m.

We therefore know that if $z'/z \in \mathbf{C}(x)$, then

$$\begin{aligned} z &= (x - a_{s+1})^{c_{s+1,1}} \cdots (x - a_m)^{c_{m,1}} \exp\left(\int S(x)\right) \\ &= q(x) \exp\left(\int S(x)\right) \end{aligned}$$

where $S(x)$ is a rational function determined up to a finite set of possibilities and $q(x)$ is a polynomial whose degree can be effectively bounded. Fix $S(x)$ and make the substitution

$$Y = V \exp\left(\int S(x)\right)$$

in $L(Y) = 0$. Dividing by $\exp(\int S(x))$ gives us a linear differential equation $\bar{L}(V) = 0$. If we substitute $V = q(x)$, where q is a polynomial with undetermined coefficients, we get a set of linear equations in these coefficients. If these can be solved, then $z = q(x) \exp(\int S(x))$ is a solution of $L(Y) = 0$ whose logarithmic derivative is a rational function. If these equations cannot be solved, for all of the finite set of possible $S(x)$, then $L(Y) = 0$ has no solution whose logarithmic derivative is a rational function. □

Before proceeding to Proposition 3.6, we need some facts about Riccati equations. Let $L(Y) = 0$ be an n^{th} order homogeneous linear differential equation. Let U be a new variable and set $Y' = UY$. If we differentiate this relation we get:

$$\begin{aligned} Y'' &= UY' + U'Y = U^2Y + U'Y \\ Y''' &= 3UU'Y + U^3Y + U''Y \\ Y^{iv} &= \text{etc.} \end{aligned}$$

Substituting these expressions in $L(Y) = 0$ and dividing by Y, we get a nonlinear differential equation, $R(U) = 0$, of order $n - 1$, called the *Riccati equation associated with L.* For $n = 2$, if $L(Y) = Y'' + pY' + qY$, then $R(U) = U' + pU + U^2 + q$. Notice that the coefficients of L always appear linearly in R. $R(U)$ has the property: y is a nonzero solution of $L(Y) = 0$ if and only if $u = y'/y$ is a solution of $R(U) = 0$.

PROPOSITION 3.6. *Let F be a finite algebraic extension of $\mathbf{Q}(x)$ and L a linear differential operator with coefficients in F. One can decide in a* *finite number of steps if $L(Y) = 0$ has a solution z with $z'/z \in \mathbf{C}(x)$ and if so find such a solution. In particular, one can effectively find integers M and N, depending on L, such that if $Lz = 0$ and $z'/z \in \mathbf{C}(x)$, then:*

1. *The degrees of the numerator and denominator of z'/z are bounded by M.*
2. *The residue of z'/z at any point has absolute value $\leq N$.*

Proof. We can assume F is a normal extension of $\mathbf{Q}(x)$ and let $F(\mathbf{C})$ be the composition of F and $\mathbf{C}$. Let σ be in the galois group G of $F(\mathbf{C})$ over $\mathbf{C}(x)$ and L^σ be the operator obtained by applying σ to the coefficients of L. Let

$$\operatorname{Tr} L = \sum_{\sigma \in G} L^\sigma.$$

$\operatorname{Tr} L$ has coefficients in $\mathbf{C}(x)$. We claim that if $Ly = 0$ and $y'/y \in \mathbf{C}(x)$, then $(\operatorname{Tr} L)(y) = 0$. To see this let $R(U) = 0$ be the Riccati equation associated with L. If $\sigma \in G$, let R^σ be the differential polynomial obtained by applying σ to the coefficients of R. If we let

$$\operatorname{Tr} R = \sum_{\sigma \in G} R^\sigma$$

then $\operatorname{Tr} R = 0$ is the Riccati equation associated with $\operatorname{Tr} L$. If y is a solution of $Ly = 0$ such that $y'/y \in \mathbf{C}(x)$ and $\sigma \in G$, then

$$0 = \sigma(R(y'/y)) = R^\sigma(y'/y)$$

Therefore $(\operatorname{Tr} R)(y'/y) = 0$ so y is a solution of $(\operatorname{Tr} L)(y) = 0$.

Since $\operatorname{Tr} L$ has coefficients in $\mathcal{C}_F(x)$ and $\mathcal{C}_F$ is a finite algebraic extension of $\mathbf{Q}$, we can apply Proposition 3.5 to $\operatorname{Tr} L$ to get our decision procedure. Furthermore $\operatorname{Tr} L$ can be effectively calculated since it is easy to see that $\operatorname{Tr} L$ is just the operator whose coefficients are the traces (in F with respect to $\mathcal{C}_F(x)$) of the coefficients of L. Proposition 3.3 allows us to calculate $\mathcal{C}_F$ and thus calculate this trace. □

c. *Auxiliary differential operators.* In the decision procedure presented in Section 4, we will need the following proposition.

PROPOSITION 3.7. *Let L be a linear differential operator and $P(Y_1, \ldots, Y_k)$ a differential polynomial, both L and P having coefficients in some algebraic extension of $\mathbf{Q}(x)$. One can effectively bound the degrees of the numerator and denominator and the absolute values of the residues* *of rational functions* $T(x) \in \mathbf{C}(x)$ *for which there exist solutions* $y_1, \ldots, y_k$ *of* $L(Y) = 0$ *such that* $P(y_1, \ldots, y_k) \neq 0$ *and* $(P(y_1, \ldots, y_k))'/P(y_1, \ldots, y_k) = T(x)$.

To prove this proposition, we will first show how to produce a differential operator L_P such that $L_P(P(y_1, \ldots, y_k)) = 0$ for all solutions $y_1, \ldots, y_k$ of $L(Y) = 0$. Proposition 3.6 then allows us to get the necessary bounds. The following lemma and propositions carry out this plan. They are similar to the results contained in Section 2C of [13], but are easier to prove since we are not concerned with Fuchsian considerations, as we were in that paper.

A differential field is said to be *constructible* if we can effectively perform the field operations and differentiation that is, if the field operations and differentiation are recursive functions. If F is a finite algebraic extension of $\mathbf{Q}(x)$, then F is a constructible differential field.

LEMMA 3.8. *Let F be a constructible differential field and let L_1 and L_2 be linear differential operators with coefficients in F. One can effectively construct linear differential operators L_3, L_4 and L_5 with coefficients in F such that*:

a. *The solution space of* $L_3(Y) = 0$ *contains* $\{y_1 y_2 \mid y_1$ *is a solution of* $L_1(Y) = 0$ *and* y_2 *is a solution of* $L_2(Y) = 0\}$,

b. *The solution space of* $L_4(Y) = 0$ *contains* $\{y_1 + y_2 \mid y_1$ *is a solution of* $L_1(Y) = 0$ *and* y_2 *is a solution of* $L_2(Y) = 0\}$.

c. *The solution space of* $L_5(Y) = 0$ *contains* $\{y'/y$ *is a solution of* $L_1(Y) = 0\}$.

Proof. a. Let L_1 be of order n_1 and L_2 be of order n_2. Let U and V be new indeterminants. If we formally differentiate UV $n_1 n_2$ times we get a system of $n_1 n_2 + 1$ equations:

$$
\begin{aligned}
UV &= UV \\
(UV)' &= U'V + UV' \\
&\;\;\vdots \\
(UV)^{(n_1 n_2)} &= \sum_{j=0}^{n_1 n_2} \binom{n_1 n_2}{j} U^{(j)} V^{(n_1 n_2 - j)}
\end{aligned}
\tag{3.8}
$$

Whenever $U^{(i)}$, $i \geq n_1$, occurs, we use the relation $L_1(U) = 0$ (and its derivatives) to replace $U^{(i)}$ with an expression only involving terms $U^{(i)}$

with $i < n_1$. We similarly use $L_2(V) = 0$ to replace the terms $V^{(i)}$, $i \geq n_2$, with expressions only involving $V^{(i)}$, $i < n_2$. In this way, the right-hand side of (3.8) gives us $n_1 n_2 + 1$ linear forms in the terms $U^{(i)} V^{(i)}$, $0 \leq i < n_1$, $0 \leq j < n_2$, with coefficients in F. These forms must therefore be linearly dependent over F. Let k be the smallest natural number such that the first k of these forms are linearly dependent over $K(x)$. We can then find $a_{k-2}(x), \ldots, a_0(x)$ in F such that

$$(UV)^{(k-1)} + a_{k-2}(x)(UV)^{(k-2)} + \cdots + a_0(x)UV = 0$$

We have therefore found a linear operator L_3 such that $L_3(y_1 y_2) = 0$ for all solutions y_1 of $L_1(Y) = 0$ and y_2 of $L_2(Y) = 0$.

b. L_4 is formed by differentiating $U + V$ $n_1 + n_2$ times and proceeding as above.

c. To construct L_5, assume L_1 is of the form

$$L_1 = \frac{d^{n_1}}{dx^{n_1}} + a_{n_1-1}(x)\frac{d^{n_1-1}}{dx^{n_1-1}} + \cdots + a_0(x).$$

If $a_0(x) = 0$, let

$$L_5 = \frac{d^{n_1-1}}{dx^{n_1-1}} + a_{n_1-1}(x)\frac{d^{n_1-2}}{dx^{n_1-2}} + \cdots + a_1(x).$$

In this case, $L_5(y) = L_1(\int y\,dx)$ which is zero iff y is the derivative of a solution of $L_1(Y) = 0$. If $a_0(x) \neq 0$. Let

$$L_5 = \frac{d^{n_1}}{dx^{n_1}} + a_{n_1-1}(x)\frac{d^{n_1-1}}{dx^{n_1-1}} + \left(\frac{da_{n_1-1}(x)}{dx}\right.$$

$$\left. + a_{n_1-2}(x)\right)\frac{d^{n_1-2}}{dx^{n_1-2}} + \cdots + \left(\frac{da_1(x)}{dx} + a_0(x)\right)$$

$$- \frac{da_0(x)}{dx}(a_0(x))^{-1}\left[\frac{d^{n_1-1}}{dx^{n_1-1}} + a_{n_1-1}(x)\frac{d^{n_1-2}}{dx^{n_1-2}} + \cdots + a_1(x)\right]$$

If y is any solution of $L_1(Y) = 0$, then $L_5(y') = (L_1(y))' = 0$. □

Proposition 3.9. *Let $P(Y_1, \ldots, Y_k)$ be a differential polynomial and L a linear differential operator, both L and P having coefficients in F, a constructible differential field. We can effectively construct a nonzero* *linear differential operator* L_P *such that if* $y_1, \ldots, y_k$ *are any solutions of* $L(Y) = 0$, *then* $L_P(P(y_1, \ldots, y_k)) = 0$.

Proof. The proof proceeds by induction on the complexity of P. If $P \in K(x)$, $P \neq 0$, let $L_P = (d/dx) - (P'/P)$. If $P = P_1 + P_2$ or $P = P_1 P_2$ or $P = P_1{}'$, then apply the induction hypothesis and Lemma 3.8. □

Proposition 3.7 now follows from Proposition 3.6 and 3.9.

d. *Elimination theory.* Our algorithm reduces the main problem to questions in elimination theory. We will then apply the general facts stated below. One way of improving the efficiency of our algorithm would be to replace the appeal to generalities by elimination techniques designed to handle the special systems we encounter. If $Y_1, \ldots, Y_k$ are indeterminants, we denote by $\mathbf{Q}\{Y_1, \ldots, Y_k\}$ the ring of differential polynomials with coefficients in $\mathbf{Q}$. The following result is due to Seidenberg [12].

PROPOSITION 3.10. *Consider a system*

$$F_1 = 0, \ldots, F_s = 0, \qquad G \neq 0 \tag{S}$$

where F_i and G are elements of $\mathbf{Q}\{c_1, \ldots, c_m; V_1, \ldots, V_n\}$. There exist a finite number of systems

$$f_{j1} = 0, \ldots, f_{js_j} = 0, \qquad g_j \neq 0 \tag{S_j}$$

where f_{jk}, g are in $\mathbf{Q}\{c_1, \ldots, c_m\}$ having the following property: for any differential field K and any values $\bar{c}_i$ in K of the c_i, the system $(\overline{S})$ obtained from (S) by replacing the c_i by $\bar{c}_i$ has a solution in some differential extension L of K if and only if for at least one j, the $\bar{c}_i$ form a solution of (S_j). Moreover, the (S_j) can be computed in a finite number of steps depending only on the F_i and G.

We shall use the proposition in conjunction with the following result of classical elimination theory (see [14]).

PROPOSITION 3.11. *Let*

$$f_1 = 0, \ldots, f_r = 0, \qquad g \neq 0 \tag{$\mathcal{S}$}$$

be a system of equations with the f_i *and* g *in* $\mathbf{Q}[c_1, \ldots, c_n]$. *One can decide in a finite number of steps if* $(\mathcal{S})$ *has a solution in* $\mathbf{C}$. *If* $(\mathcal{S})$ *has a solution in* $\mathbf{C}$, *it will have a solution in* $\overline{\mathbf{Q}}$, *the algebraic closure of* $\mathbf{Q}$, *and one can find such a solution in a finite number of steps.*

Note that Proposition 3.11 can also be deduced from Proposition 3.10.

4. The Decision Procedure.

THEOREM 4.1. *Let L be a linear differential operator with coefficients in F, a finite algebraic extension of $\mathbf{Q}(x)$. One can decide in a finite number of steps if $L(Y) = 0$ has a solution liouvillian over F. If it does, one can find u, algebraic over F, such that $y = \exp(\int u)$ satisfies $L(Y) = 0$.*

Proof. If $L(Y) = 0$ has a solution liouvillian over F, then Theorem 2.4 implies that $L(Y) = 0$ has a solution z such that z'/z is algebraic over $F(\mathbf{C})$ of degree $\leq I(n)$. Note that $u = z'/z$ is a solution of $R(U) = 0$, the Riccati equation associated with L. Therefore it is enough to decide if $R(U) = 0$ has a solution algebraic over $F(\mathbf{C})$ of degree $\leq I(n)$. If u is a solution of $R(U) = 0$ of degree $\leq I(n)$ over $F(\mathbf{C})$, then $[\mathbf{C}(x, u):\mathbf{C}(x)] = N \leq I(n)\cdot m$ where $m = [F:\mathbf{Q}(x)]$. Let

$$P(U) = U^N + a_{N-1}(x)U^{N-1} + \cdots + a_0(x)$$

be the minimum polynomial of u over $\mathbf{C}(x)$. We shall show that:

$$\text{Given } L, \text{ one can effectively bound the degrees of the numerators and denominators of the } a_i(x). \tag{4.1}$$

Let us assume that we have shown this and let M_N be such a bound. Let $F = \mathbf{Q}(x, \alpha)$ and let $T(V, x)$ be the irreducible polynomial of α over $\mathbf{Q}(x)$. Let $R(U, \alpha, x) = 0$ be the Riccati equation associated with L; where the coefficients are explicitly written as polynomials in α with coefficients in $\mathbf{Q}(x)$. If we clear denominators of T and R, we get new polynomials $\overline{T}(V, x) \in \mathbf{Q}[V, x]$ and $\overline{R}(U, \alpha, x) \in \mathbf{Q}[x, \alpha]\{U\}$. Finally let $M = \max\{M_N \mid 1 \leq N \leq I(n)\cdot m\}$. Consider the system of differential polynomials:

$$\begin{aligned} \overline{T}(V, x) &= 0 \\ c_1' &= 0 \\ c_2' &= 0 \\ &\vdots \\ c'_{n_N} &= 0 \\ x' &= 1 \\ \overline{R}(U, V, x) &= 0 \end{aligned} \tag{S}$$

$$P(U, c_1, \ldots, c_{n_N}, x) = b_N(c_0, \ldots, c_{n_1})U^N + b_{N-1}(c_{n_1+1}, \ldots, c_{n_2})U^{N-1} + \cdots + b_0(c_{n_{N-1}+1}, \ldots, c_{n_N}) = 0$$

This is a system of differential equations in the variables $(c_1, \ldots, c_{n_N}; U, V, x)$. Proposition 3.10 allows us to find systems (S_j) of differential equalities and inequalities in the variables $(c_1, \ldots, c_{n_N})$ such that for some choice of $\bar{c}_i$ for the c_i, $\bar{S}$ has a solution if and only if $(\bar{c}_1, \ldots, \bar{c}_{n_N})$ satisfy some S_j. Since the $\bar{c}_i$ must be constants, we can replace all occurrences of $c_i^{(k)}$, $k > 0$, in the S_j by 0. In this way we get a collection $\{\tilde{S}_j\}$ of systems of algebraic equations in the c_i. Proposition 3.11 allows us to decide if some $\tilde{S}_j$ has a solution in $\mathbf{C}$. If no $\tilde{S}_j$ has a solution in $\mathbf{C}$, $L(Y) = 0$ has no liouvillian solutions. If some $\tilde{S}_j$ has a solution, we can find $\bar{c}_1, \ldots, \bar{c}_{n_N}$ in $\overline{\mathbf{Q}}$ such that $P(U, \bar{c}_1, \ldots, \bar{c}_{n_N}, x) = 0$ will have a solution u such that $R(u, \alpha) = 0$. Let $\bar{F}$ be the splitting field of $P(u, \bar{c}_1, \ldots, \bar{c}_n, x) = 0$ over $F(\bar{c}_1, \ldots, \bar{c}_n)$. In $\bar{F}$ we can now find a u such that $R(u, \alpha) = 0$. For this u, $y = \exp(\int u)$ satisfies $L(Y) = 0$.

We will therefore be done if we can verify (4.1). To do this, assume that F is a normal extension of $\mathbf{Q}(x)$ and let $G = \{\sigma_1, \ldots, \sigma_m\}$ be its galois group. For $\sigma_i \in G$, let L_i be the operator obtained by applying σ_i to the coefficients of L. Let $\bar{L}(Y) = 0$ be the homogeneous linear differential equation whose solution space is $\{y_1 + \cdots + y_m \mid L_i(y_i) = 0\}$. This can be found using Lemma 3.8. Note that if u is a solution of $P(U) = 0$, then $y = \exp(\int u)$ is a solution of $\bar{L}y = 0$. Therefore all solutions of $P(U) = 0$ are of the form $u = y'/y$ where $\bar{L}(y) = 0$. Let $u_1, \ldots, u_N$ be the roots of $P(U) = 0$ and let $u_i = y_i'/y_i$ where $\bar{L}(y_i) = 0$. We see that

$$\begin{aligned} a_{N-1}(x) &= -(u_1 + \cdots + u_N) \\ &= -\left(\frac{y_1'}{y_1} + \cdots + \frac{y_N'}{y_N}\right) \\ &= -\left(\prod_{i=1}^{N} y_i\right)' \Big/ \left(\prod_{i=1}^{N} y_i\right) \end{aligned}$$

Let $P_{N-1}(Y_1, \ldots, Y_N) = \Pi_{i=1}^{N} Y_i$. The logarithmic derivative of $P_{N-1}(y_1, \ldots, y_N)$ is a rational function, $a_{N-1}(x)$, so Proposition 3.7 allows us to bound the degree of the numerator and denominator of $a_{N-1}(x)$. Later we shall also need information about the absolute values of the residues of $a_{N-1}(x)$.

Now, considering $a_{N-2}(x)$, we have:

$$a_{N-2}(x) = \sum_{1\le i<j\le N} u_i u_j$$

$$= \sum_{i\le i<j\le N} \frac{y_i' y_j'}{y_i y_j}$$

$$= [2(m-2)! \prod_{i=1}^{N} y_i]^{-1} \Sigma y'_{i_1} y'_{i_2} y_{i_3} \cdots y_{i_N}$$

where this latter sum is taken over all permutations of $(1, \ldots, N)$. Let $P_{N-2}(Y_1, \ldots, Y_N) = \Sigma Y'_{i_1} Y'_{i_2} Y_{i_3} \cdots Y_{i_N}$. Calculating, we find:

$$(P_{N-2}(y_1, \ldots, y_N))'/P_{N-2}(y_1, \ldots, y_N) = a'_{N-2}/a_{N-2} + a_{N-1} \in \mathbf{C}(x).$$

Proposition 3.7 allows us to describe the form of $(P_{N-2}(y_1, \ldots, y_N))'/P_{N-2}(y_1, \ldots, y_N)$. Writing

$$a'_{N-2}/a_{N-2} = (P_{N-2}(y_1, \ldots, y_N))'/P_{N-2}(y_1, \ldots, y_N) - a_{N-1} \tag{4.3}$$

we see that the degrees of the numerator and denominator of a_{N-2} are bounded by the sum of the absolute values of the residues of the right-hand side of (4.3). Proposition 3.7 allows us to calculate this number.

Continuing in this way we can bound the degrees of the numerators and denominators of each of the $a_i(x)$. □

THEOREM 4.2. *Let L be a linear differential operator with coefficients in F, a finite algebraic extension of $\mathbf{C}(x)$. One can find, in a finite number of steps, a basis for the vector space of liouvillian solutions of $Ly = 0$.*

Proof. We proceed by induction on the order of L. If the order of L is 1, then

$$L = \frac{d}{dx} + p$$

with $p \in F$. All solutions of L are constant multiples of $\exp(-\int p)$.

Let L have order n. Use Theorem 4.1 to decide if L has a liouvillian solution. If it does, find u, algebraic over F such that $y = \exp(\int u)$ satisfies $L(Y) = 0$. Let $E = F(u)$. In E, we can factor L as $L = L_{n-1} \circ L_1$, where $L_1 = (d/dx) - u$ and L_{n-1} is a linear operator with coefficients in E. Let $w_1, \ldots, w_k$ be a basis for the space of liouvillian solutions of $L_{n-1}(Y) = 0$ (which we can find by induction). Let

$$y_i = \exp\left(-\int u\right)\left[\int (w_i \exp \int u)\right] \qquad \text{for } i = 1, \ldots, k$$

$$y_{k+1} = \exp\left(\int u\right)$$

We claim that $y_1, \ldots, y_{k+1}$ is a basis for V, the space of liouvillian solutions of $L(Y) = 0$. To see that $y_1, \ldots, y_{k+1}$ are linearly independent over $\mathbf{C}$, let $\Sigma_{i=1}^{k+1} c_i y_i = 0$, $c_i \in \mathbf{C}$. Applying L_1 to this equation, we get

$$L_1\left(\sum_{i=1}^{k+1} c_i y_i\right) = \sum_{i=1}^{k} c_i w_i = 0.$$

Therefore $c_1 = \cdots = c_k = 0$ and so $c_{k+1} = 0$. To see that $y_1, \ldots, y_{k+1}$ span V, let $y \in V$. $L_1 y$ is a liouvillian solution of $L_{n-1}(Y) = 0$, so there are $c_1, \ldots, c_k$ in $\mathbf{C}$ such that $L_1(y) = \Sigma_{i=1}^{k} c_i w_i$. Solving this equation we get $y = \Sigma_{i=1}^{k} c_i y_k + c y_{k+1}$ for some $c \in \mathbf{C}$. Finally, one can check that $y_i \in V$ for $i = 1, \ldots, k+1$. □

COROLLARY 4.3. *Let L be a linear differential operator with coefficients in F, a finite algebraic extension of $\mathbf{Q}(x)$. One can decide if all solutions of $L(Y) = 0$ are algebraic functions.*

Proof. (c.f. [13]). Use Theorem 4.2 to find a basis for the space of all liouvillian solutions of $L(Y) = 0$. Now use the algorithm developed by Rothstein and Caviness ([10]) to decide if each of the elements of this basis is algebraic over $\mathbf{C}(x)$. □

After this paper was submitted, we discovered that Proposition 2.4 and Corollary 2.7 follow immediately from results of Platonov and Malcev (Corollary 10.11, p. 142 and Theorem 3.6 p. 45 in *Infinite Linear Groups* by B. A. F. Wehrfritz, Ergebnisse der Mathematik und ihrer Grenzgebiete, Band 76, Springer-Verlag, New York, 1973).

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