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Liouvillian Solutions of Linear Differential Equations with Liouvillian Coefficients

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Let $L(y)=b$ be a linear differential equation with coefficients in a differential field K. We discuss the problem of deciding if such an equation has a non-zero solution in K and give a decision procedure in case K is an elementary extension of the field of rational functions or is an algebraic extension of a transcendental liouvillian extension of the field of rational functions. We show how one can use this result to give a procedure to find a basis for the space of solutions, liouvillian over K, of $L(y)=0$ where K is such a field and $L(y)$ has coefficients in K.

1. Introduction

In this paper the following two questions will be considered. Let K be a differential field and let $a_{n-1}, \ldots, a_0$, $b \in K$. Let $L(y)=y^{(n)}+a_{n-1}y^{(n-1)}+\cdots+a_0 y$.

QUESTION 1. When does $L(y)=b$ have non-zero solutions in K and how can one find all such solutions?

QUESTION 2. When does $L(y)=0$ have a non-zero solution y such that $y'/y \in K$ and how does one find all such solutions?

An algorithm is presented to answer these questions when K is an elementary extension of $C(x)$ or K is an algebraic extension of a purely transcendental liouvillian extension of $C(x)$, where C is a computable algebraically closed field of characteristic zero. We will discuss why these are important questions and how they are related to each other. Before starting, let us recall some definitions. A field K is said to be a *differential field* with derivation $D: K \to K$ if D satisfies $D(a+b)=D(a)+D(b)$ and $D(ab)=(Da)b+a(Db)$ for all $a, b \in K$. The set $C(K)=\{c \mid Dc=0\}$ is a subfield called the *field of constants* of K. We will usually denote the derivation by $'$, i.e. $a'=Da$. A good example to keep in mind is the field of rational functions $\mathbb{C}(x)$ with derivation d/dx ($\mathbb{C}$ denotes the complex numbers). All fields in this paper, without further mention, are of characteristic zero. We say K is a *liouvillian extension* of k if there is a tower of fields $k=K_0 \subset K_1 \subset \cdots \subset K_n = K$ such that for each $i=1,\ldots,n$, $K_i=K_{i-1}(t_i)$ where either (a) $t_i' \in K_{i-1}$ or (b) $t_i'/t_i \in K_{i-1}$ or (c) t_i is algebraic over K_{i-1}. For example $\mathbb{C}(x, e^{x^2}, e^{\int e^{x^2}})$ is a liouvillian extension of $\mathbb{C}(x)$. We say K is an *elementary extension* of k if there is a tower of fields $k=K_0 \subset K_1 \subset \cdots \subset K_n=K$ such that for each $i=1,\ldots,n$, $K_i=K_{i-1}(t_i)$ where either (a) for some $u_i \neq 0$ in K_{i-1}, $t_i'=u_i'/u_i$ or (b) for some u_i in K_{i-1}, $t_i'/t_i=u_i'$ or (c) t_i is algebraic over K_{i-1}. For example, $\mathbb{C}(x, \log x, e^{(\log x)^2})$ is an elementary extension

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of $\mathbb{C}(x)$. The example following the definition of liouvillian extension is not an elementary extension of $\mathbb{C}(x)$ since $\int e^{x^2}$ lies in no elementary extension of $\mathbb{C}(x)$ (Rosenlicht, 1972). We say that w is *liouvillian* (*elementary*) over k if w belongs to a liouvillian (elementary) extension of k.

Algorithms to answer questions 1 and 2 would be useful in solving two other problems. First of all, an answer to question 1 would have a bearing on the Risch Algorithm. In a series of papers (Risch, 1968; 1969; 1970), Risch gave a procedure to answer the following question: Given α in an elementary extension K of $C(x)$ (C a finitely generated extension of the rational numbers $\mathbb{Q}$ and $C(K) = C$), decide if $\int \alpha$ lies in an elementary extension of K. Liouville's Theorem (Rosenlicht, 1972) states that if α has an anti-derivative in an elementary extension of K, then $\alpha = v_0' + \sum c_i(v_i'/v_i)$ where $v_0 \in K$, $v_1, \ldots, v_n \in \bar{C}K$ and $c_i \in \bar{C}$, where $\bar{C}$ is the algebraic closure of C. Risch's algorithm gives a procedure to decide if such elements exist. As a corollary of Liouville's Theorem, one can show that if α is of the form $f\,e^g$ with f and g in K, then α has an elementary anti-derivative if and only if $y' + g'y = f$ has a solution y in K (i.e. if and only if there is a y in K such that $(y\,e^g)' = f\,e^g$). In general, Risch's Algorithm forces one to deal, again and again, with this same question: given f and g in an elementary extension K of $C(x)$, decide if $y' + g'y = f$ has a solution in K. When K is a purely transcendental extension of $C(x)$, one may write $K = E(t)$ with $t' \in E$ or $t'/t \in E$ and t transcendental over E. Letting

$$y = \sum_{i=1}^{m} \sum_{j=1}^{n_i} \frac{a_{ij}(t)}{(p_i(t))^j} + h(t)$$

be the partial fraction decomposition of y, one can plug this expression into $y' + g'y = f$. Equating powers and using the uniqueness of partial fraction decompositions, one can find a finite number of candidates for the p_is and bound the degree of h. This allows one to find all possible solutions y. (In fact there are now improvements on this idea. Rothstein (1976) showed how one can use "Hermite Reduction" to postpone, as much as possible, the need to factor polynomials.) When K is not a purely transcendental extension of $C(x)$, but involves algebraics in the tower, things are more complicated. In the purely transcendental case, partial fractions gives us a global normal form that captures all the necessary local information (e.g. the factors of the denominators and the powers to which they appear). When algebraics occur, one does not have this normal form. If $K = E(t, \gamma)$ with γ algebraic of degree n over $E(t)$, one may write $y = b_0 + b_1\gamma + \cdots + b_{n-1}\gamma^{n-1}$ with the $b_i \in E(t)$. To find the b_i, one is forced to work with puiseux expansions (a local normal form) at each place of the function field $E(t, \gamma)$. Although Risch showed that this approach does yield an algorithm, it is much more complex than the purely transcendental case (Bronstein (1990) has made significant improvements in the Risch algorithm and can avoid puiseux expansions in many situations, but he is still forced to consider them in certain cases). One would like to reduce the question of deciding if $y' + g'y = f$ has a solution in $E(t, \gamma)$ to a similar question in $E(t)$, where one could apply partial fraction techniques and a suitable induction hypothesis. In section 3, we shall see that we can reduce the problem of solving such an equation in an algebraic extension of a field to solving linear differential equations (more than one and possibly of order greater than one) in that field. We are then forced to answer question 1 for that field.

The second place these questions arise is in the general problem of finding liouvillian solutions of linear differential equations with liouvillian coefficients. In Singer (1981) it was shown that given a homogeneous linear differential equation $L(y) = 0$ with coefficients in F, a finite algebraic extension of $\mathbb{Q}(x)$, one can find in a finite number of steps, a basis for the vector space of liouvillian solutions of $L(y)=0$. I would like to extend this result to find, given a homogeneous linear differential equation with coefficients in a liouvillian extension K of $\mathbb{Q}(x)$, a basis for the liouvillian solutions of $L(y)=0$. One can show that to solve this problem, it is sufficient to find one non-zero liouvillian solution. An inductive procedure would then allow one to find all such solutions (see Lemma 2.5(iii) below). To see how problem 2 fits into this, I will outline the procedure to decide if a given $L(y)=0$ with coefficients in K has a non-zero liouvillian solution. It is known (Singer, 1981) that if $L(y)=0$ has a non-zero liouvillian solution, then there is a solution y such that $u=y'/y$ is algebraic over K of degree bounded by an integer N that depends only on the order of $L(y)$. Furthermore there are effective estimates for N. Therefore, for some $m \leq N$, u satisfies an irreducible equation of the form $f(u)=u^m+a_{m-1}u^{m-1}+\cdots+a_0=0$ with the $a_i \in K$. We must now find the possible $a_i \in K$ and test to see if, for such a choice of a_i, $e^{\int u}$ satisfies $L(y)=0$. For example, let us try to determine the possible a_{m-1}. If $u=u_1,\ldots,u_m$ are the roots of $f(u)=0$ and $y_1=e^{\int u_1}$ satisfies $L(y)=0$, then for $i=2,\ldots,m$, $y_i=e^{\int u_i}$ also satisfies $L(y)=0$. We have

$$a_{m-1}=-(u_1+\cdots+u_m)=-\left(\frac{y_1'}{y_1}+\cdots+\frac{y_m'}{y_m}\right)=-\left(\frac{(y_1\cdots y_m)'}{y_1\cdots y_m}\right).$$

One can show that the product $y=y_1\cdots y_m$ satisfies a homogeneous linear differential equation $L_m(y)=0$ and that $y'/y \in K$. Finding all such solutions is just problem 2 above. Theorem 4.2 below states that for certain liouvillian extensions K, we can fill in the details of the above argument and give a procedure to find a basis for the vector space of all solutions of $L(y)=0$ that are liouvillian over K.

Finally, we note that it appears that to answer one of these two questions we need to be able to answer the other. The rest of the paper is organized as follows. Section 2 is devoted to showing how one can algorithmically reduce question 2 to question 1. Section 3 contains procedures to answer question 1 in certain cases. Section 4 contains some final comments and open problems. The results of this paper were announced in Singer (1989).

2. Reducing Question 2 to Question 1

In this section we shall consider fields of the form $E(t)$, where either $t' \in E$, $t'/t \in E$ or t is algebraic over E and where E satisfies certain hypotheses. We shall show that for these fields, if we can answer question 1 algorithmically then we can answer question 2 algorithmically. This is made precise in Proposition 2.1, but first we need some definitions. We call a differential field K a *computable differential field* if the field operations and the derivation are recursive functions and if we can effectively factor polynomials over K. We say that we can *effectively solve homogeneous linear differential equations over K* if for any homogeneous linear differential equation $L(y)=0$ with coefficients in K, we can effectively find a basis for the vector space of all $y \in K$ such that $L(y)=0$. We say that we can *effectively find all exponential solutions of homogeneous linear differential equations over K* if for any homogeneous linear differential equation $L(y)=0$ with coefficients in K, we can effectively find $u_1,\ldots,u_m$ in K such that if $L(e^{\int u})=0$ for some $u \in K$, then $e^{\int u}/e^{\int u_i} \in K$ for some i.

The main result of this section is:

PROPOSITION 2.1. *Let $E \subset E(t)$ be computable differential fields with $C(E)=C(E(t))$, an algebraically closed field, and assume that either $t' \in E$, $t'/t \in E$, or t is algebraic over E.*

Assume that we can effectively solve homogeneous linear differential equations over $E(t)$ and that we can effectively find all exponential solutions of homogeneous linear differential equations over E. Then we can effectively find all exponential solutions of homogeneous linear differential equations over $E(t)$.

We will deal with each of the three cases for t separately in the following propositions and lemmas. We start by defining and reviewing some facts about the Riccati equation. If u is a differential variable and $y = e^{\int u}$, formal differentiation yields $y^{(i)} = P_i(u, u', \ldots, u^{(i-1)})\, e^{\int u}$, where the P_i are polynomials with integer coefficients satisfying $P_0 = 1$ and $P_i = P'_{i-1} + uP_{i-1}$. If $L(y) = y^{(n)} + A_{n-1}y^{(n-1)} + \cdots + A_0 y = 0$ is a linear differential equation, then $y = e^{\int u}$ satisfies $L(y) = 0$ if and only if u satisfies $R(u) = P_n(u, \ldots, u^{(n-1)}) + A_{n-1}P_{n-1}(u, \ldots, u^{(n-2)}) + \cdots + A_0 = 0$. This latter equation is called *the Riccati equation associated with* $L(y) = 0$. We will need the following technical lemma.

Lemma 2.2. *Let $E(t)$ be a differential field with t transcendental over E and either $t' \in E$ or $t'/t \in E$. Let $p(t)$ be an irreducible polynomial in $E[t]$ where $p \neq t$ if $t'/t \in E$.*

(i) *Let $u \in E(t)$ have p-adic expansion of the form $u = u_\gamma/p^\gamma$ + higher order terms, where $\gamma > 0$, $u_\gamma \neq 0$, and $\deg_t u_\gamma < \deg_t p$. If $\gamma > 1$ then for $i \geq 1$, $P_i(u, \ldots, u^{(i-1)}) = v_{i\gamma}/p^{i\gamma}$ + higher order terms, where $v_{i\gamma} \equiv (u_\gamma)^i \bmod p$. If $\gamma = 1$ then for $i \geq 1$, $P_i(u, \ldots, u^{(i-1)}) = v_i/p^i$ + higher order terms where $v_i \equiv \prod_{j=0}^{i-1} (u_1 - jp') \bmod p$.*

(ii) *Assume that $t' \in E$ and that $u \in E(t)$ has $(1/t)$-adic expansion of the form $u = u_\gamma t^\gamma$ + higher powers of $1/t$, $u_\gamma \neq 0$. If $\gamma > 0$ then the $(1/t)$-adic expansion of $P_i(u, \ldots, u^{(i-1)}) = u_\gamma^i t^{i\gamma}$ + higher powers of $1/t$. If $\gamma = 0$, then the $(1/t)$-adic expansion of $P_i(u, \ldots, u^{(i-1)}) = P_i(u_0, \ldots, u_0^{(i-1)})$ + higher powers of $1/t$.*

(iii) *Assume that $t'/t \in E$ and that $u \in E(t)$. If u has t-adic expansion of the form $u = u_\gamma/t^\gamma$ + higher powers of t, $\gamma > 0$, $u_\gamma \neq 0$, then $P_i(u, \ldots, u^{(i-1)}) = u_\gamma^i/t^{i\gamma}$ + higher powers of t. If u has $(1/t)$-adic expansion $u = u_\gamma t^\gamma$ + higher powers of $1/t$, $u_\gamma \neq 0$, then $P_i(u, \ldots, u^{(i-1)}) = u_\gamma^i t^{i\gamma}$ + higher powers of $1/t$ if $\gamma > 0$ and $P(u, \ldots, u^{(i-1)}) = \gamma_i(u_0, \ldots, u_0^{(i-1)})$ + higher powers of $1/t$ if $\gamma = 0$.*

Proof. We proceed in all cases by induction.

(i): Note that for p as above, p does not divide p'. First assume that $\gamma > 1$. If $i = 1$, $P_1 = u$, so $v_{1\gamma} = u_\gamma$. If $i > 0$, then

$$\begin{aligned} P_{i+1} = P'_i + uP_i &= \left(\frac{-i\gamma v_{i\gamma}p'}{p^{i\gamma+1}} + \cdots\right) + \left(\frac{u_\gamma}{p^\gamma} + \cdots\right)\left(\frac{v_{i\gamma}}{p^{i\gamma}} + \cdots\right) \\ &= \left(\frac{u_\gamma v_{i\gamma}}{p^{(i+1)\gamma}} + \cdots\right) \qquad \text{since } (i+1)\gamma > i\gamma + 1 \\ &= \frac{v_{(i+1)\gamma}}{p^{(i+1)\gamma}} + \cdots \qquad \text{where } v_{(i+1)\gamma} \equiv (u_\gamma)^{i+1} \bmod p. \end{aligned}$$

Now assume that $\gamma = 1$. If $i = 1$ then the result is obvious. For $i > 0$,

$$\begin{aligned} P_{i+1} = P'_i + uP_i &= \left(\frac{-iv_ip'}{p^{i+1}} + \cdots\right) + \left(\frac{u_1}{p} + \cdots\right)\left(\frac{v_i}{p^i} + \cdots\right) \\ &= \frac{v_i(u_1 - ip')}{p^{i+1}} + \cdots \\ &= \frac{v_{i+1}}{p^{i+1}} + \cdots \qquad \text{where } v_{i+1} \equiv \prod_{j=0}^{i} (u_1 - jp') \bmod p. \end{aligned}$$

(ii) and (iii): The proofs are similar to (i), proceeding by induction and comparing leading terms.

PROPOSITION 2.3. *Let $E \subset E(t)$ be computable differential fields with $C(E) = C(E(t))$ and assume that either $t' \in E$ or $t'/t \in E$ and that t is transcendental over E. Furthermore, assume that we can effectively solve homogeneous linear differential equations over $E(t)$ and that we can effectively find all exponential solutions of homogeneous linear differential equations over E. Then we can decide if a homogeneous linear differential equation $L(y) = 0$ with coefficients in $E(t)$ has a solution $e^{\int u}$ with $u \in E(t)$.*

PROOF. Assume that $t' \in E$. We wish to decide if there is a u in $E(t)$ such that $R(u) = 0$ where $R(u)$ is the Riccati equation associated with $L(y) = 0$. We shall try and determine the possible partial fraction decomposition for such a u. Let $p(t)$ be a monic irreducible polynomial in $E[t]$ and let $u = u_\gamma/p^\gamma + u_{\gamma-1}/p^{\gamma-1} + \cdots$, where $\deg_t u_i < \deg_t p$ and $\gamma > 1$. I claim that one can find γ and u_γ up to some finite set of choices. The following method is very similar to the Newton polygon process used to expand algebraic functions. Let $L(y) = y^{(n)} + A_{n-1}y^{(n-1)} + \cdots + A_0 y$ and $A_i = a_{i\alpha_i}/p^{\alpha_i} + \cdots$. The leading powers in $R(u) = P_n + A_{n-1}P_{n-1} + \cdots + A_0$ must cancel. The leading term of A_iP_i is $(a_{i\alpha_i}v_{i\gamma})/p^{\alpha_i + i\gamma}$ (using the notation of Lemma 2.2). Therefore for some i, j, $i \neq j$, we have $\alpha_i + i\gamma = \alpha_j + j\gamma$ or $\gamma = \alpha_i - \alpha_j/(i-j)$. Fix a value of γ and a corresponding j such that $\alpha_k + k\gamma \le \alpha_j + j\gamma$ for all other k (of course we only consider such γ that are integers > 1). Summing over all h such that $\alpha_h + h\gamma = \alpha_j + j\gamma$ we have $\sum a_{h\alpha_h} v_{h\gamma} = 0$. Lemma 2.2 implies that $\sum a_{h\alpha_h} u_\gamma^h = 0 \bmod p$. Since $\deg_t u_\gamma < \deg_t p$, this latter equation determines u_γ up to some finite set (to find u_γ we factor $\sum a_{h\alpha_h} Y^h$ in $(E(t)/p)[Y]$ and consider the linear factors). We now alter our original $L(y)$. Let $\tilde{L}(y) = L(y\, e^{\int(u_\gamma/p^\gamma)})/e^{\int(u_\gamma/p^\gamma)}$. We now look for solutions of $\tilde{L}(y) = 0$ of the form $e^{\int \tilde{u}}$ with $\tilde{u} \in E$ and $\tilde{u} = \tilde{u}_\delta/p^\delta + \cdots$ with δ an integer. We proceed now as above, except we only consider those δ with $\delta < \gamma$. Note that if $u = u_\delta/p^\delta + \cdots$ satisfies $R(u) = 0$ with $\delta > 1$, then p must occur in the denominator of some A_i. Therefore, we continue until we can assume that u is of the form $\sum u_{j1}/p_j + s$, where $s \in E[t]$. Some of the p_j occur in denominators of the A_i and some do not. Let $p = p_j$ occur in the denominator of some A_i and let $u_1 = u_{j1}$. We then look for cancellation as before. Fixing a value of i and summing over all h such that $\alpha_h + h = \alpha_i + i$, we have that $\sum a_{h\alpha_h} v_h = 0$. We have that $v_h \equiv \prod_{j=0}^{h-1} (u_1 - jp') \bmod p$ by Lemma 2.2, so u_1 will satisfy $\sum a_{h\alpha_h}(\prod_{j=0}^{h-1} (u_1 - jp')) \equiv 0 \bmod p$. This equation is a non-zero polynomial in u_1, and u_1 is assumed to have degree less than the degree of p, so we can determine u_1 up to some finite set of choices, as before. We can alter $L(y)$ as before and assume that u is of the form $u = \sum u_{j1}/p_j + s$, where this sum is over all p_j that do not occur in the denominator of some A_i. For such a p_j (which we again refer to as p), the leading term in the p-adic expansion of $R(u)$ is v_n/p^n (by Lemma 2.2), so $v_n = 0$ and so $\prod_{j=0}^{n-1} (u_i - jp') \equiv 0 \bmod p$. Therefore $u_1 = jp'$ for some j, $1 \le j \le n-1$. This allows us to assume that u is of the form $u = \sum (n_j p_j')/p_j + s$ where the n_j are integers and s and the p_j are polynomials not yet determined. We now proceed to determine $s = s_m t^m + \cdots + s_0$. First assume that $m > 1$. Expanding u in decreasing powers of t, we have $u = s_m t^m +$ smaller powers of t. Lemma 2.2(ii) implies that $P_i(u) = s_m^i t^{im} +$ lower powers of t. Writing $A_i = a_i t^{\alpha_i} +$ lower powers of t, we see that for cancellation to occur in $R(u)$ we must have $\alpha_i + im = \alpha_j + jm$ for some $i \neq j$. Therefore m can be determined up to some finite set of possibilities by considering the possible integer $\alpha_i - \alpha_j/(j-i)$. We fix such a value of m and a j such that $\alpha_k + km \le \alpha_j + jm$ for all other k. Summing over all h such that $\alpha_h + hm = \alpha_j + jm$, we have $\sum a_h s_m^h = 0$.

Therefore s_m is determined up to a finite set of possibilities. We can again alter $L(y)$ until we are in a position to assume that $u=u_0+\sum (n_ip_i')/p_i$. Looking for cancellation in $R(u)=0$, we have, by Lemma 2.2(ii), that $\sum a_iP_i(u_0,\ldots,u_0^{(i-1)})=0$, where the summation is over all i with $\alpha_i=\max_j(\alpha_j)$. Therefore $e^{\int u_0}$ satisfies $\hat{L}(y)=0$, where $\hat{L}(y)=\sum a_iy^{(i)}$, the summation being over all i with $\alpha_i=\max_j(\alpha_j)$. Since we can effectively find all exponential solutions of homogeneous linear differential equations over E, we can find a finite set $\{v_0,\ldots,v_r\}$ such that $e^{\int u_0}/e^{\int v_i}=r_i\in E(t)$ for some i. For each i, we form $L_i(y)=L(y\,e^{\int v_i})/e^{\int v_i}$. We then have that $y=r_i\exp\int(\sum(n_ip_i')/p_i)=r_i\prod p_i^{n_i}$ will satisfy some $L_i(y)$. Since we can effectively solve homogeneous linear differential equations over $E(t)$, we can find such a solution, and so reconstruct an exponential solution of our original differential equation.

We now deal with the case when $t'/t\in E$. We again try to determine the possible partial fraction expansions for solutions of $R(u)=0$. Let p be a monic irreducible polynomial in $E[t]$ and *assume* $p\neq t$. If p occurs in the denominator of u to a power larger than 1, then p must occur in the denominator of some A_i. For these p, we can proceed with the reduction used above. We can therefore assume that $u=\sum u_{j1}/p_j+s$, where the p_j are monic irreducible polynomials, $p_j\neq t$ and $s=s_m/t^m+\cdots+s_0+\cdots+s_Mt^M$. We can eliminate those p_j that appear in the denominator of some A_i as before and so assume that the p_j that appear do not occur in the denominator of any A_i. Fix some p_i, say p. The leading term in $R(u)=P_n(u)+A_{n-1}P_{n-1}(u)+\cdots+A_0$ is (using the notation Lemma 2.2(i)) v_n/p^n, where $v_n\equiv\prod_{j=0}^{n-1}(u_1-jp')\bmod p$ and u_1 is the leading coefficient in the p-adic expansion of u. Since $v_n=0$, we must have $u_1\equiv jp'\bmod p$ for some j, $1\le j\le n-1$. Since p is monic and the degree of p is the same as the degree of p' (say N), we have that $u_1=jp'-Nj\zeta p$ where $\zeta=t'/t$. Therefore $u=\sum((n_ip_i'+m_i\zeta p_i)/p_i)+s$, where the n_i and m_i are integers and $\deg_t(n_ip_i'+m_i\zeta p_i)<\deg_t p_i$. We now will determine the coefficients in $s=s_m/t^m+\cdots+s_Mt^M$. If $A_i=a_{i\alpha_i}/t^{\alpha_i}+$higher powers of t, then the leading term in the t-adic expansion of A_iP_i is $(a_{i\alpha_i}s_m^i)/t^{\alpha_i+mi}$. To get cancellation in $R(u)=0$, we must have two such terms being equal. This determines m up to some finite set of choices. Selecting an m and a j such that $km+\alpha_k\le jm+\alpha_j$ for all other k and summing over all h such that $hm+\alpha_h=jm+\alpha_j$, we have $\sum a_{h\alpha_h}s_m^h=0$. Therefore s_m is determined up to some finite set of possibilities. We can determine s_M in a similar way. We can alter $L(y)$ as before until we are in a position to assume that $u=u_0+\sum(n_ip_i'+m_i\zeta p_i)/p_i$. Looking for cancellation in the $(1/t)$-adic expansion of $R(u)$, we have by Lemma 2.2(iii) that $\sum a_{h\alpha_h}P_h(u_0,\ldots,u_0^{(i-1)})=0$ where $A_i=a_{i\alpha_i}t^{\alpha_i}+$higher powers of $1/t$ and the summation is over all h such that $\alpha_h=\max_j(\alpha_j)$. Therefore $e^{\int u_0}$ satisfies $\hat{L}(y)=0$ where $\hat{L}(y)=\sum a_{i\alpha_i}y^{(i)}$, the summation being as before. Since we can effectively find all exponential solutions of homogeneous linear differential equations over E, we can find a finite set $\{v_0,\ldots,v_r\}$ such that $e^{\int u_0}/e^{\int v_i}=w_i\in E(t)$ for some i. For each i, $y_i=y\,e^{-\int v_i}=w_i\exp(\int\sum(n_ip_i'+m_i\zeta p_i)/p_i)=w_it^{(\sum m_i)}\prod p_i^{n_i}\in E(t)$. y_i also satisfies the linear differential equation $L_i(y)=L(y\,e^{\int v_i})/e^{\int v_i}=0$. Since we can effectively solve homogeneous linear differential equations over $E(t)$, we can decide if this equation has a non-zero solution in $E(t)$. If not, then $L(y)=0$ has no solution of the desired form and if so, we can reconstruct a solution of the desired form.

Examples are now given to illustrate Proposition 2.3.

EXAMPLE 2.3.1. Let $E=\mathbb{Q}(x)$ and $t=\log x$. We shall consider the differential equation

$$L(y)=y''-\frac{1}{x(\log x+1)}y'-(\log x+1)^2y=0$$

and decide if it has solutions of the form $e^{\int u}$ with $u \in E(t)$. We shall assume that the hypotheses of the theorem are satisfied by E (this will be shown later). The associated Riccati equation is

$$R(u) = (u' + u^2) - \frac{1}{x(\log x + 1)} u - (\log x + 1)^2 = 0.$$

Assume that u is a solution of $R(u) = 0$ in $E(t) = \mathbb{Q}(x, \log x)$. If $p(t) \neq t+1$ is irreducible in $E[t]$, then as we have noted above the order of u at $p(t)$ is bigger than or equal to -1. At $t+1 = \log x + 1$, we may write

$$u = \frac{u_\gamma}{(\log x + 1)^\gamma} + \frac{u_{\gamma-1}}{(\log x + 1)^{\gamma-1}} + \cdots.$$

Substituting this expression in $R(u)$ and comparing leading terms, one sees that if $\gamma > 1$, then the leading term in $R(u)$ is $u_\gamma (\log x + 1)^{2\gamma}$. If $\gamma = 1$, then the leading term (after some cancellation) is $u_1^2 (\log x + 1)^2$. This means that u cannot have a pole at $\log x + 1$. We therefore have that $u = \sum p_i'/p_i + s$ where the p_i are irreducible polynomials in $E[t]$, not equal to $t+1$ and s is a polynomial in $E[t]$. We now proceed to determine $s(t) = s_m t^m + \cdots + s_0$. Plugging into $R(u)$ and comparing terms we see that $m = 1$ and $s_1 = \pm 1$ and so $s(t) = \pm t + s_0 = \pm \log x + s_0$. We therefore alter $L(y)$ in two ways. Let

$$\begin{aligned} L_1(y) &= L(y\, e^{-\int \log x}) / e^{-\int \log x} \\ &= y'' + \frac{-2x \log^2 x - 2x \log x - 1}{x \log x + x} y' + \frac{-2x \log^2 x - 3x \log x - x - 1}{x \log x + x} y. \end{aligned}$$

Let

$$\begin{aligned} L_2(y) &= L(y\, e^{\int \log x}) / e^{\int \log x} \\ &= y'' + \frac{2x \log^2 x + 2x \log x - 1}{x \log x + x} y' + \frac{-2x \log^2 x - 3x \log x - x + 1}{x \log x + x} y. \end{aligned}$$

To determine the possible s_0 we consider L_1 and L_2 separately. In both cases we are looking for solutions of this equation of the form $y = e^{\int s_0 + (\sum p_i'/p_i)}$ with s_0 in E. For L_1, if we expand the coefficients in decreasing powers of $\log x$, we get

$$L_1(y) = y'' + (2 \log x + \cdots) y' + (-2 \log x + \cdots) y = 0.$$

$e^{\int s_0}$ will satisfy $\hat{L}_1(y) = 2y' - 2y = 0$. By the hypotheses, we can find exponential solutions of this latter equation over $E = \mathbb{Q}(x)$. In fact, e^x is the only such solution, i.e. the only possibility for s_0 is 1. We now modify $L_1(y)$ and form

$$\begin{aligned} \bar{L}_1(y) &= L_1(y\, e^x)/e^x \\ &= y'' + \frac{2x \log^2 x + 4x \log x + 2x - 1}{x \log x + x} y'. \end{aligned}$$

We are looking for solutions of this latter equation of the form $r(\exp(\int (\sum p_i'/p_i)))$ with r in $E(t)$, that is, solutions in $E(t)$. A partial fractions argument shows that the only such solutions are constants. This implies that our original equation has a solution of the form $e^{\int (\log x + 1)} = e^{x \log x}$. Repeating this procedure for $L_2(y)$ would yield a solution of our original equation of the form $e^{\int (-\log x - 1)} = e^{-x \log x}$.

EXAMPLE 2.3.2. Let $E = \mathbb{Q}(x)$ and $t = e^x$. We shall consider the differential equation

$$L(y) = y'' + (-2e^x - 1)y' + e^{2x}y = 0.$$

The associated Riccati equation is

$$R(u) = (u' + u^2) + (-2e^x - 1)u + e^{2x} = 0.$$

One easily shows that all solutions in $E(t)$ of $R(u) = 0$ must be of the form $u = s + \sum (n_i p_i' + m_i p_i)/p_i$ where p_i are irreducible in $E[t]$ and not equal to t, and $s = s_m/t^m + \cdots + s_M t^M$. One easily sees that $m = 0$. Substituting u in $R(u)$ and expanding in powers of t, we have

$$Ms_M t^M + \cdots + s_M^2 t^{2M} + \cdots - 2Ms_M t^{M+1} + \cdots + t^2 = 0.$$

Therefore $M = 1$ and $s_M = 1$. Therefore $u = t + s_0 + \sum (n_i p_i' + m_i p_i)/p_i$. We alter the equation $L(y) = 0$ to get $L_1(y) = L(y\, e^{\int e^x})/e^{\int e^x} = y'' - y'$. We are looking for solutions of the form $e^{\int s_0 + \sum (n_i p_i' + m_i p_i)/p_i}$. We find that $s_0 = 1$ or 0. We now form the equations $L_{11}(y) = L_1(y\, e^{\int 1})/e^{\int 1} = y'' + y'$ and $L_{12}(y) = L_1(y\, e^{\int 0})/e^{\int 0} = y'' - y'$ and look for solutions of these equations that lie in $E(t)$. These have solutions $e^{-x}, 1$ and $e^x, 1$ respectively. Therefore the original equation $L(y)$ has solutions e^{e^x} and e^{x+e^x}.

Note that in these last two examples we have found all exponential solutions of $L(y) = 0$, not just a single one. The algorithm described in Proposition 2.3 can be modified to do this, but we would rather do this task in the following

LEMMA 2.4. *Let K be a computable differential field.*

(i) *Assume that for any homogeneous linear differential equation $L(y) = 0$ with coefficients in K we can decide if there exists a $u \in K$ such that $L(e^{\int u}) = 0$ and if so find such an element. Then we can effectively find all exponential solutions of homogeneous linear differential equations over K.*

(ii) *Assume that we can effectively solve homogeneous linear differential equations over K and that we can find all exponential solutions of homogeneous linear differential equations over K. If $L(y) = 0$ is a homogeneous linear differential equation with coefficients in K then one can find u_i, $1 \le i \le r$ and v_{ij}, $1 \le i \le r$, $1 \le j \le n_j$, such that if $u \in K$ and $L(e^{\int u}) = 0$ then there exists an i, $1 \le i \le r$ and constants c_{ij} such that $e^{\int u} = (\sum_j c_{ij} u_{ij})\, e^{\int u_i}$.*

PROOF. (i) We proceed by induction on the order of the linear differential equation. Let $L(y) = 0$ be a homogeneous linear differential equation of order n with coefficients in K. Decide if there exists a $u \in K$ such that $L(e^{\int u}) = 0$. If no such element exists, we are done. Otherwise find such an element. Let $L_1(y) = L(y\, e^{\int u})/e^{\int u}$. $L_1(y)$ has no term of order zero, so we may write $L_1(y) = \tilde{L}(y')$, where $\tilde{L}(y)$ has order $n - 1$. By induction we can find $u_1, \ldots, u_r$ in K such that if v is in K and $\tilde{L}(e^{\int v}) = 0$, then $e^{\int v}/e^{\int u_i}$ is in K. Let $w \in K$ satisfy $L(e^{\int w}) = 0$. We then have $0 = \tilde{L}((e^{\int w-u})') = \tilde{L}(e^{\int w - u + (w'-u')/(w-u)})$. Therefore $e^{\int w-u}/e^{\int u_i} \in K$ or $(e^{\int w-u})' = 0$. We can conclude that if $e^{\int w}$ satisfies $L(y) = 0$, then either $e^{\int w}/e^{\int u_i + u} \in K$ or $e^{\int w}/e^{\int u} \in K$.

(ii) Let $L(y) = 0$ be a homogeneous linear differential equation with coefficients in K. We can find $u_1, \ldots, u_r$ such that if $u \in K$ and $L(e^{\int u}) = 0$, the $e^{\int u}/e^{\int u_i} \in K$. For each i, form $L_i(y) = L(y\, e^{\int u_i})/e^{\int u_i}$ and find a basis $\{u_{ij}\}$ for the vector space of solutions in K of $L_i(y) = 0$. This choice of u_i and u_{ij} satisfies the conclusion of the lemma.

EXAMPLE 2.4.1. We consider the same equation as in Example 2.3.2, $L(y) = y''+(-2\,e^x-1)y'+e^{2x}y=0$. $e^{\int u}$ is a solution of this equation where $u=e^x$, so we form $L_1(y)=L(y\,e^{e^x})/e^{e^x}=y''-y'$. Therefore $\tilde{L}(y)=y'-y$. This latter equation has solution $e^{\int u}$ where $u=x$. Therefore if $w\in K=\mathbb{Q}(x, e^x)$ and $L(e^{\int w})=0$ then either $e^{\int w}/e^{e^x}\in K$ or $e^{\int w}/e^{e^x+x}\in K$.

LEMMA 2.5. *Let K be a computable differential field with an algebraically closed field of constants and $L(y)=0$ a homogeneous linear differential equation with coefficients that lie in a finitely generated algebraic extension E of K. Assume that one can effectively find all solutions of homogeneous linear differential equations over K and effectively find all exponential solutions of homogeneous linear differential equations over K. Then*

(i) *One can decide if there exists an element u algebraic over K such that $L(e^{\int u})=0$ and if so find a minimal polynomial of u over K.*

(ii) *One can find an algebraic extension F of E and elements u_i, u_{ij} in F such that if u is an element in F and $L(e^{\int u})=0$, then there exist an i and constants c_{ij} such that $e^{\int u}=(\sum_j c_{ij}u_{ij})\,e^{\int u_i}$.*

(iii) *One can find elements $y_1,\ldots,y_r$, liouvillian over K, that span the space of all solutions of $L(y)=0$ that are liouvillian over K.*

PROOF. (i) This follows from the techniques and results of Singer (1981). For the convenience of the reader we outline the proof here. We know from Theorem 2.4 of Singer (1981) that if $L(y)=0$ has a solution of the prescribed form then it has one where u is algebraic over E of degree bounded by an integer N that depends only on the order n of L. Furthermore, there is a recursive function $I(n)$ such that $N\le I(n)$. Therefore u will satisfy a polynomial equation over F of degree at most $I(n)[E:F]$. Fix an integer $m\le I(n)[E:F]$. We wish to decide if there exist $a_{m-1},\ldots,a_0$ in K such that if $f(u)=u^m+a_{m-1}u^{m-1}+\cdots+a_0=0$, then $L(e^{\int u})=0$. We shall first determine the possible a_{m-1} that can occur. We may assume that E is a normal extension of F and let $G=\{\sigma_1,\ldots,\sigma_t\}$ be the galois group of E over K. For each $\sigma_i\in G$, let $L_i(y)=0$ be the homogeneous linear differential equation obtained by applying σ_i to the coefficients of $L(y)=0$. Let $\tilde{L}(y)=0$ be the homogeneous linear differential equation whose solution space is $\{y_1+\cdots+y_m \mid L_i(y_i)=0,\ i=1,\ldots,m\}$ (see Lemma 3.8 of Singer (1981)). Assuming that f is irreducible, we have that $\tilde{L}(e^{\int u_i})=0$ for all roots u_i of $f(u)=0$. Let $y_i=e^{\int u_i}$. We see that

$$\begin{aligned}a_{m-1}&=-(u_1+\cdots+u_m)\\&=-\left(\frac{y_1'}{y_1}+\cdots+\frac{y_m'}{y_m}\right)\\&=-\left(\frac{(y_1\cdots y_m)'}{y_1\cdots y_m}\right).\end{aligned}$$

Let $L_1(y)=0$ be the homogeneous linear differential equation with coefficients in E, satisfied by $z_m=y_1\cdots y_m$. This can be calculated from $\tilde{L}(y)$ using Lemma 3.8 of Singer (1981). Since $z_m'/z_m\in K$, $\hat{L}_1(z_m)=0$ where $\hat{L}_1(y)=\sum_{i=1}^m (L_1(y))^{\sigma}$. $\hat{L}_1(y)$ has coefficients in K, so by Lemma 2.4(ii), we can find v_i and v_{ij} in K such that $z_m=(\sum c_{ij}v_{ij})\,e^{\int v_i}$ for some constants c_{ij}. Therefore, for some i, $a_{m-1}=v_i+(\sum c_{ij}v_{ij})'/(\sum c_{ij}v_{ij})$. We can conclude that we can construct a finite number of rational functions $R_{m-1,i}(c_{i,j})$ with coefficients in K such that for some choice of constants c_{ij} and i, $a_{m-1}=R_{m-1,i}(c_{i,j})$. To compute a_{m-2}, note that

$$\begin{aligned} a_{m-2} &= \sum_{1\le i,j\le m} u_i u_j \\ &= \sum_{1\le i,j\le m} \frac{y_i' y_j'}{y_i y_j} \\ &= c\left(\prod y_j\right)^{-1} \sum y_{i_1}' y_{i_2}' y_{i_3} \cdots y_{i_m} \end{aligned}$$

where this latter sum is taken over all permutations of $(1,\ldots,m)$. Let $P(Y_1,\ldots,Y_m)=\sum Y_{i_1}' Y_{i_2}' Y_{i_3}\cdots Y_{i_m}$. We can construct a linear differential equation $L_2(y)$ with coefficients in E, such that for any solutions $y_1,\ldots,y_m$ of $L(y)=0$, $L_2(P(y_1,\ldots,y_m))=0$. Note that $\prod y_j=\sum_j c_{ij}v_{ij}\,e^{\int v_i}$ for some i and constants c_{ij} (as above). Therefore, for some i, we have that $(\prod y_j)a_{m-2}=\sum c_{ij}v_{ij}\,e^{\int v_i}a_{m-2}=P(y_1,\ldots,y_m)$ and so $e^{-\int v_i}P(y_1,\ldots,y_m)$ is in K. Let $L_{2,i}(y)=L_2(e^{\int v_i}y)/e^{\int v_i}$ and let $\hat{L}_{2,i}(y)=\sum_{i=1}^{l}(L_{2,i}(y))^{\sigma_i}$. Since $L_{2,i}(e^{\int v_i}P(y_1,\ldots,y_n))=0$, and $e^{-\int v_i}P(y_1,\ldots,y_m)\in K$, we have $\hat{L}_{2,i}(e^{-\int v_i}P(y_1,\ldots,y_m))=0$. By assumption, we can find $\{w_{ij}\}$ in K such that for each i, $\{w_{i,j}\}$ forms a basis of the vector space of solutions of $\hat{L}_{2,i}(y)=0$ in K. Therefore, for some i and constants $c_{ij}, d_{ij}, a_{m-2}=c(\prod y_i)^{-1}P(y_1,\ldots,y_n)=(\sum_j c_{ij}v_{ij}\,e^{\int v_i})^{-1}P(y_1,\ldots,y_n)=(\sum_j c_{ij}v_{ij})^{-1}(\sum_j d_{ij}w_{ij})$. We denote this rational function as $R_{2,i}$. In a similar way we can find, for the other a_{n-1}, expressions $R_{h,i}$ that are rational functions of known quantities with unknown constant coefficients. For all possible choices of $i=(i_0,\ldots,i_{m-1})$ we form

$$f_i(u)=u^m+R_{m-1,i_{m-1}}u^{m-1}+\cdots+R_{0,i_0}.$$

We wish to determine if there is a choice of constants $c_{ij}, d_{ij},\ldots$ such that any solution of $f_i(u)=0$ is a solution of $R(u)=0$, where $R(u)=0$ is the Riccati equation associated with $L(y)=0$. If we reduce $R(u)$ with respect to $f_i(u)$ as in Ritt (1966, p. 6), we get a remainder $H_i(u)$ that must vanish identically. This forces a collection of polynomials (with coefficients in E) in the $c_{ij}, d_{ij},\ldots$ to vanish identically. Since we are looking for constant solutions, there is an equivalent set of polynomials with constant coefficients. We can then decide if there exist constants that satisfy these polynomial equations. If such a set of constants do not exist then $L(y)=0$ does not have a solution of the desired form. If such a set does exist, then we factor $f_i(u)$ to find a minimal polynomial for u.

(ii) We proceed by induction on the order of $L(y)$. If the order is 1, then $L(y)=y'+ay$, for some $a\in K$. We then let $F=E$ and note that for any u in F such that $L(e^{\int u})=0$, we have $e^{\int u}=c\,e^{\int -a}$, for some constant c. Now assume that $L(y)$ has order $n>1$. By part (i) of this lemma, we can decide if there is a u algebraic over K such that $L(e^{\int u})=0$. Let $L_1(y)=L(y\,e^{\int u})/e^{\int u}$. $L_1(y)$ has no term of order zero so we may write $L_1(y)=\tilde{L}(y')$, where $\tilde{L}(y)$ has order $n-1$ and coefficients in $E(u)$. By induction, there exists an algebraic extension F of $E(u)$ and elements v_i in F such that if v is in F and $\tilde{L}(e^{\int v})=0$, then $e^{\int v}/e^{\int v_i}\in F$. If $w\in F$ and $L(e^{\int w})=0$, then $\tilde{L}((e^{\int w-u})')=0$, so $(w-u)\,e^{\int w-u}/e^{\int v_i}\in F$ or $e^{\int w}=c\,e^{\int u}$ for some constant c. Therefore $e^{\int w}/e^{\int v_i+u}\in F$ or $e^{\int w}/e^{\int u}\in F$. Let $u_1=v_1+u,\ldots,u_r=v_r+u$, $u_{r+1}=u$ and $L_i(y)=L(y\,e^{\int u_i})/e^{\int u_i}$ for $i=1,\ldots,r+1$. Each L_i has coefficients in F, an explicitly given algebraic extension of K. In Proposition 3.1 we shall see that we can effectively solve homogeneous linear differential equations in F. Therefore, we can find u_{ij} such that, for each i, $\{u_{ij}\}$ forms a basis for the set of solutions of $L_i(y)=0$ in F. We have then found the desired u_i and u_{ij}.

(iii) We again proceed by induction on the order of $L(y)$. When $L(y)$ has order 1, then $L(y)=y'+ay$ for some a in K. Therefore $y_1=e^{-\int a}$ will satisfy the conclusion of the lemma. For $n>1$, Theorem 2.4 of Singer (1981) implies that if $L(y)=0$ has a solution liouvillian over K, then there exists an element u algebraic over K such that $L(e^{\int u})=0$. Part (i) of this lemma lets us decide if this is the case and if so, find such an element. For such a u, let $L_1(y)=L(y\, e^{\int u})/e^{\int u}$. Since L_1 has no term of order 0, we may write $L_1(y)=\tilde{L}(y')$, where $\tilde{L}(y)$ has order $n-1$. By induction, we can find $z_1, \ldots, z_r$ that span the space of solutions of $\tilde{L}(y)=0$ liouvillian over $K(u)$. $e^{\int u}$, $e^{\int u}\int z_1, \ldots, e^{\int u}\int z_r$ then span the solutions of $L(y)=0$ liouvillian over K.

EXAMPLE 2.5.1. Let $K=\mathbb{Q}(x, \log x)$ and consider the linear differential equation

$$L(y)=y''+\frac{4x\log x+2x}{4x^2\log x}y'-\frac{1}{4x^2\log x}y=0.$$

We will find all liouvillian solutions of this equation. We start by looking for all solutions of the form $e^{\int u}$ where u is algebraic over K of degree at most 2. In general, we would have to decide if there is such a solution with u algebraic over K of degree bounded by some computable function of the order of L, but in this case we will see that the number 2 is enough. u satisfies an equation of the form $f(u)=u^2+au+b=0$ with $a, b\in K$. We will furthermore assume that $f(u)$ is irreducible. We then have that $a=-(y_1'/y_1+y_2'/y_2)$ where y_1 and y_2 are solutions of $L(y)=0$. We now construct a linear differential equation $L_2(y)$ satisfied by all elements y_1y_2 where y_1 and y_2 are solutions of $L(y)=0$. An algorithm for this is given in Singer (1981). We have

$$L_2(y)=y'''+\frac{6x\log x+3x}{2x^2\log x}y''+\frac{2\log x+1}{2x^2\log x}y'=0.$$

We need to find all solutions y of this latter equation such that $y'/y\in K$. An algorithm for this is given in Lemma 2.3 and Lemma 2.4. We find that the only such solutions are constants. This implies that $a=0$. To determine b, we note that $b=(y_1'y_2')/y_1y_2$, so $y_1y_2b=y_1'y_2'$. Since y_1y_2 must be a constant, $y_1'y_2'$ must be in K. We again construct a linear differential equation $L_3(y)=0$ satisfied by all expressions of the form $y_1'y_2'$. We find

$$\begin{aligned}L_3(y)=y'''&+\frac{18x^2\log^2 x+9x^2\log x}{2x^3\log^2 x}y''+\frac{38\log^2 x+43\log x+6x}{2x^3\log^2 x}y'\\&+\frac{16\log^2 x+32\log x+10}{2x^3\log^2 x}y\\&=0.\end{aligned}$$

We must find all solutions of this latter equation in K. An algorithm for this is given in Proposition 3.10. We find that the only such solutions are constant multiples of $1/(4x^2\log x)$. Therefore $f(u)$ must be of the form $u^2+c/(4x^2\log x)$ for some constant c. If $f(u)=0$, then $u^2=-c/(4x^2\log x)$ and $u'=-\frac{1}{2}[8x\log x+4x/(4x^2\log x)]\cdot u$. Substituting these expressions in

$$R(u)=u^2+u'+\frac{4x\log x+2x}{4x^2\log x}u-\frac{1}{4x^2\log x}=0$$

we see that $c=-1$. Therefore $f(u)=u^2-1/(4x^2\log x)$ so $L(y)$ has solutions of the form $y=e^w$ where $w=\pm(\log x)^{1/2}$. These two solutions form a basis for the space of all solutions of $L(y)=0$.

PROPOSITION 2.6. *Let $E \subset E(t)$ be computable differential fields and assume that t is algebraic over E and that $C(E) = C(E(t))$ is algebraically closed. Assume that we can effectively solve homogeneous linear differential equations over E and that we can effectively find all exponential solutions of homogeneous linear differential equations over E. Then we can decide if a homogeneous linear differential equation $L(y) = 0$ with coefficients in $E(t)$ has a solution $e^{\int u}$ with $u \in E(t)$.*

PROOF. Let F, u_i, u_{ij} be as in Lemma 2.5(ii) where $E(t) \subset F$. If $L(e^{\int u}) = 0$ for some $u \in E(t)$, then there exists an i and constants c_{ij} such that $u = u_i + (\sum c_{ij} u_{ij})'/(\sum c_{ij} u_{ij})$. Therefore we need to decide if there exist constants c_{ij} such that $u_i + (\sum c_{ij} u_{ij})'/(\sum c_{ij} u_{ij}) \in E(t)$. If we write this in terms of a basis of F over $E(t)$, this is equivalent to a system of polynomials in the c_{ij}, with coefficients in $E(t)$ vanishing. There is an equivalent polynomial system with constant coefficients and we can decide if this has a solution in the subfield of constants.

PROOF OF PROPOSITION 2.1. This follows immediately from Propositions 2.3 and 2.6 and Lemma 2.4(i).

3. Question 1

In this section we discuss the problem of answering question 1 for fields of the form $E(t)$ where E satisfies a suitable hypothesis and either $t'/t \in E$, $t' \in E$ or t is algebraic over E. We actually deal with a slightly more general question related to the following definition. Let K be a differential field. We say that we can *effectively solve parameterized linear differential equations over* K if given $a_{n-1}, \ldots, a_0, b_m, \ldots, b_0$ in K, one can effectively find $h_1, \ldots, h_r$ in K and a system $\mathscr{L}$ in $m+r$ variables with coefficients in $C(K)$ such that $y^{(n)} + a_{n-1} y^{(n-1)} + \cdots + a_0 y = c_1 b_1 + \cdots + c_m b_m$ for $y \in K$ and c_i in $C(K)$ if and only if $y = y_1 h_1 + \cdots + y_r h_r$ where the $y_i \in C(K)$ and $c_1, \ldots, c_m, y_1, \ldots, y_r$ satisfy $\mathscr{L}$. Obviously, if K is computable and we can effectively solve parameterized linear differential equations over K, then we can effectively solve homogeneous linear differential equations over K. Propositions 3.1 and 3.4 can be proved if both the hypotheses and conclusions regarding solving parameterized linear differential equations are replaced by the weaker statement that we can effectively solve homogeneous linear differential equations. In Proposition 3.9, we need the stronger statement to make the induction work. We prove these stronger statements with the hope that they will be more useful in applications.

We first deal with the field $E(t)$ where t is algebraic over E. Let $E[D]$ be the ring of differential operators with coefficients in E. This is the set of expressions of the form $a_n D^n + \cdots + a_0$ where multiplication corresponds to composition of these operators. In general, this is not a commutative ring, since $Da = D(a) + aD$. It is known that this ring has a right and left division algorithm (Poole, 1960, p. 31), so we can row and column reduce any matrix with coefficients in $E[D]$ (Poole 1960, p. 39).

PROPOSITION 3.1. *Let E be a computable differential field and t an element algebraic over E. If we can effectively solve parameterized linear differential equations over E then we can effectively solve parameterized linear differential equations over $E(t)$.*

PROOF. Let $1, t, \ldots, t^N$ form a vector space basis of $E(t)$ over E and let $y = y_0 + y_1 t + \cdots + y_N t^N$ where $y_0, \ldots, y_N$ are new variables. Using the fact that t' may be explicitly written as an element of $E(t)$, we may then write

$$L(y)=y^{(n)}+a_{n-1}y^{(n-1)}+\cdots+a_0y=c_1b_1+\cdots+c_mb_m$$

as

$$L_0(y_0,\ldots,y_N)+L_1(y_0,\ldots,y_N)t+\cdots+L_N(y_0,\ldots,y_N)t^N$$
$$=B_0(c_1,\ldots,c_m)+B_1(c_1,\ldots,c_m)t+\cdots+B_N(c_1,\ldots,c_m)t^N$$

where the L_i are linear differential equations in the y_j with coefficients in E and the B_i are linear polynomials in the c_j with coefficients in K. We can write this latter expression in matrix form $AY=B$ where A is an $N+1\times N+1$ matrix with entries in $E[D]$, $Y=(y_0,\ldots,y_N)^T$ and $B=(B_0,\ldots,B_N)^T$. Using row and column reduction, we can find matrices U and V with entries in $E[D]$ such that U has a left inverse, V has a right inverse and $UAV=C$ where

$$C=\begin{bmatrix}\tilde{L}_0 & 0 & 0 & \cdots & 0\\ 0 & \tilde{L}_1 & 0 & \cdots & 0\\ \cdot & \cdot & \cdot & & \cdot\\ 0 & \cdot & \cdot & \cdots & \tilde{L}_N\end{bmatrix}$$

and the $\tilde{L}_i$ are in $E[D]$. Y is a solution of $AY=B$ if and only if $W=V^{-1}Y$ is a solution of $CW=UB$. Solving this latter system is equivalent to solving $N+1$ equations $\tilde{L}_i(w_i)=\sum c_j\tilde{b}_{ij}$, where the $\tilde{b}_j$ are in E. Since we can effectively solve parameterized linear differential equations in E we can find appropriate h_{ij} in E and systems of linear equations $\mathscr{L}_i$. Using these we can construct elements h_j in $E(t)$ and a system $\mathscr{L}$ of linear equations satisfying the conditions for $L(y)=\sum c_ib_i$ in the definition of effectively solving parameterized linear differential equations.

An example illustrating the above proposition is given in Davenport & Singer (1986, p. 242). We now turn to fields of the form $E(t)$ where $t'/t\in E$ or $t'\in E$.

LEMMA 3.2. *Let $E\subset E(t)$ be computable differential fields with $C(E)=C(E(t))$, t transcendental over E and either $t'/t\in E$ or $t'\in E$. Assume*:

(i) *we can effectively solve parameterized linear differential equations over E,*

(ii) *if $t'/t\in E$ and $A_n,\ldots,A_0$, $B_m,\ldots,B_1$ are in $E[t,t^{-1}]$, we can effectively find an integer M such that if $Y=y_\gamma/t^\gamma+\cdots+y_0+\cdots+y_\delta t^\delta$ with $y_i\in E$, $y_\delta y_\gamma\neq 0$, satisfies $A_nY^{(n)}+\cdots+A_0Y=c_mB_m+\cdots+c_1B_1$ for some $c_i\in C(E)$, then $\gamma\leq M$ and $\delta\leq M$.*

(iii) *if $t'\in E$ and $A_n,\ldots,A_0$, $B_m,\ldots,B_1\in E[t]$, we can effectively find an integer M such that if $Y=y_0+\cdots+y_\gamma t^\gamma$ with $y_i\in E$, $y_\gamma\neq 0$, satisfies $A_nY^{(n)}+\cdots+A_0Y=c_mB_m+\cdots+c_1B_1$ for some c_i in $C(E)$, then $\gamma\leq M$.*

Then we can effectively solve parameterized linear differential equations over $E(t)$.

PROOF. We first consider the case where $t'/t\in E$. Let

$$L(y)=y^{(n)}+a_{n-1}y^{(n-1)}+\cdots+a_0y=c_1b_1+\cdots+c_mb_m \tag{1}$$

with the a_i, $b_i \in E(t)$. Let p be a monic irreducible polynomial in $E[t]$, $p \neq t$, and let

$$y = \frac{y_\alpha}{p^\alpha} + \cdots$$

$$a_i = \frac{a_{i\alpha_i}}{p^{\alpha_i}} + \cdots$$

$$b_i = \frac{b_{i\beta_i}}{p^{\beta_i}} + \cdots$$

be the p-adic expansions of these elements (for convenience we define $a_n = 1$ so $a_{n0} = 1$). Differentiating, we see that

$$y^{(j)} = \frac{u_j}{p^{\alpha+j}} + \cdots$$

where $u_j \equiv \pm\alpha(\alpha+1)\cdots(\alpha+j-1)y_\alpha(p')^j \bmod p$. Note that p' and p are relatively prime so that $u_j \neq 0$. If $\alpha > 0$, then some $\alpha_i > 0$ or some $\beta_i > 0$. Therefore only $p \neq t$ that occur to negative powers in the partial fraction decomposition of a solution of (1) have this property. We shall first try to bound α for such a p. In order for cancellation to occur in (1), we must have that either $\max_i(\alpha + i + \alpha_i) \leq \max_i \beta_i$, in which case we can bound α or $\gamma = \max_i(\alpha + i + \alpha_i) > \max_i \beta_i$. In this latter case we must have $\sum a_{i\alpha_i} u_i \equiv 0 \bmod p$, where the sum is over all i such that $\gamma = \alpha + i + \alpha_i$. This latter equation can be rewritten as $\sum a_{i\alpha_i}(\pm\alpha(\alpha+1)\cdots(\alpha+i-1)y_\alpha(p')^i) \equiv 0 \bmod p$. We can divide by y_α and get $\sum a_{i\alpha_i}(\pm\alpha(\alpha+1)\cdots(\alpha+i-1)(p')^i) \equiv 0 \bmod p$. Since p' and p are relatively prime and, for each i, $a_{i\alpha_i}$ and p are relatively prime, this latter equation gives a non-zero polynomial that α must satisfy. α is therefore determined up to some finite set of choices and so we can effectively find a bound α^*. Set $y = Y/p_1^{\alpha_1^*} \cdots p_k^{\alpha_k^*}$, where the p_j are those monic irreducible polynomials ($\neq t$) appearing in the denominators of some a_i or b_i and the α_j^* are the bounds calculated above. Substitute this into $L(y) = c_1 b_1 + \cdots + c_m b_m$ and clear denominators to get

$$A_n Y^{(n)} + A_{n-1} Y^{(n-1)} + \cdots + A_0 Y = c_m B_m + \cdots + c_m B_m, \tag{2}$$

where

$$Y = y_\gamma / t^\gamma + \cdots + y_0 + \cdots + y_\delta t^\delta$$

with the y_i and the $a_{i\alpha_i}$ in E and $A_n, \ldots, A_0$, $B_1, \ldots, B_m$ in $E[t, t^{-1}]$. By our hypotheses, we can find an M such that $\delta \leq M$ and $\gamma \leq M$.

We now wish to determine the y_j. Substituting our expression for Y into (2) and writing this in terms of powers of t, we have

$$\begin{aligned} L_{N_1}(y_\gamma, \ldots, y_\delta)t^{-N_1} + \cdots + L_{N_2}(y_\gamma, \ldots, y_\delta)t^{N_2} \\ = C_{N_3}(c_1, \ldots, c_m)t^{-N_3} + \cdots + C_{N_4}(c_1, \ldots, c_m)t^{N_4} \end{aligned}$$

for some $N_1 \leq N_2$ and $N_3 \leq N_4$ integers, where the L_i are linear differential equations in the y_j with coefficients in E and the C_j are linear in the c_i with coefficients in E. If $N_3 > N_1$, we set $C_{N_3} = \cdots = C_{N_1+1} = 0$ and get a system of linear equations $\mathcal{L}_1$ for the c_i. We similarly can get a system of linear equations $\mathcal{L}_2$ if $N_2 > N_4$. For $N_1 \leq i \leq N_2$, we have the equations $L_i(y_\alpha, \ldots, y_\gamma) = C_i(c_1, \ldots, c_m)$. This system can be written as $AY = B$, where A is an $[N_2 + N_1 + 1] \times [N_2 + N_1 + 1]$ matrix with coefficients in $E[D]$, $Y = (y_\gamma, \ldots, y_\delta)^T$ and $B = (C_\gamma, \ldots, C_\delta)^T$. We can find (as in Proposition 3.1) an equivalent diagonal system $CW = UB$ and apply the hypotheses of this proposition to find linear systems $\mathcal{L}_i$ in the c_j and appropriate h_{ij}. Transforming these back to our system $AY = B$

and then substituting into $y = y_\gamma t^{-\gamma} + \cdots + y_\delta t^\delta$ gives us the appropriate h_i for the conclusion of this proposition. We may take $\mathscr{L} = \mathscr{L}_1 \cup \mathscr{L}_2 \cup (\cup \mathscr{L}_i)$.

The proof when $t' \in E$ follows in a similar manner and will be omitted.

LEMMA 3.3. *Let $E \subset E(t)$ be computable differential fields with $C(E) = C(E(t))$, t transcendental over E and $t'/t \in E$. Assume:*

(i) *we can effectively find all exponential solutions of homogeneous linear differential equations over E, and*

(ii) *for any u in E, we can decide if $y' + uy$ has a non-zero solution in $E(t)$ and find such a solution.*

Then given any $A_n, \ldots, A_0$, $B_m, \ldots, B_1$ in $E[t, t^{-1}]$, we can effectively find an M such that if $Y = y_\gamma / t^\gamma + \cdots + y_\delta t^\delta$ with $y_i \in E$, $y_\gamma y_\delta \neq 0$, satisfies $A_n Y^{(n)} + \cdots + A_0 Y = c_m B_m + \cdots + c_1 B_1$ for some $c_i \in C(E)$, then $\gamma \leq M$ and $\delta \leq M$.

PROOF. We first show how to bound γ. Let

$$A_i = \frac{a_{i\alpha_i}}{t^{\alpha_i}} + \cdots + a_{i\beta_i} t^{\beta_i}$$

$$c_1 B_1 + \cdots + c_m B_m = \frac{b_\mu}{t^\mu} + \cdots + b_\nu t^\nu$$

with the a_{ij} in E and the b_i linear in the c_j with coefficients in E. Note that (if $\gamma > 0$) we have

$$Y^{(i)} = \frac{u_i}{t^\gamma} + \cdots \qquad \text{where } u_i = \left(\frac{y_\gamma}{t^\gamma}\right)^{(i)} t^\gamma \in E.$$

Furthermore, $u_i \neq 0$, since otherwise t would be algebraic over E. Substituting the above expression for Y into

$$A_n Y^{(n)} + \cdots + A_0 Y = c_m B_m + \cdots + c_1 B_1 \tag{3}$$

and equating coefficients, we see that $\bar{\alpha} = \max_i(\gamma + \alpha_i) < \max_i \beta_i$, in which case γ can be bounded, or $\bar{\alpha} > \max_i \beta_i$. In this latter case, the leading term on the left hand side of (3) is $\sum a_{i\alpha_i} u_i / t^{\gamma + \alpha_i}$ where the summation is over all i such that $\gamma + \alpha_i = \bar{\alpha}$. We then will have

$$0 = t^{-\gamma} \sum a_{i\alpha_i} u_i = \sum a_{i\alpha_i} \left(\frac{y_\gamma}{t^\gamma}\right)^{(i)}.$$

Therefore, $Z = y_\gamma / t^\gamma$ is a solution of $L(Z) = \sum a_{i\alpha} Z^{(i)} = 0$. By our assumptions we can find u_j and u_{ij} in E such that for some j, $y_\gamma t^{-\gamma} = \sum d_i u_{ij} \, e^{\int u_i}$ for some constants d_i. This implies that for some j, $y' - u_j y = 0$ has a solution in $E(t)$. Finding all such solutions allows us to bound γ. We can bound δ in a similar way.

PROPOSITION 3.4. *Let $E \subset E(t)$ be computable differential fields with $C(E) = C(E(t))$, t transcendental over E and $t'/t \in E$. Assume that we can effectively find all exponential solutions of homogeneous linear differential equations over E and that for any u in E decide if $y' + uy = 0$ has a non-zero solution in $E(t)$ and find all such a solution if it exists. Then we can effectively solve parameterized linear differential equations over $E(t)$.*

PROOF. Immediate from Lemma 3.2 and Lemma 3.3.

EXAMPLE 3.4.1. Let $E=\mathbb{Q}$ and $t=e^x$. Consider the linear differential equation

$$L(y)=y''+\frac{-24e^x-25}{4e^x+5}y'+\frac{20e^x}{4e^x+5}y=0.$$

We wish to find all solutions of this equation in $\mathbb{Q}(e^x)$. Using p-adic expansions for $p\neq t$, one can easily show that any solution must be of the form $y_\gamma/t^\gamma+\cdots+y_\delta t^\delta$. We therefore clear denominators in the above differential equation and consider

$$(4t+5)y''+(-24t-25)y'+20ty=0. \tag{4}$$

Comparing highest powers of t, we see that $y_\delta t^\delta$ satisfies $4y''-24y'+20y=0$. This latter equation has solutions e^{5x} and e^x that are exponential over $E=\mathbb{Q}$. Both of these are in $\mathbb{Q}(e^x)$. Therefore $\delta\leq 5$. Comparing lowest powers of t, we see that y_γ/t^γ satisfies $5y''-25y'+20y=0$. This latter equation has solutions e^{4x} and e^x in $\mathbb{Q}(e^x)$. Since $\gamma\geq 0$, we conclude that either $\gamma=0$ or $y_\gamma=0$. Therefore $y=y_5t^5+\cdots+y_0$ for some y_i constants. If we substitute this expression in (4) we get the following

$$-12y_4t^5+(-20y_4-16y_3)t^4+(-30y_3-12y_2)t^3+(-30y_2)t^2+(20y_0-20y_1)t=0.$$

Equating powers of t to 0 and solving gives us that $y_2=y_3=y_4=0$ and $y_0=y_1$. Therefore, solutions of (4) in $E(t)$ are of the form $c_1e^{5x}+c_2(e^x+1)$ where c_1 and c_2 are arbitrary constants.

A few words need to be said about the assumption in the previous proposition that for $u\in E$ we can decide if $y'+uy=0$ has a solution in $E(t)$. A priori, this is stronger than the assumption that we can decide effectively find all exponential solutions or all solutions of homogeneous linear differential equations over E. Since $t'/t\in E$, it is known (Rosenlicht, 1976, Theorem 2) that any solution in $E(t)$ of $y'+uy=0$ must be of the form y_nt^n for some integer n. y_n will then satisfy $y_n'+(u+n(t'/t))y_n=0$. We are therefore asking to decide if there is some integer n such that this latter equation has a non-zero solution in E. Similar problems come up in the Risch algorithm for integration in finite terms (we are asking if $\int u=\log y_n+n\log t$ for some y_n and integer n). We do not know how to reduce this question to the assumptions that we can effectively find all exponential solutions or effectively solve homogeneous linear differential equations. The following lemma shows that there are classes of fields for which this hypothesis is true.

LEMMA 3.5. *Let $E\subset E(t)$ be computable differential fields with $C(E)=C(E(t))$ and assume t is transcendental over E with $t'/t\in E$ or $t'\in E$.*

(i) *If E is an elementary extension of $C(x)$, $x'=1$, and $u\in E$, then one can decide if $y'+uy=0$ has a non-zero solution in $E(t)$ and find such a solution.*

(ii) *If E is a purely transcendental liouvillian extension of $C(x)$, $x'=1$, and $u\in E$, then one can decide if $y'+uy=0$ has a non-zero solution in $E(t)$ and find such a solution.*

PROOF. In this proof we shall rely heavily on the results of Rothstein & Caviness (1979) and the appendix of Singer *et al.* (1985). If $t'\in E$, then the Corollary to Theorem 1 of Rosenlicht (1976) implies that any solution u of $y'+uy=0$ in $E(t)$ is actually in E. If E is an elementary extension of $C(x)$, the result follows from Risch (1968). If E is a purely transcendental liouvillian extension of $C(x)$, the result follows from Theorem $A1(b)$ of Singer *et al.* (1985) and the fact that we can effectively embed such an extension in a log-explicit extension. We now assume that $t'/t\in E$ and let $t'/t=v$.

(i) Assume that E is an elementary extension of $C(x)$. We can use the Risch Algorithm (Risch, 1968) to decide if v has an elementary anti-derivative. If it does, then we can find $v_1, \ldots, v_r$ in E such that $E(\int v) \subset E(\log v_1, \ldots, \log v_r)$. Since, for each i, $E_i = E(\log v_1, \ldots, \log v_i)$ is an elementary extension of E, we can inductively decide if $\log v_{i+1}$ is algebraic over E_i (and so in E_i) or transcendental over E_i. Therefore we can assume that $E_r = E(\log v_1, \ldots, \log v_r)$ is a computable differential field. The corollary to Theorem 1 of Rosenlicht (1976) implies that t is transcendental over E_r. $E_r(t)$ is a generalized log-explicit extension of C and we can write $E_r(t) = C(t_1, \ldots, t_n)$ as in (Rothstein & Caviness, 1979, Theorem 3.1). It is enough to decide, for a given u in E, if $y' + uy = 0$ has a solution in $E_r(t)$, since the corollary to Theorem 1 of Rosenlicht (1976) implies that such a solution will lie in $E(t)$. To decide if $y' + uy$ has a solution in $E_r(t)$, we use Corollary 3.2 of Rothstein & Caviness (1979). According to this result, if such a solution existed then

$$u = c + \sum_{i \in \mathscr{L}} r_i t_i + \sum_{i \in \mathscr{E}} r_i a_i$$

where c is a constant, $\mathscr{L} = \{i \mid t_i' = a_i'/a_i$, for some $a_i \in C(t_1, \ldots, t_{i-1})\}$, and $\mathscr{E} = \{i \mid t_i'/t_i = a_i'$ for some $a_i \in C(t_1, \ldots, t_{i-1})\}$. Writing this last equation as $u' = \sum r_i t_i' + \sum r_i a_i'$, and expanding in terms of a $\mathbb{Q}$-basis of $C(t_1, \ldots, t_n)$, we can find a rational solution $\{r_i\}$ if one exists. If such a solution exists, then $y = e^{\int u} = d \prod_{i \in L} a_i^{r_i} \prod_{i \in E} t_i^{r_i}$, for some constant d. This means that for some integer N (that can be determined from the r_i) $(y/(d^{1/N}))^N \in E_r(t)$. $E_r(t)$ is a computable field, so to determine if $y \in E$, we need only factor $Y^N - (\prod_{i \in L} a_i^{r_i} \prod_{i \in E} t^{r_i})^N$ over $E_r(t)$.

If $\int v$ is not elementary over E, then $E(\int v, t)$ is a log explicit extension of C and we can proceed as above.

(ii) Either $\int v$ is in E or it is transcendental over E. Lemma 3.4 of Rothstein & Caviness (1979) and Theorem A1 of Singer *et al.* (1985) imply that one can effectively embed $E(t)$ into a regular (i.e. purely transcendental) log-explicit extension F of C. Furthermore F will be of the form $E(t_1, \ldots, t_n)$, with the t_i' in E. The corollary to Theorem 1 of Rosenlicht (1976) implies that t is transcendental over F. Given u in E it is enough to decide if $y' + uy = 0$ has a solution in $F(t)$, since the corollary to Theorem 1 of Rosenlicht (1976) will imply this solution lies in E. Therefore, let us assume that E is a regular log-explicit extension of C. Theorem A1(b) now allows us to decide if $y' + uy = 0$ has a solution in $E(t)$ and find such a solution if it does.

We will now prove a result similar to Lemma 3.3 for fields of the form $E(t)$ with $t' \in E$. This lemma will describe an algorithm to find a certain integer M that bounds the degree of solutions in $E[t]$ of linear differential equations. To show the algorithm is correct, we need to consider more general extensions of E and we will prove two simple lemmas about these extensions.

Let $E \subset E(t)$ be countable differential fields, $C(E) = C(E(t))$, t transcendental over E and $t' \in E$. Since $C(E)$ is countable we may assume that $C(E) \subset \mathbb{C}$. Let $F = \mathbb{C} \otimes_{C(E)} E$. We first note that t is transcendental over F. If not, then $t^n + a_{n-1} t^{n-1} + \cdots a_0 = 0$ for some $a_i \in F$. Differentiating this equation, we have $nt^{n-1}t' + a_{n-1}' t^{n-1} + \cdots = 0$ so $t' = (1/n) a_{n-1}'$. Therefore there exists a $u \in F$ such that $u' = t'$. Let $\{\gamma_i\} \subset \mathbb{C}$ be an E-basis of F and write $u = \sum \gamma_i u_i$ for some $u_i \in E$. We then have $\sum \gamma_i u_i' = t' \in E$. Therefore for some i, $\gamma_i = 1$ and $t' = u_i'$. This implies that in $E(t)$, $(u - t)' = 0$ so $t \in E$, a contradiction.

We now consider the field $K = F((t^{-1}))$, the field of formal Laurent series in t^{-1} with coefficients in F. We can extend the derivation on F to K by defining

$$\left(\sum_{i \le n_0} a_i t^i\right)' = a'_{n_0} t^{n_0} + \sum_{i \le n_0} (i a_i t' + a'_{i-1}) t^{i-1}.$$

Let $\kappa \in \mathbb{C} - \mathbb{Q}$ and define an extension $K(u)$ of K where u is transcendental over K and $u'/u = \kappa t'/t$. We then have

Lemma 3.6. (i) $C(F) = C(K)$. (ii) $C(K) = C(K(u))$.

Proof. (i) Let $(\sum_{i \le n_0} a_i t^i)' = 0$. First assume that $n_0 \ne 0$. We then have $a'_{n_0} = n_0 a_{n_0} t' + a'_{n_0-1} = 0$. Therefore $t' = (a_{n_0-1}/n_0 a_{n_0})'$, so $t - (a_{n_0-1}/n_0 a_{n_0}) \in C(F)$ contradicting the fact that t is transcendental over F. If $n_0 = 0$, let $n_1 < n_0$ be the largest integer such that $a_{n_1} \ne 0$. We then have $a'_{n_0} = a'_{n_1} = 0$ and $n_1 a_{n_1} t' + a'_{n_1-1} = 0$ and we get a similar contradiction as above.

(ii) If $C(K)$ is properly contained in $C(K(t))$ then there exists an integer n and a $v \in K$ such that $v'/v = n\kappa t'/t$ (Risch, 1969). If we write $v = a_{n_0} t^{n_0} + a_{n_0-1} t^{n_0-1} + \cdots$, then

$$\frac{v'}{v} = \frac{a'_{n_0} t^{n_0} + (n_0 a_{n_0} t' + a'_{n_0-1}) t^{n_0-1} + \cdots}{a_{n_0} t^{n_0} + \cdots} = n\kappa \frac{t'}{t}.$$

Therefore, $a'_{n_0} = 0$ and $(n_0 a_{n_0} t' + a'_{n_0-1})/a_{n_0} = n\kappa t'$. This implies that $(n\kappa - n_0)t' = (a_{n_0-1}/a_{n_0})'$. Since $t \notin K$, we must have $n\kappa - n_0 = 0$, contradicting the fact that $\kappa \notin \mathbb{Q}$.

We need one more lemma before we can prove that the algorithm described in Lemma 3.8 terminates.

Lemma 3.7. *Let $K \subset F$ be differential fields and assume that we can solve parameterized linear differential equations over K. Let $A_0, \ldots, A_n, B_1, \ldots, B_m \in K$ and let $\mathcal{L}$ be a set of homogeneous linear equations with coefficients in $C(K)$ and $z_1, \ldots, z_r$ be elements of K such that $A_n y^n + \cdots + A_0 y = c_m B_m + \cdots + c_1 B_1$ for $y \in K$, $c_i \in C(K)$ if and only if $y = \sum h_i z_i$ for some $h_i \in C(K)$ and $c_1, \ldots, c_m, h_1, \ldots, h_r$ satisfy $\mathcal{L}$. Then for $y \in K \cdot C(F)$ and $c_i \in C(F)$, we have $A_n y^{(n)} + \cdots + A_0 y = c_m B_m + \cdots + c_1 B_1$ if and only if $y = \sum h_i z_i$ for some $h_i \in C(F)$ and $c_1, \ldots, c_m, h_1, \ldots, h_r$ satisfy $\mathcal{L}$.*

Proof. The proof follows by expanding $y \in K \cdot C(F)$ in a K-basis and noting that all equations (both differential and algebraic) involved are linear.

Lemma 3.8. *Let $E \subset E(t)$ be computable differentiable fields with $C(E) = C(E(T))$, t transcendental over E and $t' \in E$. Assume that we can effectively solve parameterized linear differential equations over E. Let $A_n, \ldots, A_0, B_m, \ldots, B_1 \in E[t]$. Then we can effectively find an integer M such that if $Y = y_0 + \cdots + y_\gamma t^\gamma$, $y_\gamma \ne 0$, is a solution of*

$$A_n Y^{(n)} + \cdots + A_0 Y = c_m B_m + \cdots + c_1 B_1 \tag{5}$$

for some $c_i \in C(E)$ then $\gamma < M$.

Proof. We shall describe a procedure that successively attempts to compute $y_\gamma, y_{\gamma-1}, \ldots$. We shall then show that for some i, in the process of computing $y_{\gamma-i}$, we shall find a bound for γ. This bound will be independent of the c_is. At present we have no way of giving an a priori estimate for the i such that the computation of $y_{\gamma-i}$ gives us the bound for γ.

Let

$$A_i = a_{i\alpha}t^{\alpha} + \cdots + a_{i0}$$

$$B_i = b_{i\beta}t^{\beta} + \cdots + b_{i0}$$

where some $a_{i\alpha} \neq 0$ and some $b_{i\beta} \neq 0$. We replace Y in (5) by $Y = y_\gamma t^\gamma + \cdots + y_0$ and equate powers of t. We first consider the highest power of t, that is $t^{\gamma+\alpha}$. There are two cases: either $\gamma + \alpha \leq \beta$ or $\gamma + \alpha > \beta$ and

$$L_\gamma(y_\gamma) = \sum_{i=0}^{n} a_{i\alpha} y_\gamma^{(i)} = 0.$$

By our hypotheses, we can find $z_{\gamma 1}, \ldots, z_{\gamma r_\gamma}$ in E such that any solution y_γ of $L_\gamma(y_\gamma) = 0$ in E is of the form $y_\gamma = \sum_i c_{\gamma i} z_{\gamma i}$ for some $c_{\gamma i}$ in $C(E)$. If there are no non-zero solutions of $L_\gamma(y_\gamma) = 0$, we stop and have $\gamma \leq \beta - \alpha$. Otherwise, we now replace y_γ in (5) by $\sum c_{\gamma i} z_{\gamma i}$ (where the $c_{\gamma i}$ are indeterminants) and consider the coefficients of $t^{\gamma+\alpha-1}$. Either $\gamma + \alpha - 1 \leq \beta$ or $\gamma + \alpha - 1 > \beta$ and the coefficient of $t^{\gamma+\alpha-1}$ is

$$L_{\gamma-1}(y_{\gamma-1}) = \sum a_{i\alpha} y_{\gamma-1}^{(i)} - \left(\sum c_{\gamma j} e_{\gamma j} + \sum \gamma c_{\gamma j} f_{\gamma j}\right) = 0$$

where the $e_{\gamma j}$ and $f_{\gamma j}$ are known elements of E. By our hypotheses we can find $z_{\gamma-1,1}, \ldots, z_{\gamma-1,r_{\gamma-1}}$ in E and a linear system $\mathcal{L}_{\gamma-1}$ in $r_{\gamma-1} + r_\gamma$ variables with coefficients in $C(E)$ such that $y_{\gamma-1} = \sum c_{\gamma-1,i} z_{\gamma-1,i}$ is a solution of $L_{\gamma-1}(y_{\gamma-1}) = 0$ for some choice of $c_{\gamma,1}, \ldots, c_{\gamma,r_\gamma}, \gamma$ if and only if $(c_{\gamma-1,1}, \ldots, c_{\gamma-1,r_{\gamma-1}}, c_{\gamma,1}, \ldots, c_{\gamma,r_\gamma}, \gamma c_{\gamma,1}, \ldots, \gamma c_{\gamma,r_\gamma})$ satisfies $\mathcal{L}_{\gamma-1}$. We can replace $\mathcal{L}_{\gamma-1}$ with a linear system $\mathcal{L}^*_{\gamma-1}$ having coefficients in $C[\gamma]$ such that $y_{\gamma-1} = \sum c_{\gamma-1,i} z_{\gamma-1,i}$ is a solution of $L_{\gamma-1}(y_{\gamma-1}) = 0$ for some choice of $c_{\gamma,1}, \ldots, c_{\gamma,r_\gamma}$ if and only if $(c_{\gamma-1,1}, \ldots, c_{\gamma-1,r_{\gamma-1}}, c_{\gamma,1}, \ldots, c_{\gamma,r_\gamma})$ satisfies $\mathcal{L}^*_{\gamma-1}$. Using elimination theory, we can effectively find systems $\mathcal{S}_1^{(1)}, \ldots, \mathcal{S}_{n_1}^{(1)}$ where each $\mathcal{S}_i^{(1)} = \{f_{i,1}^{(1)} = 0, \ldots, f_{i,m_i}^{(1)} = 0, g_i^{(1)} \neq 0\}$ where $f_{i,j}^{(1)}, g_i^{(1)} \in C(E)[\gamma]$ such that for γ in some algebraically closed extension field k of $C(E)$, γ satisfies some $\mathcal{S}_i^{(1)}$ if and only if $\mathcal{L}^*_{\gamma-1}$ has a solution $(c_{\gamma-1,1}, \ldots, c_{\gamma-1,r_{\gamma-1}}, c_{\gamma,1}, \ldots, c_{\gamma,r_\gamma})$ with $(c_{\gamma,1}, \ldots, c_{\gamma,r_\gamma}) \neq (0, \ldots, 0)$. We shall deal with two cases:

Case 1. Each $\mathcal{S}_i^{(1)}$ has only a finite number of solutions γ. In this case we can bound γ by $\gamma \leq \max(\beta - \alpha$, integer solutions of the $\mathcal{S}_i^{(1)})$.

Case 2. Some $\mathcal{S}_i^{(1)}$ has an infinite number of solutions. In this case, such an $\mathcal{S}_i^{(1)}$ is of the form $\{0 = 0, g_i^{(1)} \neq 0\}$. When this happens we continue and attempt to calculate $y_{\gamma-2}$ in the following way.

We now replace $y_{\gamma-1}$ by $\sum c_{\gamma-1,j} z_{\gamma-1,j}$ in (5), where the $c_{\gamma-1,j}$ are undetermined coefficients and consider the coefficient of $t^{\gamma-2}$. This will be of the form

$$L_{\gamma-2}(y_{\gamma-2}) = \sum a_{i\alpha} y_{\gamma-2}^{(i)} - M_{\gamma-2}(c_{\gamma,j}, \gamma c_{\gamma,j}, \gamma(\gamma-1)c_{\gamma,j}, c_{\gamma-1,j}, \gamma c_{\gamma-1,j})$$

where $M_{\gamma-2}$ is a linear form in the $c_{\gamma,j}, \gamma c_{\gamma,j}, \gamma(\gamma-1)c_{\gamma j}, \ldots$ with known coefficients from E. By our hypotheses, we can find $z_{\gamma-2,1}, \ldots, z_{\gamma-2,r_{\gamma-2}}$ in E and a system of linear equations $\mathcal{L}_{\gamma-2}$ with coefficients in $C(E)$ such that $y_{\gamma-2} = \sum c_{\gamma-2,i} z_{\gamma-2,i}$ is a solution of $L_{\gamma-2}(y_{\gamma-2}) = 0$ for some choice of $(c_{\gamma,j}, \gamma c_{\gamma,j}, \gamma(\gamma-1)c_{\gamma,j}, c_{\gamma-1,j}, \gamma c_{\gamma-1,j})$ if and only if $(c_{\gamma,j}, \gamma c_{\gamma,j}, \gamma(\gamma-1)c_{\gamma,j}, c_{\gamma-1,j}, \gamma c_{\gamma-1,j}, c_{\gamma-2,j})$ satisfies $\mathcal{L}_{\gamma-2}$. As before, we can absorb γ into the coefficients and produce a system of homogeneous linear equations $\mathcal{L}^*_{\gamma-2}$ with coefficients in $C(E)[\gamma]$ such that $(c_{\gamma,j}, \gamma c_{\gamma,j}, \gamma(\gamma-1)c_{\gamma,j}, c_{\gamma-1,j}, \gamma c_{\gamma-1,j}, c_{\gamma-2,j})$ is a solution of $\mathcal{L}_{\gamma-2}$ if and only if $(c_{\gamma,j}, c_{\gamma-1,j}, c_{\gamma-2,j})$ is a solution of $\mathcal{L}^*_{\gamma-2}$. Again there exist systems $S_1^{(2)}, \ldots, S_{n_2}^{(2)}$ where each $S_i^{(2)} = \{f_{i,1}^{(2)} = 0, \ldots, f_{i,m_i}^{(2)} = 0, g_i \neq 0\}$ with $f_{i,j}^{(2)}, g_i^{(2)} \in C(E)[\gamma]$, such that for any γ in some algebraically closed extension k of $C(E)$, $\mathcal{L}^*_{\gamma-2} \cup \mathcal{L}^*_{\gamma-1}$ has a solution

$(c_{\gamma,j}, c_{\gamma-1,j}, c_{\gamma-2,j})$ in k with the first r_γ coordinates not identically zero if and only if γ satisfies $\mathcal{S}_i^{(2)}$ for some i. We again have two cases:

Case 1. Each $\mathcal{S}_i^{(2)}$ has only a finite number of solutions. In this case we can bound γ by $\gamma \le \max(\beta-\alpha-1$, integer solutions of the $\mathcal{S}_i^{(2)})$.

Case 2. Some $\mathcal{S}_i^{(2)}$ has an infinite number of solutions. In this case such an $\mathcal{S}_i^{(2)}$ is of the form $\{0=0, g_i^{(2)} \ne 0\}$.

If we encounter case 2, we continue this process, otherwise we stop. Assume that we do not encounter case 1 before the kth repetition of the process. We have at this point found $z_{\gamma,1}, \ldots, z_{\gamma,r_\gamma}, \ldots, z_{\gamma-k+1,1}, \ldots, z_{\gamma-k+1,r_{\gamma-k+1}}$ and systems of linear equations $\mathcal{L}^*_{\gamma-1}, \ldots, \mathcal{L}^*_{\gamma-k+1}$, with coefficients in $C(E)[\gamma]$ such that for some c_i in $C(E)$ if $y = y_\gamma t^\gamma + \cdots$ is a solution of (5) with $y_\gamma \ne 0$ and $\gamma > \beta-\alpha+k-1$, then there exist $c_{i,\gamma-j} \in C(E)$, $1 \le i \le r_{\gamma-j}$, $0 \le j \le k-1$ such that $y_{\gamma-j} = \sum c_{i,\gamma-j} z_{i,\gamma-j}$ and $\{c_{i,\gamma-j}\}$ satisfy $\mathcal{L}^*_{\gamma-1} \cup \cdots \cup \mathcal{L}^*_{\gamma-k+1}$. Furthermore, there are systems $\mathcal{S}_1^{(k-1)}, \ldots, \mathcal{S}_{n_{k-1}}^{(k-1)}$ such that $\mathcal{L}^*_{\gamma-1} \cup \cdots \cup \mathcal{L}^*_{\gamma-k+1}$ has a solution with $c_{\gamma,1} \cdots c_{\gamma,r_\gamma}$ not all zero if and only if γ satisfies some $\mathcal{S}_i^{(k-1)}$. We can continue if and only if some $\mathcal{S}_i^{(k-1)}$ is of the form $\{0=0, g_i^{(k-1)} \ne 0\}$. We shall show that for some k, we have that no $\mathcal{S}_i^{(k-1)}$ is of this form. This will show that the algorithm terminates.

We argue by contradiction, so assume the process continues indefinitely. We now think of $C(E)$ as being embedded in $\mathbb{C}$ and fix some $\kappa \in \mathbb{C}$ transcendental over $C(E)$ (note that $C(E)$ is countable and so this can be done). For each k, We are assuming that there is an $\mathcal{S}_i^{(k-1)}$ of the form $\{0=0, g_i^{(k-1)} \ne 0\}$. Clearly κ satisfies $\mathcal{S}_i^{(k-1)}$. Therefore, for this κ we can solve $\mathcal{L}^*_{\gamma-1} \cup \cdots \cup \mathcal{L}^*_{\gamma-k+1}$ in $\mathbb{C}$ with non-zero $c_{\gamma,1}, \ldots, c_{\gamma,r_\gamma}$. Note that for fixed k the set V_k of $(c_{\gamma,1}, \ldots, c_{\gamma,r_\gamma})$ in $\mathbb{C}^{r_\gamma}$ such that $\mathcal{L}^*_{\gamma-1} \cup \cdots \cup \mathcal{L}^*_{\gamma-k+1}$ has a solution is a vector space. Notice that $V_k \supset V_{k+1}$ and $V_k \ne 0$. Therefore, for some k, we have $V_k = V_{k+1} = \cdots \ne 0$. This implies (using Lemma 3.7) that there exist $c_{i,\kappa-j} \in \mathbb{C}$, $1 \le i \le r_{\kappa-j}$, $0 \le r_{\kappa-j} < \infty$ such that

$$w_\kappa = \sum_{j=0}^{\infty} \left(\sum_{i=1}^{r_{\kappa-j}} c_{i,\kappa-j} z_{i,\kappa-j} \right) t^{\kappa-j}$$

is a solution of $A_n y^{(n)} + \cdots + A_0 y = 0$ with $c_{\kappa,1}, \ldots, c_{\kappa,r_\kappa}$ not all zero. We can repeat the above argument for $\gamma = \kappa-1, \ldots, \gamma = \kappa-n$ and produce $n+1$ solutions $w_\kappa, \ldots, w_{\kappa-n}$ in $E((t^{-1}))(t^\kappa)$ of $A_n y^{(n)} + \cdots + A_0 y = 0$. Note that by looking at leading terms, we can see that these solutions are linearly independent over $C(E)$ and therefore (by Lemma 3.6) over $C(E((t^{-1}))(t^\kappa)$. Since a homogeneous linear differential equation can have at most n solutions linearly independent over the constants (Kaplansky, 1957), this yields a contradiction. Therefore the process described above terminates.

Proposition 3.9. *Let $E \subset E(t)$ be computable differential fields with $C(E) = C(E(t))$, t transcendental over E and $t' \in E$. Assume that we can effectively solve parameterized linear differential equations over E. Then we can effectively solve parameterized linear differential equations over $E(t)$.*

Proof. Immediate from Lemma 3.2 and Lemma 3.9.

Example 3.9.1. Let $E = \mathbb{Q}(x)$ and $t = \log x$. Let

$$L(y) = (x^2 \log^2 x) y'' + (x \log^2 x - 3x \log x) y' + 3y = 0$$

We will look for solutions y of $L(y) = 0$ in $E(t) = \mathbb{Q}(x, \log x)$. Considering y as a rational function of t, we see that the only possible irreducible factor of the denominator is $t=\log x$. If we expand y in powers of $\log x$ and write $y=y_\alpha/(\log x)^\alpha+\cdots$, we see that the leading coefficient in $L(y)$ is $y_\alpha[\alpha(\alpha+1)-3(-\alpha)+3]$. Since this must equal zero, we have that $(\alpha+3)(\alpha+1)=0$. This means that any solution of $L(y)=0$ in $E(t)$ is actually in $E[t]$. We let $y=y_\gamma t^\gamma+y_{\gamma-1}t^{\gamma-1}+\cdots$ and substitute into $L(y)=0$. Calculating the coefficients of powers of t, we get the following:

l	Coefficient of t^l
$\gamma+2$	$L_\gamma(y_\gamma)=x^2y''_\gamma+xy'_\gamma$
$\gamma+1$	$L_{\gamma-1}(y_{\gamma-1})=x^2y''_{\gamma-1}+xy'_{\gamma-1}+(2\gamma x-3x)y'_\gamma$
γ	$L_{\gamma-2}(y_{\gamma-2})=x^2y''_{\gamma-2}+xy'_{\gamma-2}+(2\gamma x-5x)y'_{\gamma-1}+(\gamma^2-4\gamma+3)y_\gamma$

It is easy to see that $L_\gamma(y_\gamma)=0$ has only constant solutions in E. Replacing y_γ by $c_{\gamma,1}\cdot 1$ in $L_{\gamma-1}(y_{\gamma-2})$ yields the equation $x^2y''_{\gamma-1}+xy'_{\gamma-1}=0$ for $y_{\gamma-1}$. This new equation has only constant solutions in E and places no restrictions on γ. We let $y_{\gamma-1}=c_{\gamma-1,1}\cdot 1$ and substitute in the expression $L_{\gamma-2}(y_{\gamma-2})$. We obtain

$$x^2y''_{\gamma-2}+xy'_{\gamma-2}+(\gamma^2-4\gamma+3)c_{\gamma,1}=0.$$

Since $c_{\gamma,1}\neq 0$, this latter equation has a solution in E if and only if $\gamma^2-4\gamma+3=0$. This implies that $\gamma\leq 3$. Therefore $y=y_3t^3+y_2t^2+y_1t+y_0$. Substituting this expression into $L(y)=0$ and calculating the coefficients of powers of t, we find:

l	Coefficient of t^l
5	$L_3(y_3)=x^2y''_3+xy'_3$
4	$L_2(y_2)=x^2y''_2+xy'_2+3xy'_3$
3	$L_1(y_1)=x^2y''_1+xy'_1+xy'_2$
2	$L_0(y_0)=x^2y''_0+xy'_0-xy'_1-4y_2$
1	$-3xy'_0$
0	$3y_0$

Successively setting these expressions equal to zero and finding solutions in E yields that y_3 and y_1 are arbitrary constants and y_2 and y_0 are 0. Therefore all solutions of $L(y)=0$ in $\mathbb{Q}(x,\log x)$ are of the form $c_1(\log x)^3+c_2\log x$.

4. Final Comments

Using the results of the last two sections, we can answer questions 1 and 2 for certain classes of fields.

THEOREM 4.1. *Let C be an algebraically closed computable field and assume that either:*

(i) *K is an elementary extension of $C(x)$ with $x'=1$ and $C(K)=C$, or*

(ii) *K is an algebraic extension of a purely transcendental liouvillian extension of C with $C(K)=C$.*

Then one can effectively find exponential solutions of homogeneous linear differential equations over K and effectively solve parameterized linear differential equations over K.

PROOF. It is easy to see that one can find exponential solutions of homogeneous linear differential equations and effectively solve parameterized linear differential equations over C. Using Propositions 2.1, 3.1, 3.4, 3.9 and Lemma 3.5, one can prove this theorem by induction on the number of elements used to define the tower leading to K.

As a consequence of this and Lemma 2.5(ii), one can generalize the results of Singer (1981) in the following way:

THEOREM 4.2. *Let C and K be as in Theorem 4.1. If $L(y)=0$ is a homogeneous linear differential equation with coefficients in K, then one can find a basis for the space of solutions of $L(y)=0$ liouvillian over K.*

There remain several open problems and directions for further research.

(a) The algorithms presented above are certainly not very efficient. Efficiency could certainly be improved by using (where possible) Hermite reduction techniques (cf. Bronstein, 1990). We also have sometimes assumed that the field of constants is algebraically closed. For actual computations one has a finitely generated field and one is forced to compute the necessary algebraic extension. Work needs to be done efficiently to find minimal algebraic extensions that are sufficient and also incorporate the D^5 method (Della Dora *et al.*, 1985; Dicrescenzo & Duval, 1989).

(b) There should be a more direct algorithm to solve the problem stated in Proposition 2.6. In particular, one should not have to first decide if there exists a u algebraic over $E(t)$ such that $L(e^{\int u})=0$ in order to decide if there is a u in $E(t)$ satisfying this property. A procedure just working in $E(t)$ would be preferable and would possibly avoid the need to assume that the field of constants is algebraically closed.

(c) We do not have a priori bounds on how many cycles are required in the procedure presented in Lemma 3.8. Is there a simple function $f(n)$ (where n is the order of the differential equation) such that the algorithm terminates after $f(n)$ steps?

(d) We would like to extend Theorems 4.1 and 4.2 to other classes of fields, in particular liouvillian extensions of C (not just purely transcendental liouvillian extensions). At present this would require extending Lemma 3.5 to such fields. This seems to be related to the problem of parameterized integration in finite terms mentioned in Davenport & Singer (1986).

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