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THE MODEL THEORY OF ORDERED DIFFERENTIAL FIELDS

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In this paper, we show that the theory of ordered differential fields has a model completion.[1] We also show that any real differential field, finitely generated over the rational numbers, is isomorphic to some field of real meromorphic functions. In the last section of this paper, we combine these two results and discuss the problem of deciding if a system of differential equations has real analytic solutions. The author wishes to thank G. Stengle for some stimulating and helpful conversations and for drawing our attention to fields of real meromorphic functions.

§ **1. Real and ordered fields.** A *real field* is a field in which -1 is not a sum of squares. An *ordered field* is a field F together with a binary relation $<$ which totally orders F and satisfies the two properties: (1) If $0 < x$ and $0 < y$ then $0 < xy$. (2) If $x < y$ then, for all z in F, $x + z < y + z$. An element x of an ordered field is positive if $x > 0$. One can see that the square of any element is positive and that the sum of positive elements is positive. Since -1 is not positive, an ordered field is a real field. Conversely, given a real field F, it is known that one can define an ordering (not necessarily uniquely) on F [2, p. 274]. An ordered field F is a *real closed field* if: (1) every positive element is a square, and (2) every polynomial of odd degree with coefficients in F has a root in F. For example, the real numbers form a real closed field. Every ordered field can be embedded in a real closed field. It is also known that, in a real closed field K, polynomials satisfy the intermediate value property, i.e. if $f(x) \in K[x]$ and $a, b \in K$, $a < b$, and $f(a)f(b) < 0$ then there is a c in K such that $f(c) = 0$.

In the language whose nonlogical symbols are $(+, \cdot, -, 0, 1, ^{-1}, <)$, one can axiomatize the concepts of ordered field and real closed field. It is known that the theory of real closed fields is the model completion of the theory of ordered fields, [6, p. 92]. In fact, since the theory of ordered fields is a universal theory,

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[1] After this paper was submitted for publication, the referee brought to my attention the independent work of Lacava [1a] on the model completion of the theory of ordered differential fields. His paper, though, seems to contain several errors. His Axiom 5 on p. 325 is inconsistent with his other axioms, since this axiom would imply that the system $(v')^2 + v^2 = 0$, $v > 0$, has a solution, (let $x = 0$ to satisfy this axiom), an impossibility in an *ordered* differential field. He also makes the error of assuming that algebraic type + cut = elementary type on pp. 324 and 325.

the theory of real closed fields admits elimination of quantifiers, [6, p. 67]. We will use the following two consequences of these facts throughout this paper. By a polynomial system in $x_1, \ldots, x_n$ with coefficients in an ordered field k, we mean a finite collection of equations of the form $p(x_1, \ldots, x_n) = 0$ or $q(x_1, \ldots, x_n) > 0$, where $p, q \in k[x_1, \ldots, x_n]$.

PROPOSITION 1. *Let k be an ordered field and let S be a polynomial system in $x_1, \ldots, x_n$ with coefficients in k. The system S has a solution in some ordered extension field of k if and only if it has a solution in all real closed fields containing k.*

PROPOSITION 2. *Let S be a polynomial system in $x_1, \ldots, x_n, y_1, \ldots, y_m$ with coefficients in the rational numbers, and let k be an ordered field. One can effectively find a collection $\{S_1, \ldots, S_k\}$ of polynomial systems in $x_1, \ldots, x_n$ with coefficients in the rationals which have the following property: If $c_1, \ldots, c_n$ are any elements of k, then the system S^*, gotten from S by substituting c_i for x_i, $i = 1, \ldots, n$, has a solution in some ordered extension of k if and only if the c_i's satisfy one of the S_j's.*

An application of Propositions 1 and 2 is the following result needed in § 3.

PROPOSITION 3. *Let K be a real field, finitely generated over the rationals. Then K is isomorphic to a subfield of the real numbers.*

PROOF. Let $K = Q(a_1, \ldots, a_n, b)$ where $a_1, \ldots, a_n$ are algebraically independent over Q and b is algebraic over $Q(a_1, \ldots, a_n)$. Let $f(x_1, \ldots, x_n, y)$ be an irreducible polynomial in $Q[x_1, \ldots, x_n, y]$ such that $f(a_1, \ldots, a_n, b) = 0$. Consider the system $S = \{f(x_1, \ldots, x_n, y) = 0\}$ and let $S_1, \ldots, S_k$ be the systems whose existence is guaranteed by Proposition 2. Since K is a real field, we can order it in some way and consider it as an ordered field. The equation $f(a_1, \ldots, a_n, y) = 0$ has a solution in K so, by Proposition 2, we know that $a_1, \ldots, a_n$ must satisfy one of the S_j's, say $S_0 = \{f_1 = 0, \ldots, f_r = 0, g_1 > 0, \ldots, g_s > 0\}$. Since $a_1, \ldots, a_n$ are algebraically independent over Q, each f_i must be identically zero. Therefore this system is equivalent to $\{g_1 > 0, \ldots, g_s > 0\}$. This system has a solution in an ordered extension of Q, namely K, so by Proposition 1, it must have a solution in the real numbers R. We can conclude from this that $g_1 > 0, \cdots, g_s > 0$ defines a nonempty open set in R^n. Pick $(c_1, \ldots, c_n)$ in this open set such that $c_1, \ldots, c_n$ are algebraically independent over Q. By Proposition 2, $f(c_1, \ldots, c_n, y) = 0$ has a solution in some ordered extension of $Q(c_1, \ldots, c_n)$ and so by Proposition 1, has a solution d in R. Since $c_1, \ldots, c_n$ are algebraically independent over Q, $f(c_1, \ldots, c_n, y)$ is irreducible over $Q(c_1, \ldots, c_n)$. Therefore, there is an isomorphism from $Q(a_1, \ldots, a_n, b)$ onto $Q(c_1, \ldots, c_n, d) \subset R$.

Finally, we mention some conventions we shall use in what follows. Let $F \subset E$ be ordered fields. For x in E, we let $|x| = x$ if $x \geq 0$ and $|x| = -x$ if $x < 0$. If t is in E and $0 \leq |t| < |a|$ for all nonzero a in F, we say that t is infinitesimal with respect to F. If x is in E and $|x| < |z|$ for some z in F, we say x is finite with respect to F. The set of elements of E finite with respect to F forms a subring of E and the set of infinitesimals with respect to E forms an ideal in this ring. Finally we say that x is infinitely close to y if $x - y$ is an infinitesimal with respect to F.

§2. The theory of ordered differential fields and its model completion. A. Robinson showed, [5], that the theory of differential fields of characteristic zero has a model completion. L. Blum [1], [6, p. 295] reproved this result and gave simple axioms for this model completion. We will use Blum's techniques to give axioms for the model completion of the theory of ordered differential fields.

A *differential ring* is a ring R with a function $D: R \to R$ (called a derivation) such that $D(x+y) = Dx + Dy$ and $D(xy) = (Dx)y + x(Dy)$. A differential field is a field which is also a differential ring. All fields in this paper are assumed to be of characteristic 0. A differential polynomial with coefficients in a differential field F is a polynomial $f(y) = f(y, y', \dots, y^{(n)})$ in the variables $y, y', \dots, y^{(n)}, \dots$ with coefficients in F. The ring of differential polynomials is denoted by $F\{y\}$. Similarly, one can define differential polynomials in several variables and call this ring $F\{y_1, \dots, y_n\}$. There is a natural derivation $'$ on $F\{y_1, \dots, y_n\}$ given by $(y_j^{(i)})' = y_j^{(i+1)}$ and $a' = Da$ for $a \in F$. If $f \in F\{y\}$, the largest n such that $y^{(n)}$ appears in f is called the order of f. If z is in F then we define $f(z) = f(z, Dz, \dots, D^n z)$, where $D^{n+1}z = D(D^n z)$. If $k \subset K$ are differential fields and S a subset of K, we denote by $k\langle S\rangle$ the smallest differential subfield of K containing k and S. Elements $z_1, \dots, z_n$ of K are said to be differentially algebaically dependent if there is an $f \in k\{y_1, \dots, y_n\}$, $f \neq 0$, such that $f(z_1, \dots, z_n) = 0$. $z_1, \dots, z_n$ are said to be differentially algebraically independent if they are not dependent. In the former case, when $n = 1$, we say that z is differentially algebraic over k. Assume $z \in K$ is differentially algebraic over k and among all differential polynomials in $k\{y\}$ which vanish at z, pick one of smallest order, say n; of smallest degree in $y^{(n)}$; which is also irreducible. We call this a minimal polynomial of z over k. If u is an element of some differential extension field of k, and u is a zero of this polynomial, but of no differential polynomial of lower order, then $k\langle z\rangle$ is isomorphic to $k\langle u\rangle$ via an isomorphism which sends z to u and leaves k fixed. If a minimal polynomial f of z over k has order n, then the (abstract-algebraic) field $k(z, Dz, \dots, D^n z)$ is a differential field. To see that $D^{n+1}z$ lies in this field, just apply D to the relation $f(z) = 0$ and solve for $D^{n+1}z$. Note that $z, \dots, D^{n-1}z$, are algebraically independent over k. Conversely, given an irreducible polynomial f of order n, we can form a field containing a zero of f. Let $k(u_0, \dots, u_{n-1})$ be a purely transcendental extension of k and let u_n be a root of $f(u_0, \dots, u_{n-1}, X) = 0$, (where u_i is substituted for $y^{(i)}$ for $0 \leq i \leq n-1$ and X is substituted for $y^{(n)}$. Let the differential structure on $k(u_0, \dots, u_n)$ be determined by $Du_i = u_{i+1}$, $i = 0, \dots, n-1$.

As seen above, it is sometimes convenient to treat a differential polynomial as an algebraic polynomial by considering the $y^{(i)}$'s as variables having nothing to do with one another. We warn the reader now that we will have occasion to do this again.

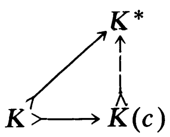
An *ordered differential field* is a differential field which is also an ordered field. In fact, any field of real meromorphic functions of one variable can be ordered. If it is also closed under differentiation, it can be considered as an ordered differential field. Other examples can be found in [5a]. It is easy to see that there is a universal theory in a language, whose nonlogical symbols are

$(0, 1, ^{-1}, D, +, -, \cdot, <)$, whose models are precisely the ordered differential fields. Call this theory ODF. We define the *theory of closed ordered differential fields* (CODF) to be the theory whose axioms include the axioms for ODF, the axioms for real closed fields, and for each n, the following axiom:

Let $f, g_1, \ldots, g_m$ be differential polynomials in one variable y, with $n =$ the order of $f \geq$ the order of any of the g_i. If there exist elements $c_0, \ldots, c_n$ such that $f(c_0, \ldots, c_n) = 0$, $(\partial f / \partial y^{(n)})(c_0, \ldots, c_n) \neq 0$, and $g_i(c_0, \ldots, c_n) > 0$ (here we are considering these polynomials as algebraic polynomials), then there is a z such that $f(z) = 0$, $g_i(z) > 0$ (here the polynomials are considered differential polynomials).

I will show that CODF is the model completion of ODF. The proof of this fact hinges on the following criterion (due to Blum). For a proof (and definitions of the notions involved) see [6, p. 89].

PROPOSITION 4. *Let T and T^* be theories in the same language such that $T \subset T^*$, T is universal, and every model of T can be extended to some model of T^*. Then T^* is the model completion of T if and only if every diagram of the following sort can be completed as shown*:



where K and $K(c)$ are models of T and K^* is a $(\operatorname{card} K)^+$*-saturated model of* T^*.

THEOREM. *The theory of closed ordered differential fields is the model completion of the theory of ordered differential fields.*

PROOF. Blum's criterion requires us to perform two tasks. First we must show that every ordered differential field can be embedded in a closed ordered differential field, and second, that we can complete diagrams of the above type.

To tackle our first task, we will show that if F is an ordered differential field, $f, g_1, \ldots, g_m \in F\{y\}$ with the order of $f \geq$ the orders of the g_i, and if there are elements $c_0, \ldots, c_n$ in F such that $f(c_0, \ldots, c_n) = 0$, $(\partial f / \partial y^{(n)})(c_0, \ldots, c_n) \neq 0$, $g_i(c_0, \ldots, c_n) > 0$, then we can find an ordered differential extension E of F and an element z in E such that $f(z) = 0$, and $g_i(z) > 0$. We can assume that f is an irreducible polynomial in $y, \ldots, y^{(n)}$. Let $F(t_0, \ldots, t_{n-1})$ be a purely transcendental extension of F ordered in such a way that the t_i's are infinitesimals with respect to F. Let K be the real closure of $F(t_0, \ldots, t_{n-1})$. Let $u_i = c_i + t_i$, $i = 0, \ldots, n-1$. Now consider f as an (algebraic) polynomial in $y, \ldots, y^{(n)}$ and replace $y^{(i)}$ by u_i for $i = 0, \ldots, n-1$ and $y^{(n)}$ by $c_n + h$, where h is a new variable. Assume $(\partial f / \partial y^{(n)})(c_0, \ldots, c_n) > 0$, (the argument is the same, with appropriate changes if this expression is < 0). Since $u_0, \ldots, u_{n-1}$ are infinitely close to $c_0, \ldots, c_{n-1}$, we have $(\partial f / \partial y^{(n)})(u_0, \ldots, u_{n-1}, c_n) > 0$. Writing down the Taylor expansion of $f(u_0, \ldots, u_{n-1}, c_n + h)$ with respect to h, we have $f(u_0, \ldots, u_{n-1}, c_n + h) = f(u_0, \ldots, u_{n-1}, c_n) + (\partial f / \partial y^{(n)})(u_0, \ldots, u_{n-1}, c_n) h +$ higher order terms. Since $u_0, \ldots, u_{n-1}$ are infinitely close to $c_0, \ldots, c_{n-1}$, $f(u_0, \ldots, u_{n-1}, c_n)$ is infinitely close to 0. Therefore, for small positive

h *in* F, $f(u_0, \ldots, u_{n-1}, c_n + h)$ is positive, and for small negative h *in* F, $f(u_0, \ldots, u_{n-1}, c_n + h)$ is negative. By the intermediate value property, and the fact that $F(X) = f(u_0, \ldots, u_{n-1}, c_n + X)$ has only finitely many roots, there is some t_n in K, t_n infinitessimal with respect to F, such that $f(u_0, \ldots, u_n) = 0$, where $u_n = c_n + t$. Each u_i is infinitely close to each c_i, so $g_i(u_0, \ldots, u_n) > 0$. We now extend the derivation D on F to a derivation on $F(u_0, \ldots, u_n)$ by setting $Du_i = u_{i+1}$, $i = 1, \ldots, n-1$. Let $E = F(u_0, \ldots, u_n) = F\langle u_0 \rangle$ and $z = u_0$, and we get our desired element. By transfinite induction and taking real closures, we can find a closed ordered differential field containing F.

Now, to our second task. Let K be an ordered differential field and $K\langle c \rangle$ a simple ordered differential extension. Assume first that c is differentially algebraic over K. The isomorphism class of c over K is determined by a minimal polynomial of c over K, and by a collection of inequalities of the form $g_i(c, \ldots, D^n c) > 0$, $g_i \in K\{y\}$ where n is the order of a minimal polynomial for c over K. Therefore the element c is determined by the facts that $f(c) = 0$, for some irreducible $f \in K\{y\}$ of order n; c satisfies no differential polynomial of lower order; and $g_i(c) > 0$ for some collection of differential polynomials $g_i \in K\{y\}$ of order $\leq n$.

Since f is a minimal polynomial for c, we have $(\partial f / \partial y^{(n)})(c) \neq 0$. Let g_i be some finite collection of differential polynomials of order $\leq n$ such that $g_i(c) > 0$, and consider the system $\{f = 0, \partial f / \partial y^{(n)} \neq 0, g_i > 0\}$. This system has a solution in an ordered extension of K, namely $y^{(i)} = D^i c$. Therefore, this system has a solution in K^*, since K^* is a real closed field. Since K^* is a closed ordered differential field, there is a u in K^* such that $f(u) = 0$ and $g_i(u) > 0$. Since K^* is $(\text{card } K)^+$-saturated, there is a z in K^* such that $K\langle c \rangle$ is isomorphic to $K\langle z \rangle \subset K^*$.

Now let $K\langle c \rangle$ be a simple extension of K and assume that c satisfies no differential polynomial over K. This extension of K is determined by the fact that for any differential polynomial $g \in K\{y\}$, we have either $g(c) > 0$ or $g(c) < 0$. Let $\{g_i\}$ be any finite collection of differential polynomials in y over K, and assume $g_i(c) > 0$ for all i. Let n be the largest order of the g_i and consider the system $\{y^{(n+1)} = 0, g_i > 0\}$. This system, viewed as a polynomial system in $y, \ldots, y^{(n+1)}$, has a solution in $K\langle c \rangle$, namely $y^{(i)} = D^i c$, for $i = 0, \ldots, n$, and $y^{(n+1)} = 0$. Now continue as above.

The fact that the theory of closed ordered differential fields is the model completion of the theory of ordered differential fields lets us deduce (by general model theoretic considerations) several facts. In particular, one can conclude, [6, p. 67], that CODF admits elimination of quantifiers. We shall use this fact in the following form.

A differential system in $y_1, \ldots, y_n$ over k is a finite collection of equalities and inequalities $\{p_i(y_1, \ldots, y_n) = 0, q_i(y_1, \ldots, y_n) > 0\}$, where each p_i and q_i is in $k\{y_1, \ldots, y_n\}$.

COROLLARY. *Let* $S = \{p_i = 0, q_i > 0, h \neq 0\}$ *be a finite collection of equalities and inequalities,* $p_i, q_i, h \in Q\{x_1, \ldots, x_r, y_1, \ldots, y_s\}$.

We can effectively find a collection $S_1, \ldots, S_t$ *of differential systems in* $x_1, \ldots, x_r$ *over Q which has the following property: Let k be an ordered differential field and*

$z_1, \ldots, z_r$ elements of k. The system S with the x_i replaced by the z_i, has a solution in some ordered differential extension of k if and only if $z_1, \ldots, z_r$ satisfy one of the S_i's.

Besides giving simple axioms for the theory of differentially closed fields, Blum showed that this latter theory is totally transcendental. From this, she used results of Shelah and Morley to conclude that every differential field has a differential closure, unique up to isomorphism. By the closure of a differential field k, we mean a differentially closed field K such that if T is any isomorphism of k into a differentially closed field E, we can extend T to an isomorphism of K into E. Since we are working with *ordered* differential fields, we could not possibly hope to show that the theory of closed ordered differential fields is totally transcendental. In fact, we will show that the principal points are not dense in the Stone space, S_1 (CODF), of CODF. It is known that this implies that CODF does not have a prime model, [6, p. 125] and therefore that Q cannot have an "ordered differential closure". Let $A(y)$ be the formula $(y' = 1)$. We claim that $A(y)$ defines an open set in S_1 (CODF) which contains no isolated points. To do this it is enough to show that if $B(y)$ is any consistent formula with one free variable such that CODF $\vdash (B(y) \rightarrow A(y))$, then there exist ordered differential extensions of Q, say $Q\langle u \rangle$ and $Q\langle v \rangle$ such that u satisfies B in any model of CODF containing $Q\langle u \rangle$, and similarly for v, but such that $Q\langle u \rangle$ is not isomorphic to $Q\langle v \rangle$. We know that CODF admits elimination of quantifiers, so we can assume that B is quantifier free. This implies that B says "y satisfies one of the differential systems $\{S_1, \ldots, S_n\}$" for some set of differential systems. Since B is consistent with CODF, there exists a z in some closed ordered differential extension of Q which satisfies B and therefore one of the S_i's, say S_1. Let $S_1 = \{f_1(y) = 0, \ldots, f_s(y) = 0, g_1(y) > 0, \ldots, g_t(y) > 0\}$. Since z satisfies B, we have $z' = 1$, so we can assume that the differential polynomials in S_1 do not involve y' or higher derivatives of y. Also, z must be (algebraically) transcendental over Q. Therefore, S_1 is equivalent to a system of *polynomial* inequalities $g_1(y) > 0, \ldots, g_t(y) > 0$. This has a solution in some ordered extension of Q, so it must have a solution in the real numbers. Therefore, this system defines a nonempty open set in R. Let u and v be two algebraically independent real numbers in this open set. Let $Q(u)$ and $Q(v)$ have their orderings induced by R and turn them into differential fields by setting $Du = Dv = 1$. Since there is a rational number between u and v, $Q(u)$ is not isomorphic to $Q(v)$ as ordered fields, and so they are not isomorphic as ordered differential fields.

§3. An embedding theorem. In this section, we will show that if K is a real differential field, finitely generated (in the differential sense) over the rationals, then K is isomorphic to a field of real meromorphic functions in some neighborhood of the origin. The proof is essentially the same as that given by Seidenberg in [8], [9], where he showed that if K is any differential field, finitely generated over the rationals, then K is isomorphic to a field of (possibly complex) meromorphic functions in some neighborhood of the origin. We will follow Seidenberg's proof very closely but include complete proofs for the convenience of the reader.

LEMMA. *Let Q be the rational numbers and $K = Q\langle u_1, \ldots, u_n \rangle$ be a differential extension of Q. Let σ be an abstract-field isomorphism of K into a field K_1, and let $\sigma(D^j u_i) = c_{ij} \in K_1$. Let the field of formal power series $K_1((x))$ be converted into a differential field by placing $D_1(\Sigma a_j x^j/j!) = \Sigma a_j x^{j-i}/(j-1)$. Then the assignment $\varphi : u_i \to \bar{u}_i = \Sigma c_{ij} x^j/j!$ defines a differential isomorphism of K onto $Q\langle \bar{u}_1, \ldots, \bar{u}_n \rangle$.*

PROOF. Let $A = Q\{u_1, \ldots, u_n\}$ be the smallest differential ring containing Q and $u_1, \ldots, u_n$. We will show that φ is a differential isomorphism on A and so therefore must be a differential isomorphism on K. Let $P(U_1, \ldots, U_n)$ be a differential polynomial in $U_1, \ldots, U_n$ over Q. Letting $P^{(k)}$ denote the kth derivative of P in $Q\{U_1, \ldots, U_n\}$, we see that $P^{(k)}(u_1, \ldots, u_n) = D^k(P(u_1, \ldots, u_n))$. Since σ is a field theoretic isomorphism, we get $P(u_1, \ldots, u_n) = 0$ if and only if $P^{(k)}(c_{ij}) = 0$ for all k (where $P^{(k)}$ is treated here as an algebraic polynomial in $U_j^{(i)}$ and the $U_j^{(i)}$ are replaced by c_{ij}). Since $P(\bar{u}_1, \ldots, \bar{u}_n) = \Sigma P^{(k)}(c_{ij}) x^k/k!$, we get $P(u_1, \ldots, u_n) = 0$ if and only if $P(\bar{u}_1, \ldots, \bar{u}_n) = 0$. Therefore φ is well defined and bijective. We will now check that $\varphi(u + v) = \varphi(u) + \varphi(v)$ for u, v in A. Let $u = P(u_1, \ldots, u_n)$ and $v = Q(u_1, \ldots, u_n)$ with P, Q in $Q\{U_1, \ldots, U_n\}$. Then

$$\begin{aligned}\varphi(u + v) &= \Sigma (P + Q)^k (c_{ij}) x^k/k! = \Sigma P^{(k)}(c_{ij}) x^k/k! + \Sigma Q^{(k)}(c_{ij}) x^{(k)}/k! \\ &= \varphi(u) + \varphi(v).\end{aligned}$$

The other properties making φ a differential isomorphism check as easily.

EMBEDDING THEOREM. *Let $K = Q\langle u_1, \ldots, u_n \rangle$ be a real differential extension of Q. Then K is isomorphic to $Q\langle \bar{u}_1, \ldots, \bar{u}_n \rangle$, where each u_i is a real function of x, analytic in some neighborhood of 0, and where the derivation on $Q\langle \bar{u}_1, \ldots, \bar{u}_n \rangle$ is d/dx.*

PROOF. We can assume that K contains an element u such that $Du \neq 0$, otherwise we would be done by Proposition 3. The primitive element theorem [7] says that we can then assume that $K = Q\langle u_1, \ldots, u_n \rangle$ where $u_1, \ldots, u_{n-1}$ are differentially algebraically independent over Q and u_n satisfies a differential polynomial over $Q\langle u_1, \ldots, u_{n-1} \rangle$ of order r and no less. Let $G(U_n)$ be such a polynomial, irreducible over $Q\langle u_1, \ldots, u_{n-1} \rangle$. Replace each $D^i u_j$, $i = 1, \ldots, n-1$, by an indeterminate $U_i^{(j)}$ and consider the resulting polynomial $\tilde{G}(U_i^{(j)})$. The idea of the proof is to let $\tilde{G}$ determine real numbers c_{ij} such that the map sending $D^j u_i$ to c_{ij} induces an isomorphism of $Q(D^j u_i)$ onto $Q(c_{ij})$ and $\bar{u}_i = \Sigma c_{ij} x^j/j!$ converges in some neighborhood of 0.

Let $S = \{D^j u_i\}$ be the finite set of the u_i's $(1 \leq i \leq n-1)$ and their derivatives which appear in G, and let $E = Q(S, u_n, \ldots, D^r u_n)$. E is a finitely generated real extension of Q, so by Proposition 3, there is an isomorphism ψ mapping E into the real numbers. Let $\psi(D^j u_i) = c_{ij}$. Since $\partial G/\partial U_n^{(r)} \neq 0$, we have $(\partial \tilde{G}/\partial U_n^{(r)})(c_{ij}) \neq 0$. Now consider G as a real analytic function of the $U_i^{(j)}$ in some neighborhood of the c_{ij}. Since $(\partial G/\partial U_n^{(r)})(c_{ij}) \neq 0$, we can apply the implicit function theorem [4, p. 13], and write

$$U_n^{(r)} - c_{nr} = T(U_i^{(j)} - c_{ij}, U_n - c_{n0}, \ldots, U_n^{(r-1)} - c_{nr-1}),$$

where T is a real analytic function in some neighborhood of the c_{ij}'s. Now we construct $\bar{u}_1, \ldots, \bar{u}_{n-1}$ by finding c_{ij}'s (including those already chosen) such that: (1) all the c_{ij}'s, $i = 1, \ldots, n-1$ and $c_{n0}, \ldots, c_{nr-1}$ are algebraically independent, and (2) $\bar{u}_i = \Sigma c_{ij} x^j / j!$ converge in some neighborhood of 0. To determine $\bar{u}_n$: substitute $\bar{u}_1, \ldots, \bar{u}_{n-1}$ in T and get

$$U_n^{(r)} - c_{nr} = T(x, U_n - c_{n0}, \ldots, U_n^{(r)} - c_{nr-1}).$$

This is an analytic function in some neighborhood of the origin. By the existence theorem for ordinary differential equations [4, p. 43], there is a unique real analytic solution $\bar{u}_n = \Sigma c_{nj} x^j / j!$ with initial conditions $\bar{u}_n(0) = c_{n0}, \ldots, (d^r u / dx^r) n(0) = c_{nr-1}$. Clearly the map sending $D^j u_i$ to c_{ij} defines a field isomorphism, so by the lemma we are done.

Seidenberg goes on to prove a stronger theorem than the one above. He shows that if $K \subset K_1$ are finitely generated differential extensions of the rational numbers and $\tau: K \to K^*$ is an isomorphism of K onto a field of meromorphic functions on some domain $\mathscr{G}$, then τ can be extended to an isomorphism τ_1 of K_1 onto K_1^*, a field of meromorphic functions on a domain $\mathscr{G}_1 \subset \mathscr{G}$, such that for u in K, $\tau(u) = \tau(u)$ restricted to $\mathscr{G}_1$. The obvious counterpart for real differential fields is false, as can be seen by letting $K = K^* = Q(x)$, $\mathscr{G} = (-1, 1)$, $\tau =$ identity, and $K_1 = Q(x, \sqrt{x-2})$. Since we cannot find a function f defined anywhere in $(-1, 1)$ such that $f^2 = x - 2$, we cannot extend τ.

§4. **Some decidability questions.** In this section we will show that there are algorithms to decide the following questions:

(1) Given $p_1(y_1, \ldots, y_n), \ldots, p_m(y_1, \ldots, y_n), h(y_1, \ldots, y_n)$ differential polynomials in $Q\{y_1, \ldots, y_n\}$, do there exist real analytic functions $u_1, \ldots, u_n$, defined in some neighborhood of 0 such that $p_1(u_1, \ldots, u_n) = \cdots = p_m(u_1, \ldots, u_n) = 0$, $h(u_1, \ldots, u_n)$?

(2) Let $Q(x)$ be the differential field of rational functions in x, where $Dx = 1$. Given $p_1, \ldots, p_m$, h, in $Q(x)\{y_1, \ldots, y_n\}$, do there exist real analytic functions $u_1, \ldots, u_n$ defined in *some* open subset of the real line such that $p_1(u_1, \ldots, u_n) = \cdots = p_n(u_1, \ldots, u_n) = 0$, $h(u_1, \ldots, u_n) \neq 0$?

We will also show that there is *no* algorithm to decide the following question:

(3) Given $p_1, \ldots, p_m, h$, in $Q(x)\{y_1, \ldots, y_n\}$, do there exist real analytic functions $u_1, \ldots, u_n$, defined in a neighborhood of 0, such that $p_1(u_1, \ldots, u_n) = \cdots = p_m(u_1, \ldots, u_n) = 0$, $h(u_1, \ldots, u_n) \neq 0$?

(1) First note that any field of real meromorphic functions is a real field, and therefore can be ordered. Combining this fact with the embedding theorem, we see that $p_1 = \cdots = p_m = 0$, $h \neq 0$, has a real analytic solution $u_1, \ldots, u_n$ in some neighborhood of 0, if and only if this system has a solution in some ordered differential extension of Q. Using Corollary 1 in §2, we see that this happens if and only if the coefficients of the p_i's satisfy one of a collection of differential systems. Since $Dc = 0$ for all c in Q, we can find equivalent polynomial systems. So, to decide if $p_1 = \cdots = p_m = 0$, $h \neq 0$, has an analytic

solution in some neighborhood of 0, we must check if the coefficients of the p_i's and h satisfy one of a given collection of polynomial equalities and inequalities. This can certainly be done.

(2) Note that $Q(x)$ is a real differential field, which does not have a prescribed ordering. Therefore we must be careful when using the above technique. By Corollary 1 of §2, we know that we can find differential systems $S_1, \ldots, S_r$ such that $p_1 = \cdots = p_m = 0$, $h \neq 0$, has a solution in some ordered differential extension of $Q(x)$ if and only if we can order $Q(x)$ in such a way that the coefficients of the p_i satisfy one of the S_j. I claim that there is an algorithm to decide if $Q(x)$ can be ordered so that the coefficients of the p_i and h satisfy one of the S_j, say S_1. Plug the coefficients of the p_i's and h into the differential polynomials appearing in S_1, perform the indicated differentiation and rearrange terms until we get a system $\{A_i = 0, B_i > 0\}$ of *polynomial* equalities and inequalities in x. One can decide if the A_i's are equal to 0. Note that there is an ordering of $Q(x)$ which makes the $B_i > 0$ if and only if there is a real number which, when substituted for x makes the $B_i > 0$. The implication from left to right follows from Proposition 1. Conversely, if there is a real number r such that $B_i(r) > 0$ for all i, then the inequalities define a nonempty open set in the real numbers. Let a be a real transcendental number in this set and order $Q(x)$ by giving x the order type of a. $Q(x)$ then is ordered in such a way that the $B_i > 0$. Using Proposition 2, one can decide if there is a real number r such that $B_i(r) > 0$ for all i, so we can decide if there is an ordering on $Q(x)$ such that the coefficients of the p_i's and h satisfy one of the S_j's. Thus we can decide if the equations have a solution in some real differential extension of $Q(x)$.

We now show that a system of differential equations with coefficients in $Q(x)$ has a solution in some real differential extension of $Q(x)$ if and only if this system has solutions analytic in some region in the real line. The implication from right to left is clear. Now assume that $u_1, \ldots, u_n$ are solutions of the given equations in some real differential extension of $Q(x)$. The Embedding Theorem tells us that $Q\langle x, u_1, \ldots, u_n\rangle$ is isomorphic to $Q\langle \bar{x}, \bar{u}_1, \ldots, \bar{u}_n\rangle$, where $\bar{x}, \bar{u}_1, \ldots, \bar{u}_n$ are analytic in some neighborhood of 0. Since the isomorphism is a differential isomorphism we must have $\bar{x} = x + c$ for some real number c. Let $v_i = \bar{u}_i(x - c)$, then each v_i is analytic in some neighborhood of c and the v_i satisfy $p_i = 0$, $h \neq 0$.

(3) To see that there is no algorithm to decide this question we will use the fact that Hilbert's Tenth Problem is undecidable [3]. Let $P(x_1, \ldots, x_n)$ be a polynomial with integer coefficients and consider the following system of differential equations: $\{y_1' = 0, \ldots, y_n' = 0, xy_{n+1}' = y_1 y_{n+1}, \ldots, xy_{2n}' = y_n y_{2n},\ P(y_1, \ldots, y_n) = 0, y_{n+1} y_{n+2} \cdots y_{2n} \neq 0\}$. We claim that this system has power series solutions if and only if $P = 0$ has nonnegative integer solutions. First assume $P(m_1, \ldots, m_n) = 0$ for some integers $m_1, \ldots, m_n$. Letting $y_i = m_i$, $y_{n+i} = x^{m_i}$, $i = 1, \ldots, n$, we get a solution to our system of differential equations. Conversely, let $u_1, \ldots, u_{2n}$ be power series solutions of our system. The first n equations tell us that $u_1, \ldots, u_n$ must be constants and the last inequality tells us that $u_{n+1}, \ldots, u_{2n}$ must be nonzero. Let $u_{n+1} = \Sigma a_i x^i$ and plug this expression into

$xy'_{n+1} = y_1 y_{n+1}$. Comparing leading coefficients, we see that y_1 must be a nonnegative integer. Similarly we can see that $y_2, \ldots, y_n$ are all nonnegative integers. Since there is no algorithm to decide if a polynomial with integer coefficients has nonnegative integer solutions, we see that there is no algorithm to decide question (3). A similar argument also shows that one cannot decide if a system of equations $p_1(y_1, \ldots, y_n) = 0, \ldots, p_m(y_1, \ldots, y_n) = 0$, $h(y_1, \ldots, y_n) = 0$, $p_i, h \in Q(x)\{y_1, \ldots, y_n\}$, has complex analytic or even infinitely differentiable solutions in a neighborhood of the origin.

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