WALKS IN THE QUARTER PLANE, GENUS ZERO CASE

THOMAS DREYFUS, CHARLOTTE HARDOUIN, JULIEN ROQUES, AND MICHAEL F. SINGER

Abstract. In the present paper, we use Galois theory of difference equations to study the nature of the generating series of (weighted) walks in the quarter plane with genus zero kernel. Using this approach, we are able to prove that the generating series do not satisfy any nontrivial nonlinear algebraic differential equation with rational coefficients.

Contents

Introduction

The nature of the generating series of lattice walks in the quarter plane has garnered much interest in recent years. In [DHRS17a] we introduced a new method that allowed us to determine, in a large number of cases, which of these are differentially algebraic, that is, satisfy a nontrivial polynomial differential equation with rational function coefficients, and which are differentially transcendental, that is not differentially algebraic. The present paper is a continuation of this research.

In [BMM10], Bousquet-Mélou and Mishna considered the 256 lattice walks in the quarter plane whose step set is a subset of $\{-1,0,1\}^2\backslash\{(0,0)\}$ (see also [Mis09]). After taking symmetries into consideration and eliminating those equivalent to walks on the half plane, they considered the remaining 79 walks. Following [FIM99], the authors associated to each walk an algebraic curve together with a group of birational automorphisms and classified the walks accordingly. They found that 23 of these walks were associated with a finite group and showed that for all but one of these, the generating series was holonomic; the remaining one was shown to have the same property by Bostan, van Hoeij and Kauers in [BvHK10]. In their paper, Bousquet-Mélou and

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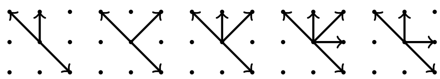
Mishna conjectured that the 56 walks whose associated group is infinite are not holonomic. These 56 walks may be divided into 5 whose associated curve has genus zero and 51 whose associated curve has genus one. In [MR09], Mishna and Reichnitzer showed that the generating series for two of the genus zero walks are not holonomic and in [MM14] Melczer and Mishna showed that this remained true for all 5 of the genus zero walks (see also [FR11]). In [KR12], Kurkova and Raschel showed that the 51 genus one walks with infinite group have nonholonomic generating series (see also [BRS14, Ras12]). Recently, Bernadi, Bousquet-Mélou and Raschel [BBMR15, BBMR17] have shown that 9 of these 51 have differentially algebraic generating series despite the fact that they are not holonomic.

In [DHRS17a], we introduced a new approach to these problems that allowed us to show that, except for the 9 exceptional walks of [BBMR15, BBMR17], the generating series of genus one walks with infinite groups are x- and y-differentially transcendental, that is, satisfy no polynomial partial differential equation involving only x-derivatives or only y-derivatives with coefficients that are rational functions of all the independent variables. This reproves and generalizes the results of [KR12]. Furthermore our results allowed us to show that the 9 exceptional series are not holonomic but are x- and y-differentially algebraic, recovering some of the results of [BBMR15, BBMR17].

In the present paper we consider the 5 remaining walks corresponding to genus zero walks with infinite group and we show that these are also x- and y-differentially transcendental, reproving and generalizing the work of Melczer and Mishna, see Theorem 4.1. In fact, we consider in full generality weighted walks for these 5 cases and show that this conclusion is true for these. These walks arise from the following 5 sets of steps:



Our strategy of proof is to associate to each of the generating series of these walks a function meromorphic on $\mathbb{C}$. These associated functions satisfy first order difference equations of the form $y(qx) - y(x) = b(x)$ for a suitable $q \in \mathbb{C}$ and $b(x)$ a *rational function* on $\mathbb{C}$. The associated functions are differentially transcendental if and only if the generating series are differentially transcendental. We then use criteria that state that if these associated functions were differentially algebraic then the $b(x)$ must themselves satisfy $b(x) = h(qx) - h(x)$ for some rational functions $h(x)$ on $\mathbb{C}$. This latter condition puts severe limitations on the poles of the $b(x)$ and, by analyzing the $b(x)$ that arise, we show that these restrictions are not met. Therefore the generating series are not differentially algebraic, see Theorem 4.1. Note that some models of unweighted walks in dimension 3, happen to be, after projection, equivalent to models of 2D weighted walks [BBMKM16, DHW16]. We apply our theorem in this setting as well. We note that finding the difference equation $y(qx) - y(x) = b(x)$ and the remaining calculations involve only algebraic computations as is true in [DHRS17a]. The general approach followed in the present work is inspired by [DHRS17a] but the details are quite different and justify an independent exposition.

The rest of the paper is organized as follows. In Section 1 we present basic notions concerning generating series for walks, their associated curves and classify those walks whose associated curves have genus zero. Curves of genus zero can be parameterized by birational maps from $\mathbb{P}^1(\mathbb{C})$. In Section 2 we present parameterizations suitable for our needs. The generating series converge for small values and for these values can be restricted to the curve of the walk. Using our parameterizations we can pull these back to analytic functions on an open set on $\mathbb{C}$. In Section 3, we show how these can be continued to meromorphic functions on $\mathbb{C}$ and show that they satisfy simple q-difference equations of the form $y(qx) - y(x) = b(x)$ in this domain. In addition we present necessary conditions on the poles of b when these equations have differentially algebraic solutions. Finally in Section 4, we show that these necessary conditions do not hold for weighted genus zero walks.

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1. Genus zero and singularities

1.1. Weighted walks and their generating series. This paper is concerned with weighted walks with small steps in the quarter plane $\mathbb{Z}^2_{\geq 0}$. More explicitly, we let $(d_{i,j})_{(i,j)\in\{0,\pm 1\}^2}$ be a family of elements of $\mathbb{Q}\cap[0,1]$ such that $\sum_{i,j} d_{i,j}=1$ and we consider the walk in the quarter plane $\mathbb{Z}^2_{\geq 0}$ satisfying the following properties :

- it starts at $(0,0)$,
- it has steps in $\{\leftarrow,\nwarrow,\uparrow,\nearrow,\rightarrow,\searrow,\downarrow,\swarrow\}$ – these steps will be identified with pairs $(i,j)\in\{0,\pm1\}^2\backslash\{(0,0)\}$,
- it goes to the direction $(i,j)\in\{0,\pm1\}^2\backslash\{(0,0)\}$ (resp. stays at the same position) with probability $d_{i,j}$ (resp. $d_{0,0}$).

The $d_{i,j}$ are called the weights of the walk. This walk is unweighted if $d_{0,0}=0$ and if the nonzero $d_{i,j}$ all have the same value.

For any $(i,j)\in\mathbb{Z}^2_{\geq 0}$ and any $k\in\mathbb{Z}_{\geq 0}$, we let $q_{i,j,k}$ be the probability for the walk to be reach the position (i,j) from the initial position $(0,0)$ after k steps. We introduce the corresponding trivariate generating series*

$$Q(x,y,t):=\sum_{i,j,k\geq 0} q_{i,j,k}x^iy^jt^k.$$

It is easily seen that, for any $k\in\mathbb{Z}_{\geq 0}$, $|q_{i,j,k}|\leq\sum_{i,j\geq 0}|q_{i,j,k}|\leq(\sum_{i,j\geq 0}|d_{i,j}|)^k=1$. From this, one sees that $Q(x,y,t)$ converges for all $(x,y,t)\in\mathbb{C}^3$ such that $|x|<1$, $|y|<1$ and $|t|\leq 1$.

1.2. Kernel and functional equation. The *Kernel* of the walk is defined by

$$K(x,y,t):=xy(1-tS(x,y))$$

where

$$\begin{array}{rcl} S(x,y) & = & \sum_{(i,j)\in\{0,\pm1\}^2} d_{i,j}x^iy^j \\ & = & A_{-1}(x)\frac{1}{y}+A_0(x)+A_1(x)y \\ & = & B_{-1}(y)\frac{1}{x}+B_0(y)+B_1(y)x, \end{array}$$

and $A_i(x)\in x^{-1}\mathbb{Q}[x]$, $B_i(y)\in y^{-1}\mathbb{Q}[y]$. Similarly to [FIM99, Section 1], we may prove that the generating series $Q(x,y,t)$ satisfies the following functional equation:

$$K(x,y,t)Q(x,y,t)=xy-F^1(x,t)-F^2(y,t)+td_{-1,-1}Q(0,0,t) \tag{1.1}$$

where

$$F^1(x,t):=K(x,0,t)Q(x,0,t),\quad F^2(y,t):=K(0,y,t)Q(0,y,t).$$

*In several papers as [BMM10], it is not assumed that $\sum_{i,j} d_{i,j}=1$. But after a rescaling of the t variable, we may always reduce to the case $\sum_{i,j} d_{i,j}=1$.

1.3. **Nondegenerate walks.** From now on, let us fix $0 < t < 1$ with $t \notin \overline{\mathbb{Q}}^{\dagger}$. The algebraic curve $K(x, y, t) = 0$ will play a crucial role in this paper, and we need to discard some degenerate cases. Following [FIM99], we have the following definition.

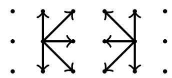
Definition 1.1. *A walk is called degenerate if one of the following holds:*

- *$K(x, y, t)$ is reducible as an element of the polynomial ring $\mathbb{C}[x, y]$,*
- *$K(x, y, t)$ the has x-degree less than or equal to 1,*
- *$K(x, y, t)$ the has y-degree less than or equal to 1.*

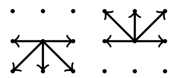
The following lemma is the analog of [FIM99, Lemma 2.3.2] in our setting.

Proposition 1.2. *A walk is degenerate if and only if at least one of the following holds:*

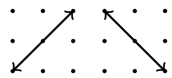
(1) *There exists $i \in \{-1, 1\}$ such that $d_{i,-1} = d_{i,0} = d_{i,1} = 0$. This corresponds to walks with steps supported in one of the following configurations*



(2) *There exists $j \in \{-1, 1\}$ such that $d_{-1,j} = d_{0,j} = d_{1,j} = 0$. This corresponds to walks with steps supported in one of the following configurations*



(3) *All the weights are 0 except maybe $\{d_{1,1}, d_{0,0}, d_{-1,-1}\}$ or $\{d_{-1,1}, d_{0,0}, d_{1,-1}\}$. This corresponds to walks with steps supported in one of the following configurations*



Proof. This proof is organized as follows. We begin by showing that (1) (resp. (2)) corresponds to $K(x, y, t)$ having x-degree ≤ 1 or x-valuation ≥ 1 (resp. y-degree ≤ 1 or y-valuation ≥ 1). In these cases, the walk is clearly degenerate. Assuming (1) and (2) do not hold, we then show that (3) holds if and only if $K(x, y, t)$ is reducible.

Cases (1) and (2). It is clear that $K(x, y, t)$ has x-degree ≤ 1 if and only if $d_{1,-1} = d_{1,0} = d_{1,1} = 0$. Similarly, $K(x, y, t)$ has y-degree ≤ 1 if and only if $d_{-1,1} = d_{0,1} = d_{1,1} = 0$. Furthermore, $d_{-1,-1} = d_{-1,0} = d_{-1,1} = 0$ if and only if $K(x, y, t)$ has x-valuation ≥ 1. Similarly, $d_{-1,-1} = d_{0,-1} = d_{1,-1} = 0$ if and only if $K(x, y, t)$ has y-valuation ≥ 1. In these cases, the walk is clearly degenerate.

Case (3).We now assume that cases (1) and (2) do not hold.

If the walk has steps supported in $\left\{\swarrow, \nearrow\right\}$ (note that this implies that $d_{1,1} \neq 0$), then the kernel

$$K(x, y, t) = -d_{-1,-1}t + xy - d_{0,0}txy - d_{1,1}tx^2y^2 \in \mathbb{C}[xy]$$

is a degree two polynomial in xy. Thus it may be factorized in the following form $K(x, y, t) = -d_{1,1}t(xy - \alpha)(xy - \beta)$ for some $\alpha, \beta \in \mathbb{C}$. If the walk has steps supported in $\left\{\nwarrow, \searrow\right\}$, then

$$K(x, y, t) = -d_{-1,1}ty^2 + xy - d_{0,0}txy - d_{1,-1}tx^2.$$

In this situation, $K(x, y, t)y^{-2} \in \mathbb{C}[x/y]$ may be factorized in the ring $\mathbb{C}[x/y]$, proving that $K(x, y, t)$ may be factorized in $\mathbb{C}[x, y]$ as well.

[†]In this paper, we have assumed that the $d_{i,j}$ belong to $\mathbb{Q}$, but everything stay true, if we assume that $d_{i,j}$ are positive real numbers and that t is transcendental over the field $\mathbb{Q}(d_{i,j})$.

Conversely, let us assume that the walk is degenerate. Remind that we have assumed that cases (1) and (2) do not hold, so $K(x,y,t)$ has x- and y-degree 2, x- and y-valuation 0, and is reducible. We have to prove that the walk has step supported by $\left\{\searrow, \nwarrow\right\}$ or $\left\{\swarrow, \nearrow\right\}$. Let us write a factorization

$$K(x,y,t) = -f_1(x,y)f_2(x,y),$$

with $f_1(x,y), f_2(x,y) \in \mathbb{C}[x,y]$ not constant.

We claim that both $f_1(x,y)$ and $f_2(x,y)$ have bidegree $(1,1)$. Suppose to the contrary that $f_1(x,y)$ or $f_2(x,y)$ does not have bidegree $(1,1)$. Since K is of bidegree at most $(2,2)$ then at least one of the f_i's has degree 0 in x or y. Up to interchange of x and y and f_1 and f_2, we may assume that $f_1(x,y)$ has y-degree 0 and we denote it by $f_1(x)$. Since $K(x,y,t) = -f_1(x)f_2(x,y)$, we find in particular that $f_1(x)$ is a common factor of the nonzero polynomials $d_{-1,-1}t + d_{0,-1}tx + d_{1,-1}tx^2$ and $d_{-1,0}t + (d_{0,0}t - 1)x + d_{1,0}tx^2$ (these polynomials are non-zero because we are not in Cases (1) and (2) of Proposition 1.2). Since t is transcendental and the $d_{i,j}$ are algebraic, we find that the roots of $d_{-1,-1}t + d_{0,-1}tx + d_{1,-1}tx^2 = 0$ are algebraic, while the roots of $d_{-1,0}t + (d_{0,0}t-1)x + d_{1,0}tx^2 = 0$ are transcendental. Therefore, they are polynomials with no common roots, and must be relatively prime, showing that $f_1(x)$ has degree 0, *i.e.*, $f_1(x) \in \mathbb{C}$. This contradicts $f_1(x,y)$ not constant and shows the claim.

We claim that $f_1(x,y)$ and $f_2(x,y)$ are irreducible in the ring $\mathbb{C}[x,y]$. If not, then we find $f_1(x,y) = (ax-b)(cy-d)$ for some $a,b,c,d \in \mathbb{C}$. Since $f_1(x,y)$ has bidegree $(1,1)$, we have $ac \neq 0$. We then have that

$$0 = K(b/a,y,t) = \frac{b}{a}y - t(\tilde{A}_{-1}(\frac{b}{a}) + \tilde{A}_0(\frac{b}{a})y + \tilde{A}_1(\frac{b}{a})y^2)$$

where $\tilde{A}_i = xA_i \in \mathbb{Q}[x]$. Equating the y^2-terms we find that $\tilde{A}_1(\frac{b}{a}) = 0$ so $\frac{b}{a} \in \overline{\mathbb{Q}}$ (note that $\tilde{A}_1(x)$ is nonzero because $K(x,y,t)$ has bidegree $(2,2)$). Equating the y-terms, we obtain that $\frac{b}{a} - t\tilde{A}_0(\frac{b}{a}) = 0$. Using $t \notin \overline{\mathbb{Q}}$ and $\frac{b}{a} \in \overline{\mathbb{Q}}$ we deduce $\frac{b}{a} = 0$. Therefore $b = 0$. This contradicts the fact that K has x-valuation 0. A similar argument shows that $f_2(x,y)$ is irreducible.

Let $\overline{f}_i(x,y)$ denote the polynomial whose coefficients are the complex conjugates of those of $f_i(x,y)$. Unique factorization of polynomials implies that since $f(x,y) = f_1(x,y)f_2(x,y) = \overline{f_1}(x,y)\overline{f_2}(x,y)$, there exists $\lambda \in \mathbb{C}^*$ such that

- either $\overline{f_1}(x,y) = \lambda f_2(x,y)$ and $\overline{f_2}(x,y) = \lambda^{-1}f_1(x,y)$;
- or $\overline{f_1}(x,y) = \lambda f_1(x,y)$ and $\overline{f}_2(x,y) = \lambda^{-1}f_2(x,y)$.

In the former case, we have $f_1(x,y) = \overline{\lambda}\,\overline{f_2}(x,y) = \overline{\lambda}\lambda^{-1}f_1(x,y)$ and so $\overline{\lambda}\lambda^{-1} = 1$. This implies that λ is real and replacing $f_1(x,y)$ by $|\lambda|^{-1/2}f_1(x,y)$ and $f_2(x,y)$ by $|\lambda|^{1/2}f_2(x,y)$, we can assume that either $\overline{f_1}(x,y) = f_2(x,y)$ and $\overline{f_2}(x,y) = f_1(x,y)$ or $\overline{f_1}(x,y) = -f_2(x,y)$ and $\overline{f_2}(x,y) = -f_1(x,y)$.

A similar computation in the latter case shows that $|\lambda| = 1$. Letting μ be a square root of λ we have $\mu^{-1} = \overline{\mu}$ so $\lambda = \mu/\overline{\mu}$. Replacing $f_1(x,y)$ by $\mu f_1(x,y)$ and $f_2(x,y)$ by $\overline{\mu}f_2(x,y)$, we can assume that $\overline{f_1}(x,y) = f_1(x,y)$ and $\overline{f_2}(x,y) = f_2(x,y)$.

To summarize, we have two possibilities:

- there exists $\epsilon \in \{\pm 1\}$ such that $\overline{f_1}(x,y) = \epsilon f_2(x,y)$, or
- $\overline{f_1}(x,y) = f_1(x,y) \in \mathbb{R}[x,y]$ and $\overline{f_2}(x,y) = f_2(x,y) \in \mathbb{R}[x,y]$.

For $i = 1,2$, let us write

$$f_i(x,y) = (\alpha_{i,4}x + \alpha_{i,3})y + (\alpha_{i,2}x + \alpha_{i,1}),$$

with $\alpha_{i,j} \in \mathbb{C}$. Equating the terms in $x^i y^j$ with $-1 \leq i,j \leq 1$, in $f_1(x,y)f_2(x,y) = -K(x,y,t)$, we find (recall that $d_{i,j} \in [0,1]$, $t \in]0,1[$)

term	coefficient in $f_1(x,y)f_2(x,y)$	coefficient in $-K(x,y,t)$
1	$\alpha_{1,1}\alpha_{2,1}$	$d_{-1,-1}t \geq 0$
x	$\alpha_{1,2}\alpha_{2,1} + \alpha_{1,1}\alpha_{2,2}$	$d_{0,-1}t \geq 0$
x^2	$\alpha_{1,2}\alpha_{2,2}$	$d_{1,-1}t \geq 0$
y	$\alpha_{1,3}\alpha_{2,1} + \alpha_{1,1}\alpha_{2,3}$	$d_{-1,0}t \geq 0$
xy	$\alpha_{1,4}\alpha_{2,1} + \alpha_{1,3}\alpha_{2,2} + \alpha_{1,2}\alpha_{2,3} + \alpha_{1,1}\alpha_{2,4}$	$d_{0,0}t - 1 < 0$
x^2y	$\alpha_{1,4}\alpha_{2,2} + \alpha_{1,2}\alpha_{2,4}$	$d_{1,0}t \geq 0$
y^2	$\alpha_{1,3}\alpha_{2,3}$	$d_{-1,1}t \geq 0$
xy^2	$\alpha_{1,4}\alpha_{2,3} + \alpha_{1,3}\alpha_{2,4}$	$d_{0,1}t \geq 0$
x^2y^2	$\alpha_{1,4}\alpha_{2,4}$	$d_{1,1}t \geq 0$

Let us treat separately two cases.

Case 1: $f_1(x,y), f_2(x,y) \notin \mathbb{R}[X,Y]$. So, in this case we have either $\overline{f_1}(x,y) = f_2(x,y)$ or $\overline{f_1}(x,y) = -f_2(x,y)$.

Let us first assume that $\overline{f_1}(x,y) = f_2(x,y)$. Then, evaluating the equality $K(x,y,t) = -f_1(x,y)f_2(x,y)$ at $x = y = 1$, we get the equality $K(1,1,t) = -f_1(1,1)f_2(1,1) = -|f_1(1,1)|^2$. But this is impossible because the left-hand term $K(1,1,t) = 1 - t\sum_{i,j\in\{-1,0,1\}^2} d_{i,j} = 1 - t$ is > 0 whereas the right-hand term $-|f_1(1,1)|^2$ is ≤ 0.

Let us now assume that $\overline{f_1}(x,y) = -f_2(x,y)$. Equating the constant terms in the equality $f_1(x,y)f_2(x,y) = -K(x,y,t)$, we get $-|\alpha_{1,1}|^2 = d_{-1,-1}t$, so $\alpha_{1,1} = \alpha_{2,1} = d_{-1,-1} = 0$. Equating the coefficients of x^2 in the equality $f_1(x,y)f_2(x,y) = -K(x,y,t)$, we get $-|\alpha_{1,2}|^2 = d_{1,-1}t$, so $\alpha_{1,2} = \alpha_{2,2} = d_{1,-1} = 0$. It follows that the y-valuation of $f_1(x,y)f_2(x,y) = -K(x,y,t)$ is ≥ 2, whence a contradiction.

Case 2: $f_1(x,y), f_2(x,y) \in \mathbb{R}[X,Y]$. We first claim that, after possibly replacing $f_1(x,y)$ by $-f_1(x,y)$ and $f_2(x,y)$ by $-f_2(x,y)$, we may assume that $\alpha_{1,4}, \alpha_{2,4}, \alpha_{1,3}, \alpha_{2,3} \geq 0$.

Let us first assume that $\alpha_{1,4}\alpha_{2,4} \neq 0$. Since $\alpha_{1,4}\alpha_{2,4} = d_{1,1}t \geq 0$, we find that $\alpha_{1,4}, \alpha_{2,4}$ belong simultaneously to $\mathbb{R}_{>0}$ or $\mathbb{R}_{<0}$. After possibly replacing $f_1(x,y)$ by $-f_1(x,y)$ and $f_2(x,y)$ by $-f_2(x,y)$, we may assume that $\alpha_{1,4}, \alpha_{2,4} > 0$. Since $\alpha_{1,3}\alpha_{2,3} = d_{-1,1}t \geq 0$, we have that $\alpha_{1,3}, \alpha_{2,3}$ belong simultaneously to $\mathbb{R}_{\geq 0}$ or $\mathbb{R}_{\leq 0}$. Then, the equality $\alpha_{1,4}\alpha_{2,3} + \alpha_{1,3}\alpha_{2,4} = d_{0,1}t \geq 0$ implies that $\alpha_{1,3}, \alpha_{2,3} \geq 0$.

We can argue similarly in the case $\alpha_{1,3}\alpha_{2,3} \neq 0$.

It remains to consider the case $\alpha_{1,4}\alpha_{2,4} = \alpha_{1,3}\alpha_{2,3} = 0$. After possibly replacing $f_1(x,y)$ by $-f_1(x,y)$ and $f_2(x,y)$ by $-f_2(x,y)$, we may assume that $\alpha_{1,4}, \alpha_{2,4} \geq 0$. The case $\alpha_{1,4} = \alpha_{1,3} = 0$ is impossible because, otherwise, we would have $d_{1,1} = d_{-1,1} = d_{0,1} = 0$, which is excluded. Similarly, the case $\alpha_{2,4} = \alpha_{2,3} = 0$ is impossible. So, we are left with the cases $\alpha_{1,4} = \alpha_{2,3} = 0$ or $\alpha_{2,4} = \alpha_{1,3} = 0$. In both cases, the equality $\alpha_{1,4}\alpha_{2,3} + \alpha_{1,3}\alpha_{2,4} = d_{0,1}t \geq 0$ implies that $\alpha_{1,4}, \alpha_{2,4}, \alpha_{1,3}, \alpha_{2,3} \geq 0$.

Arguing as above, we see that $\alpha_{1,2}, \alpha_{2,2}, \alpha_{1,1}, \alpha_{2,1}$ all belong to $\mathbb{R}_{\geq 0}$ or $\mathbb{R}_{\leq 0}$.
Using the equation of the xy-coefficients, we find that $\alpha_{1,2}, \alpha_{2,2}, \alpha_{1,1}, \alpha_{2,1}$ are all in $\mathbb{R}_{\leq 0}$.
Now, equating the coefficients of x^2y in the equality $f_1(x,y)f_2(x,y) = -K(x,y,t)$ we get $\alpha_{1,4}\alpha_{2,2} + \alpha_{1,2}\alpha_{2,4} = d_{1,0}t$. Using the fact that $\alpha_{1,4}\alpha_{2,2}, \alpha_{1,2}\alpha_{2,4} \leq 0$ and that $d_{1,0}t \geq 0$, we get $\alpha_{1,4}\alpha_{2,2} = \alpha_{1,2}\alpha_{2,4} = d_{1,0} = 0$. Similarly, using the coefficients of y, we get $\alpha_{1,3}\alpha_{2,1} = \alpha_{1,1}\alpha_{2,3} = d_{-1,0} = 0$.
So, we have

$$\alpha_{1,4}\alpha_{2,2} = \alpha_{1,2}\alpha_{2,4} = \alpha_{1,3}\alpha_{2,1} = \alpha_{1,1}\alpha_{2,3} = 0.$$

The fact that $K(x,y,t)$ has x- and y-degree 2 and x- and y-valuation 0 implies that, for any $i \in \{1,2\}$, none of the vectors $(\alpha_{i,4},\alpha_{i,3})$, $(\alpha_{i,2},\alpha_{i,1})$, $(\alpha_{i,4},\alpha_{i,2})$ and $(\alpha_{i,3},\alpha_{i,1})$ is $(0,0)$. Since $\alpha_{1,4}\alpha_{2,2}=0$, we have $\alpha_{1,4}=0$ or $\alpha_{2,2}=0$. If $\alpha_{1,4}=0$, from what precedes, we find

$$\alpha_{1,4}=\alpha_{2,4}=\alpha_{2,1}=\alpha_{1,1}=0.$$

If $\alpha_{2,2}=0$ we obtain

$$\alpha_{2,2}=\alpha_{1,2}=\alpha_{1,3}=\alpha_{2,3}=0.$$

In the first case, the walk has steps supported by $\left\{\searrow,\nwarrow\right\}$. In the second case, we find that the walk has steps supported by $\left\{\swarrow,\nearrow\right\}$. This completes the proof. $\square$

Remark 1.3. *The "degenerate walks" are called "singular" by certain authors, e.g., in* [FIM99]*. Note also that, in* [KR12]*, "singular walks" has a different meaning and refers to walks such that the associated Kernel defines a genus zero curve.*

In what follows, we always assume that the walk is not degenerate. This only discards one dimensional problems and walks in the half-plane restricted to the quarter plane that are more easy to study, as explained in [BMM10, Section 2.1].

1.4. **The algebraic curve defined by the Kernel.** The Kernel curve $\overline{E_t}$ is defined as the zero set in $\mathbb{P}^1(\mathbb{C})\times\mathbb{P}^1(\mathbb{C})$ of the following homogeneous polynomial

$$\overline{K}(x_0,x_1,y_0,y_1,t)=x_0x_1y_0y_1-t\sum_{i,j=0}^{2} d_{i-1,j-1}x_0^i x_1^{2-i} y_0^j y_1^{2-j}=x_1^2y_1^2K(\frac{x_0}{x_1},\frac{y_0}{y_1},t). \tag{1.2}$$

We will now study the Kernel curve.

For any $[x_0:x_1]$ and $[y_0:y_1]$ in $\mathbb{P}^1(\mathbb{C})$, we denote by $\Delta^x_{[x_0:x_1]}$ and $\Delta^y_{[y_0:y_1]}$ the discriminants of the degree 2 homogeneous polynomials given by $y\mapsto \overline{K}(x_0,x_1,y,t)$ and $x\mapsto\overline{K}(x,y_0,y_1,t)$ respectively, *i.e.*,

$$\begin{aligned}\Delta^x_{[x_0:x_1]}=t^2\Big((d_{-1,0}x_1^2-\frac{1}{t}x_0x_1+d_{0,0}x_0x_1+d_{1,0}x_0^2)^2\\ -4(d_{-1,1}x_1^2+d_{0,1}x_0x_1+d_{1,1}x_0^2)(d_{-1,-1}x_1^2+d_{0,-1}x_0x_1+d_{1,-1}x_0^2)\Big)\end{aligned}$$

and

$$\begin{aligned}\Delta^y_{[y_0:y_1]}=t^2\Big((d_{0,-1}y_1^2-\frac{1}{t}y_0y_1+d_{0,0}y_0y_1+d_{0,1}y_0^2)^2\\ -4(d_{1,-1}y_1^2+d_{1,0}y_0y_1+d_{1,1}y_0^2)(d_{-1,-1}y_1^2+d_{-1,0}y_0y_1+d_{-1,1}y_0^2)\Big).\end{aligned}$$

Lemma 1.4. *The following facts are equivalent:*

1. *the curve $\overline{E_t}$ is a genus zero curve;*
2. *the curve $\overline{E_t}$ has exactly one singularity $\Omega\in\overline{E_t}$;*
3. *there exists $([a:b],[c:d])\in\overline{E_t}$ such that the discriminants $\Delta^x_{[x_0:x_1]}$ and $\Delta^y_{[y_0:y_1]}$ have a root in $[a:b]\in\mathbb{P}^1(\mathbb{C})$ and $[c:d]\in\mathbb{P}^1(\mathbb{C})$ respectively;*
4. *there exists $([a:b],[c:d])\in\overline{E_t}$ such that the discriminants $\Delta^x_{[x_0:x_1]}$ and $\Delta^y_{[y_0:y_1]}$ have a double root in $[a:b]\in\mathbb{P}^1(\mathbb{C})$ and $[c:d]\in\mathbb{P}^1(\mathbb{C})$ respectively.*

If these properties are satisfied, then the singular point is $\Omega=([a:b],[c:d])$ where $[a:b]\in\mathbb{P}^1(\mathbb{C})$ is a double root of $\Delta^x_{[x_0:x_1]}$ and $[c:d]\in\mathbb{P}^1(\mathbb{C})$ is a double root of $\Delta^y_{[y_0:y_1]}$. If the previous properties are not satisfied, then the curve $\overline{E_t}$ is a genus one curve.

Proof. The curve $\overline{E_t}$ is of bidegree $(2,2)$ in $\mathbb{P}^1(\mathbb{C})\times\mathbb{P}^1(\mathbb{C})$, which we can identify with a nonsingular quadric surface in $\mathbb{P}^3(\mathbb{C})$. By [Har77, Exercise 5.6, Page 231-232 and Example 3.9.2, Page 393], the genus $g(C)$ of any irreducible curve $C\subset\mathbb{P}^1(\mathbb{C})\times\mathbb{P}^1(\mathbb{C})$ of bidegree (d_1,d_2) is given by the following formula

$$g(C)=1+d_1d_2-d_1-d_2-\sum_{P\in\mathrm{Sing}}\sum_i\frac{m_i(P)(m_i(P)-1)}{2}, \tag{1.3}$$

where $m_i(P)$ stands for the multiplicities of the proper preimages at all infinitely near points of a singular point P. Thus $\overline{E_t}$ is smooth if and only if $\overline{E_t}$ has genus one. Moreover (1.3) shows that if the curve is singular there is exactly one singular point that is a double point, and the curve has genus zero. This proves the equivalence between (1) and (2).

Let us fix $([a:b],[c:d])\in\overline{E_t}$ and let us prove (3) $\Rightarrow$ (2). Assume that the discriminant $\Delta^x_{[x_0:x_1]}$ (resp. $\Delta^y_{[y_0:y_1]}$) has a simple root in $[a:b]\in\mathbb{P}^1(\mathbb{C})$ (resp. $[c:d]\in\mathbb{P}^1(\mathbb{C})$). Let us write

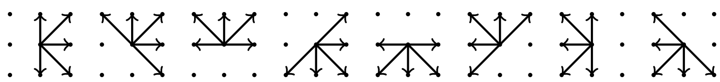
$$\begin{array}{clll}
 & \overline{K}(x_0,x_1,y_0,y_1,t) & & \\
= & e_{-1,1}(dy_0-cy_1)^2 & +e_{0,1}(bx_0-ax_1)(dy_0-cy_1)^2 & +e_{1,1}(bx_0-ax_1)^2(dy_0-cy_1)^2\\
+ & e_{-1,0}(dy_0-cy_1) & +e_{0,0}(bx_0-ax_1)(dy_0-cy_1) & +e_{1,0}(bx_0-ax_1)^2(dy_0-cy_1)\\
 & & +e_{0,-1}(bx_0-ax_1) & +e_{1,-1}(bx_0-ax_1)^2.
\end{array}$$

Since $\Delta^x_{[x_0:x_1]}$ has a simple root in $[a:b]\in\mathbb{P}^1(\mathbb{C})$ we obtain $e_{-1,0}=0$. Similarly, $\Delta^y_{[y_0:y_1]}$ has a simple root in $[c:d]\in\mathbb{P}^1(\mathbb{C})$ implies $e_{0,-1}=0$. This shows that the derivative of $\overline{K}(x_0,x_1,y_0,y_1,t)$ with respect to $[y_0:y_1]$ and $[x_0:x_1]$ at $([a:b],[c:d])$ must vanish. This proves that $([a:b],[c:d])$ is the singular point of $\overline{E_t}$.

Let us prove (2) $\Rightarrow$ (4). If $\Omega=([a:b],[c:d])$ is the singular point of $\overline{E_t}$, then $e_{-1,0}=e_{0,-1}=0$, and the discriminants $\Delta^x_{[x_0:x_1]}$ and $\Delta^y_{[y_0:y_1]}$ have a double root in $[a:b]\in\mathbb{P}^1(\mathbb{C})$ and $[c:d]\in\mathbb{P}^1(\mathbb{C})$ respectively. The proof of (4) $\Rightarrow$ (3) is clear. □

Walks associated to genus one curve have already been studied, see [BMM10, KR12], and from a Galoisian point of view, see [DHRS17a]. For this reason, we will focus in this paper on the genus zero case. Using Lemma 1.4, we see that a first step in determining nondegenerate walks of genus zero is to determine those walks whose Kernels have discriminants having double roots. Towards this end, we have the following lemma.

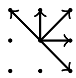
Lemma 1.5. *A walk whose discriminant $\Delta^y_{[y_0:y_1]}$ has a double zero is a walk whose steps are supported in one of the following configurations*



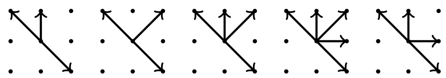
Proof. This is the result of a computation in MAPLE, see [DHRS17b]. For a Kernel with indeterminate $d_{i,j}$, one calculates the discriminant of the discriminant $\Delta^y_{[y_0:y_1]}$. This is a polynomial of degree 12 with coefficients that are polynomials in the $d_{i,j}$. Since t is transcendental, we set these polynomials equal to zero and solve. This yields 8 solutions corresponding to the above configurations. □

Using Proposition 1.2, on sees that first, third, fifth and seventh configuration in Lemma 1.5 correspond to degenerate walks. As described in [BMM10, Section 2.1], if we consider walks corresponding to the fourth and sixth configurations we are in the situation were one of the quarter plane constraints implies the other. The walks corresponding to the eighth configuration never enter the quarter-plane. Therefore the only walks that we will consider are those whose steps are supported in the second configuration of Lemma 1.5. We state this as

Assumption 1.6. *We assume that the walks under consideration have steps supported in*



and are nondegenerate. In particular, the associated curve $\overline{E_t}$ has genus zero. After eliminating duplications arising from trivial cases and the interchange of x and y, these walks arise from the following 5 sets of steps:



From now on, we assume that Assumption 1.6 is satisfied.

We need additional information about the Kernel and the zeros of $\Delta^x_{[x_0:x_1]}$ and $\Delta^y_{[y_0:y_1]}$. Note that $\Delta^x_{[x_0:x_1]}$ (resp. $\Delta^y_{[y_0:y_1]}$) is of degree 4 and so has four roots a_1, a_2, a_3, a_4 (resp. b_1, b_2, b_3, b_4) in $\mathbb{P}^1(\mathbb{C})$ (taking into consideration multiplicities). By Assumption 1.6, they both have a double root. Up to renumbering, we assume that $a_1 = a_2$ and $b_1 = b_2$. The singular point of $\overline{E_t}$ is $\Omega = (a_1, b_1)$.

Lemma 1.7. *The singular point of $\overline{E_t}$ is $\Omega = ([0:1],[0:1])$, that is, $a_1 = a_2 = [0:1]$ (resp. $b_1 = b_2 = [0:1]$) is a double root of $\Delta^x_{[x_0:x_1]}$ (resp. $\Delta^y_{[y_0:y_1]}$). The other roots are distinct and are given by*

	a_3	a_4
$\alpha_4(t) \neq 0$	$\left[\frac{-\alpha_3(t)-\sqrt{\alpha_3(t)^2-4\alpha_2(t)\alpha_4(t)}}{2\alpha_4(t)} : 1\right]$	$\left[\frac{-\alpha_3(t)+\sqrt{\alpha_3(t)^2-4\alpha_2(t)\alpha_4(t)}}{2\alpha_4(t)} : 1\right]$
$\alpha_4(t) = 0$	$[1:0]$	$[-\alpha_2(t) : \alpha_3(t)]$
	b_3	b_4
$\beta_4(t) \neq 0$	$\left[\frac{-\beta_3(t)-\sqrt{\beta_3(t)^2-4\beta_2(t)\beta_4(t)}}{2\beta_4(t)} : 1\right]$	$\left[\frac{-\beta_3(t)+\sqrt{\beta_3(t)^2-4\beta_2(t)\beta_4(t)}}{2\beta_4(t)} : 1\right]$
$\beta_4(t) = 0$	$[1:0]$	$[-\beta_2(t) : \beta_3(t)]$

where

$$\begin{aligned}
\alpha_2(t) &= 1 - 2td_{0,0} + t^2 d_{0,0}^2 - 4t^2 d_{-1,1} d_{1,-1} & \beta_2(t) &= 1 - 2td_{0,0} + t^2 d_{0,0}^2 - 4t^2 d_{1,-1} d_{-1,1} \\
\alpha_3(t) &= 2t^2 d_{1,0} d_{0,0} - 2td_{1,0} - 4t^2 d_{0,1} d_{1,-1} & \beta_3(t) &= 2t^2 d_{0,1} d_{0,0} - 2td_{0,1} - 4t^2 d_{1,0} d_{-1,1} \\
\alpha_4(t) &= t^2(d_{1,0}^2 - 4d_{1,1} d_{1,-1}) & \beta_4(t) &= t^2(d_{0,1}^2 - 4d_{1,1} d_{-1,1}).
\end{aligned}$$

Proof. We shall prove the lemma for $\Delta^y_{[y_0:y_1]}$, the proof for $\Delta^x_{[x_0:x_1]}$ being similar. Since the walk satisfies Assumption 1.6, the discriminant $\Delta^y_{[y_0:y_1]}$ has a double root at $([0:1])$ and we can write

$$\Delta^y_{[y:1]} = \beta_4(t) y^4 + \beta_3(t) y^3 + \beta_2(t) y^2.$$

Since t is transcendental and the $d_{i,j}$ are in $\overline{\mathbb{Q}}$, we see that the coefficient of y^2 is nonzero. Therefore $[0:1]$ is precisely a double root of $\Delta^y_{[y_0:y_1]}$. To see that b_3 and b_4 are distinct, we calculate the discriminant of $\Delta^y_{[y:1]}/y^2$. This is a polynomial of degree 4 in t with the following

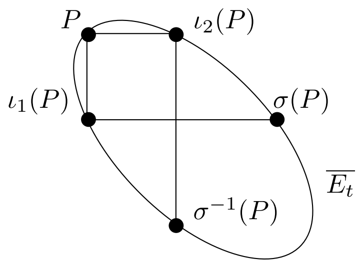


FIGURE 1. The maps ι_1, ι_2 restricted to the kernel curve $\overline{E_t}$

coefficients

term	coefficient
t^4	$-16(4d_{-1,1}d_{1,-1}d_{1,1} - d_{1,-1}d_{1,0}^2 - d_{0,0}^2 d_{1,1} + d_{0,0}d_{0,1}d_{1,0} - d_{0,1}^2 d_{1,-1})d_{-1,1}$
t^3	$-16(2d_{0,0}d_{1,1} - d_{0,1}d_{1,0})d_{-1,1}$
t^2	$16d_{-1,1}d_{1,1}$
t	0
1	0

If $\Delta^y_{[y_0:y_1]}$ has a double root different to $[0:1]$, all the above coefficients must be zero. From the coefficient of t^2 (recalling that $d_{-1,1}d_{1,-1} \neq 0$), we must have $d_{1,1} = 0$. From the coefficient of t^3, we have that $d_{0,1} = 0$ or $d_{1,0} = 0$. From the coefficient of t^4, we get in both cases $d_{0,1} = d_{1,0} = 0$. This implies that the walk would be degenerate, a contradiction. The formulas for b_3 and b_4 follow from the quadratic formula. □

Following [BMM10, Section 3] or [KY15, Section 3], we introduce the involutive birational transformations of $\mathbb{C}^2$ given by

$$i_1(x,y) = \left(x, \frac{A_{-1}(x)}{A_1(x)y}\right) \text{ and } i_2(x,y) = \left(\frac{B_{-1}(y)}{B_1(y)x}, y\right).$$

They induce birational maps $\iota_1, \iota_2 : \overline{E_t} \dashrightarrow \overline{E_t}$ given by

$$\begin{array}{rrcl} & \iota_1([x_0:x_1],[y_0:y_1]) & = & \left([x_0:x_1], \left[\dfrac{A_{-1}(\frac{x_0}{x_1})}{A_1(\frac{x_0}{x_1})\frac{y_0}{y_1}} : 1\right]\right), \\ \text{and} & \iota_2([x_0:x_1],[y_0:y_1]) & = & \left(\left[\dfrac{B_{-1}(\frac{y_0}{y_1})}{B_1(\frac{y_0}{y_1})\frac{x_0}{x_1}} : 1\right], [y_0:y_1]\right). \end{array}$$

Note that ι_1 and ι_2 are nothing but the vertical and horizontal switches of $\overline{E_t}$, see Figure 1, *i.e.*, for any $P = (x,y) \in \overline{E_t}$, we have

$$\{P, \iota_1(P)\} = \overline{E_t} \cap (\{x\} \times \mathbb{P}^1(\mathbb{C})) \text{ and } \{P, \iota_2(P)\} = \overline{E_t} \cap (\mathbb{P}^1(\mathbb{C}) \times \{y\}).$$

Proposition 1.8. *The involutive birational maps $\iota_1, \iota_2 : \overline{E_t} \dashrightarrow \overline{E_t}$ are actually involutive automorphisms of $\overline{E_t}$.*

Proof. Any point of $\overline{E_t} \setminus \{\Omega\}$ is smooth, so ι_1 and ι_2 can be uniquely extended into morphisms $\overline{E_t} \setminus \{\Omega\} \to \overline{E_t}$ still denoted by ι_1 and ι_2 ([Har77, Proposition 6.8, p. 43]). It remains to study ι_1 and ι_2 at $\Omega = ([0:1],[0:1])$. For $([x:1],[y:1]) \in \overline{E_t}$, the equation $K(x,y,t) = 0$ ensures that

$$\frac{A_{-1}(x)}{A_1(x)y} = \frac{1}{tA_1(x)} - \frac{A_0(x)}{A_1(x)} - y = \frac{x}{t\widetilde{A}_1(x)} - \frac{\widetilde{A}_0(x)}{\widetilde{A}_1(x)} - y \tag{1.4}$$

where $\widetilde{A}_0(x) = xA_0(x) = d_{-1,0} + d_{0,0}x + d_{1,0}x^2$ and $\widetilde{A}_1(x) = xA_1(x) = d_{-1,1} + d_{0,1}x + d_{1,1}x^2$. Since $d_{-1,1} \neq 0$, $\widetilde{A}_1(x)$ does not vanish at $x = 0$. So, (1.4) shows that ι_1 is regular at Ω and that $\iota_1(\Omega) = \Omega$. The argument for ι_2 is similar. □

Lemma 1.9. *A point $P = ([x_0 : x_1],[y_0 : y_1]) \in \overline{E_t}$ is fixed by ι_1 (resp. ι_2) if and only if $\Delta^x_{[x_0:x_1]} = 0$ (resp. $\Delta^y_{[y_0:y_1]} = 0$).*

Proof. This is a straightforward consequence of the fact that ι_1 and ι_2 are the vertical and horizontal switches of $\overline{E_t}$. □

We also consider the automorphism of $\overline{E_t}$ defined by

$$\sigma = \iota_2 \circ \iota_1.$$

Lemma 1.10. *Let $P \in \overline{E_t}$. The following statements are equivalent:*

(1) *P is fixed by ι_1 and ι_2;*
(2) *P is the singular point of $\overline{E_t}$;*
(3) *P is fixed by $\sigma = \iota_2 \circ \iota_1$.*

Proof. Let $P = ([a:b],[c:d]) \in \overline{E_t}$. With Lemma 1.4, P is the singular point if and only if $\Delta^x_{[x_0:x_1]}$ and $\Delta^y_{[y_0:y_1]}$ vanish at $[a:b]$ and $[c:d]$ respectively. We conclude with Lemma 1.9, that (1) is equivalent to (2).

Clearly, (1) implies (3). It remains to prove that (3) implies (1). Assume that $P = (a_1, b_1)$ is fixed by σ. Since $\iota_1(P) = (a_1, b_1')$ and $\iota_2(\iota_1(P)) = (a_1', b_1')$, it is clear that $\sigma(P) = P$ implies successively $\iota_1(P) = P$ and $\iota_2(P) = P$. □

2. Parameterization of the curve

We still assume that Assumption 1.6 holds. So, the walk is nondegenerate, and the results proved in Section 1.4 ensure that the curve $\overline{E_t}$ is irreducible, has genus zero and has a unique singular point $\Omega = (a_1, b_1) = ([0:1],[0:1])$. Moreover $\Delta^x_{[x_0:x_1]}$ has degree four with a double root at $a_1 = [0:1]$ and the remaining two roots a_3, a_4 are distinct. We let $S_3 = (a_3, *)$ and $S_4 = (a_4, *)$ be the points of $\overline{E_t}$ with first coordinates a_3 and a_4 respectively. Similarly, $\Delta^y_{[y_0:y_1]}$ has degree four with a double root at $b_1 = [0:1]$ and the remaining two roots b_3, b_4 are distinct. We let $S_3' = (*, b_3)$ and $S_4' = (*, b_4)$ be the points of $\overline{E_t}$ with second coordinates b_3 and b_4 respectively.

Since $\overline{E_t}$ has genus zero, we can find a parameterization of $\overline{E_t}$ [Har77, Example 1.3.5 page 297], *i.e.*, there exists a birational map

$$\phi = (x, y) : \mathbb{P}^1(\mathbb{C}) \dashrightarrow \overline{E_t}.$$

This ϕ is actually a surjective morphism of curves (as any non constant rational map from a smooth projective curve to a projective curve), which is injective on a Zariski-dense open subset of $\mathbb{P}^1(\mathbb{C})$. More precisely, since Ω is the unique singular point of $\overline{E_t}$, ϕ induces a bijection between $\mathbb{P}^1(\mathbb{C}) \setminus \phi^{-1}(\Omega)$ and $\overline{E_t} \setminus \{\Omega\}$. The maps $x, y : \mathbb{P}^1(\mathbb{C}) \to \mathbb{P}^1(\mathbb{C})$ are surjective morphisms of curves as well.

We let $s_3, s_4 \in \mathbb{P}^1(\mathbb{C})$ (resp. $s'_3, s'_4 \in \mathbb{P}^1(\mathbb{C})$) be such that $S_3 = \phi(s_3)$ and $S_4 = \phi(s_4)$ (resp. $S'_3 = \phi(s'_3)$ and $S'_4 = \phi(s'_4)$).

Lemma 2.1. *The morphisms $x, y : \mathbb{P}^1(\mathbb{C}) \to \mathbb{P}^1(\mathbb{C})$ have degree 2.*

Proof. This is a consequence of the fact that $\overline{E_t}$ is a biquadratic curve. Indeed, let us consider $V = \mathbb{P}^1(\mathbb{C}) \setminus \{a_1\}$. Note that the fiber of ϕ above any element of $\overline{E_t}$ of the form $(a, *)$ with $a \in V$ has one element (simply because ϕ induces a bijection between $\mathbb{P}^1(\mathbb{C}) \setminus \phi^{-1}(\Omega)$ and $\overline{E_t} \setminus \{\Omega\}$). Let U be the set of $a \in \mathbb{P}^1(\mathbb{C})$ such that the intersection of $\{a\} \times \mathbb{P}^1(\mathbb{C})$ with $\overline{E_t}$ has exactly two elements. This is also the set of $a \in \mathbb{P}^1(\mathbb{C})$ such that $\Delta_a^x \neq 0$ and, hence, U is a Zariski-dense open subset of $\mathbb{P}^1(\mathbb{C})$. Then, for any $a \in U \cap V$, $x^{-1}(a)$ has exactly two elements (indeed, we have $x^{-1}(a) = \phi^{-1}((\{a\} \times \mathbb{P}^1(\mathbb{C})) \cap \overline{E_t})$, moreover the fact that a belongs to U ensures that $(\{a\} \times \mathbb{P}^1(\mathbb{C})) \cap \overline{E_t}$ has two elements and the fact that a belongs to V ensures that $\phi^{-1}((\{a\} \times \mathbb{P}^1(\mathbb{C})) \cap \overline{E_t})$ has two elements as well). So, x has degree 2. The argument for y is similar. □

We will now follow the ideas contained in [FIM99] to produce an explicit "automorphic parameterization" of $\overline{E_t}$.

The involutive automorphisms ι_1, ι_2 of $\overline{E_t}$ induce involutive automorphisms $\tilde{\iota}_1, \tilde{\iota}_2$ of $\mathbb{P}^1(\mathbb{C})$ *via* ϕ. Similarly, σ induces an automorphism $\tilde{\sigma}$ of $\mathbb{P}^1(\mathbb{C})$. So, we have the commutative diagrams

$$\begin{tikzcd} \overline{E_t} \arrow[r, "\iota_k"] & \overline{E_t} \\ \mathbb{P}^1(\mathbb{C}) \arrow[u, "\phi"] \arrow[r, "\tilde{\iota}_k"'] & \mathbb{P}^1(\mathbb{C}) \arrow[u, "\phi"'] \end{tikzcd} \quad \text{and} \quad \begin{tikzcd} \overline{E_t} \arrow[r, "\sigma"] & \overline{E_t} \\ \mathbb{P}^1(\mathbb{C}) \arrow[u, "\phi"] \arrow[r, "\tilde{\sigma}"'] & \mathbb{P}^1(\mathbb{C}) \arrow[u, "\phi"'] \end{tikzcd}$$

We summarize some remarks in the following lemmas.

Lemma 2.2. *We have $x = x \circ \tilde{\iota}_1$ and $y = y \circ \tilde{\iota}_2$.*

Proof. We obtain $x = x \circ \tilde{\iota}_1$ by equating the first coordinates in the equality $\phi \circ \tilde{\iota}_1 = \iota_1 \circ \phi = (x, *)$ and we obtain $y = y \circ \tilde{\iota}_2$ by equating the second coordinates in the equality $\phi \circ \tilde{\iota}_2 = \iota_2 \circ \phi = (*, y)$. □

Lemma 2.3. *Let $P = \phi(s) \in \overline{E_t}$ and let $k \in \{1, 2\}$. We have :*

- *if $\tilde{\iota}_k(s) = s$ then $\iota_k(P) = P$;*
- *if $P \neq \Omega$ and $\iota_k(P) = P$ then $\tilde{\iota}_k(s) = s$.*

Proof. We have $\iota_k(P) = \iota_k(\phi(s)) = \phi(\tilde{\iota}_k(s))$. The first assertion is now clear, and the second one follows from the fact that ϕ is injective on $\overline{E_t} \setminus \phi^{-1}(\Omega)$. □

Lemma 2.4. *The fiber of ϕ above Ω has two elements: $\phi^{-1}(\Omega) = \{s_1, s_2\}$ with $s_1 \neq s_2$.*

Proof. We know that $x, y : \mathbb{P}^1(\mathbb{C}) \to \mathbb{P}^1(\mathbb{C})$ have degree 2, so $\phi^{-1}(\Omega)$ has 1 or 2 elements. Suppose to the contrary that $\phi^{-1}(\Omega) = \{s_1\}$ has 1 element. Since $\phi(\tilde{\iota}_1(s_1)) = \iota_1(\phi(s_1)) = \iota_1(\Omega) = \Omega$, we have $\tilde{\iota}_1(s_1) = s_1$. Moreover, since $S_3, S_4 \neq \Omega$ are fixed by ι_1, Lemma 2.3 ensures that s_3 and s_4 are fixed by $\tilde{\iota}_1$. Therefore, $\tilde{\iota}_1$ is an automorphism of $\mathbb{P}^1(\mathbb{C})$, *i.e.*, an homography, with at least 3 fixed points, so $\tilde{\iota}_1$ is the identity. This is a contradiction. □

Lemma 2.5. *The map $\tilde{\iota}_1$ (resp. $\tilde{\iota}_2$) has exactly two fixed points, namely s_3 and s_4 (resp. s'_3 and s'_4), and interchanges s_1 and s_2. The map $\tilde{\sigma}$ has exactly two distinct fixed points, s_1 and s_2.*

Proof. Let $s \in \mathbb{P}^1(\mathbb{C})$ be a fixed point of $\tilde{\iota}_1$. Lemma 2.3 ensures that $\phi(s)$ is fixed by ι_1. So, $\phi(s) = \Omega$, S_3 or S_4. If $\phi(s) \neq \Omega$, then $s = s_3$ or s_4 (recall that ϕ induces a bijection between $\mathbb{P}^1(\mathbb{C}) \setminus \phi^{-1}(\Omega)$ and $\overline{E_t} \setminus \{\Omega\}$) and s_3 and s_4 are indeed fixed by $\tilde{\iota}_1$. Moreover, we have $\phi(s) = \Omega$ if and only if $s = s_1$ or s_2 and the equality $\iota_1(\phi(s)) = \phi(\tilde{\iota}_1(s))$ shows that $\tilde{\iota}_1$ induces a permutation of $\phi^{-1}(\Omega) = \{s_1, s_2\}$. If s_1 and s_2 were fixed by $\tilde{\iota}_1$, then $\tilde{\iota}_1$ would be an automorphism of $\mathbb{P}^1(\mathbb{C})$, *i.e.*, an homography, with at least 4 fixed points (s_1, s_2, s_3, s_4) and, hence, would be the identity. This is a contradiction. So, $\tilde{\iota}_1$ interchanges s_1 and s_2.

The proof for $\tilde{\iota}_2$ is similar.

As any homography which is not the identity, $\tilde{\sigma}$ has at most two fixed points in $\mathbb{P}^1(\mathbb{C})$. It only remains to prove that s_1 and s_2 are fixed by $\tilde{\sigma}$, and this is indeed the case because $\tilde{\sigma} = \tilde{\iota}_2 \circ \tilde{\iota}_1$ and $\tilde{\iota}_1, \tilde{\iota}_2$ interchange s_1 and s_2. ☐

Proposition 2.6. *An explicit parameterization $\phi : \mathbb{P}^1(\mathbb{C}) \to \overline{E_t}$ such that $\tilde{\iota}_1(s) = \frac{1}{s}$ and $\tilde{\iota}_2(s) = \frac{\lambda^2}{s} = \frac{q}{s}$ for a certain $\lambda \in \mathbb{C}^*$ is given by*

$$\phi(s) = \left(\frac{4\alpha_2(t)}{\sqrt{\alpha_3(t)^2 - 4\alpha_2(t)\alpha_4(t)}(s + \frac{1}{s}) - 2\alpha_3(t)}, \frac{4\beta_2(t)}{\sqrt{\beta_3(t)^2 - 4\beta_2(t)\beta_4(t)}(\frac{s}{\lambda} + \frac{\lambda}{s}) - 2\beta_3(t)} \right).$$

Moreover, we have

$$\begin{aligned} x(0) = x(\infty) = a_1, \quad x(1) = a_3, \quad x(-1) = a_4, \\ y(0) = y(\infty) = b_1, \quad y(\lambda) = b_3, \quad y(-\lambda) = b_4. \end{aligned}$$

Proof. According to Lemma 2.5, $\tilde{\iota}_1$ is an involutive homography with fixed points s_3 and s_4, so there exists an homography h such that $h(s_3) = 1$, $h(s_4) = -1$ and $h \circ \tilde{\iota}_1 \circ h^{-1}(s) = 1/s$. Up to replacing ϕ by $\phi \circ h$, we can assume that $s_3 = 1$, $s_4 = -1$ and $\tilde{\iota}_1(s) = \frac{1}{s}$. Since $s_1 \neq s_2$, we can assume up to renumbering that $s_1 \neq \infty$. Moreover, up to replacing ϕ by $\phi \circ k$ where k is the homography given by $k(s) = \frac{s - s_1}{-s_1 s + 1}$, we can also assume that $s_1 = 0$ and $s_2 = \infty$ (note that k commutes with $\tilde{\iota}_1$, so changing ϕ by $\phi \circ k$ does not affect $\tilde{\iota}_1$). Lemma 2.1 and Lemma 2.2 ensure that the morphism $x : \mathbb{P}^1(\mathbb{C}) \to \mathbb{P}^1(\mathbb{C})$ has degree 2 and satisfies $x(s) = x(1/s)$ for all $s \in \mathbb{P}^1(\mathbb{C})$. It follows that

$$x(s) = \frac{a(s + 1/s) + b}{c(s + 1/s) + d}$$

for some $a, b, c, d \in \mathbb{C}$. We have $x(s_1) = x(0) = a_1 = 0$, $x(s_2) = x(\infty) = a_1 = 0$, $x(s_3) = x(1) = a_3$ and $x(s_4) = x(-1) = a_4$. The equality $x(\infty) = 0$ implies $a = 0$. The equalities $x(1) = a_3$ and $x(-1) = a_4$ imply

$$x(s) = \frac{4a_3a_4}{(a_4 - a_3)(s + \frac{1}{s}) + 2(a_3 + a_4)}.$$

The known expressions for a_3 and a_4 given in Lemma 1.7 lead to the expected expression for $x(s)$.

According to Lemma 2.5, $\tilde{\iota}_2$ is an homography interchanging 0 and ∞, so $\tilde{\iota}_2(s) = \frac{\lambda^2}{s}$ for some $\lambda \in \mathbb{C}^*$. Up to renumbering, we have $s'_3 = \lambda$ and $s'_4 = -\lambda$. Using the fact that the morphism $y : \mathbb{P}^1(\mathbb{C}) \to \mathbb{P}^1(\mathbb{C})$ has degree 2 and is invariant by $\tilde{\iota}_2$, and arguing as we did above for x, we see that there exist $\alpha, \beta, \gamma, \eta \in \mathbb{C}$ such that

$$y(s) = \frac{\alpha(\frac{s}{\lambda} + \frac{\lambda}{s}) + \beta}{\gamma(\frac{s}{\lambda} + \frac{\lambda}{s}) + \eta}.$$

The equality $y(\infty) = 0$ implies $\alpha = 0$. Using the equalities $y(s'_3) = y(\lambda) = b_3$ and $y(s'_4) = y(-\lambda) = b_4$, and arguing as we did above for x, we obtain the expected expression for $y(s)$. ☐

We now need to give more information of q. The following lemma determines q up to its inverse.

Proposition 2.7. *One of the two complex numbers q or q^{-1} is equal to*

$$\frac{-1+d_{0,0}t-\sqrt{(1-d_{0,0}t)^2-4d_{1,-1}d_{-1,1}t^2}}{-1+d_{0,0}t+\sqrt{(1-d_{0,0}t)^2-4d_{1,-1}d_{-1,1}t^2}}.$$

Proof. Using the explicit formulas for $x(s)$ and $y(s)$ given in Proposition 2.6, we get

$$\lim_{s\to 0}\frac{x(s)}{y(s)}=\frac{\lambda\alpha_2(t)\sqrt{\beta_3(t)^2-4\beta_2(t)\beta_4(t)}}{\beta_2(t)\sqrt{\alpha_3(t)^2-4\alpha_2(t)\alpha_4(t)}} \text{ and } \lim_{s\to 0}\frac{x(1/s)}{y(1/s)}=\frac{\alpha_2(t)\sqrt{\beta_3(t)^2-4\beta_2(t)\beta_4(t)}}{\lambda\beta_2(t)\sqrt{\alpha_3(t)^2-4\alpha_2(t)\alpha_4(t)}}.$$

But, Proposition 2.6 ensures that $\frac{x(1/s)}{y(1/s)}=\frac{x(s)}{y(\tilde{\iota}_1(s))}$. So, the above two limits imply the following :

$$\lim_{s\to 0}\frac{y(\tilde{\iota}_1(s))}{y(s)}=q.$$

Now, let us note that $y(s),y(\tilde{\iota}_1(s))$ equals to

$$\frac{-x+d_{0,0}xt+d_{1,0}x^2t\pm\sqrt{(x-d_{0,0}xt-d_{1,0}x^2t)^2-4d_{1,-1}x^2t^2(d_{-1,1}+d_{0,1}x+d_{1,1}x^2)}}{-2d_{-1,1}t-2d_{0,1}xt-2d_{1,1}x^2t},$$

with the shorthand notation $x=x(s)$. Since $x(s)$ tends to 0 when s goes to 0, we obtain the result. □

Corollary 2.8. *We have $q\in\mathbb{R}\setminus\{\pm 1\}$.*

Proof. We first claim that $(1-d_{0,0}t)^2-4d_{1,-1}d_{-1,1}t^2>0$. We know that the $d_{i,j}$ are ≥ 0, that the sum of the $d_{i,j}$ is equal to 1 and that the support of the walk is not included in $\{(0,0),(1,-1),(-1,1)\}$ (because the walk is not degenerate). Therefore, we have $1>d_{0,0}+d_{1,-1}+d_{-1,1}$, *i.e.*, $1-d_{0,0}>d_{1,-1}+d_{-1,1}$. Since $t\in]0,1[$, we have $1-d_{0,0}t>1-d_{0,0}$. Thus, $(1-d_{0,0}t)^2>(1-d_{0,0})^2>(d_{1,-1}+d_{-1,1})^2$ and, hence,

$$\begin{aligned}(1-d_{0,0}t)^2-4d_{1,-1}d_{-1,1}t^2 &> (d_{1,-1}+d_{-1,1})^2-4d_{1,-1}d_{-1,1}t^2\\ &\geq (d_{1,-1}+d_{-1,1})^2-4d_{1,-1}d_{-1,1}=(d_{1,-1}-d_{-1,1})^2\geq 0.\end{aligned}$$

This proves our claim.

Now Proposition 2.7 implies that q is a real number $\neq 1$. Moreover, it also shows that $q=-1$ if and only if $-1+d_{0,0}t=0$. But this is excluded because $1>d_{0,0}t$. □

In particular, this implies that the birational maps σ and $\tilde{\sigma}$ have infinite order (see also [FR11]) and that the group associated with these walks has infinite order.

3. Analytic continuation

3.1. **Functional equation.** We still assume that Assumption 1.6 is satisfied and $0<t<1$ is transcendental. We let $\phi=(x,y):\mathbb{P}^1(\mathbb{C})\to\overline{E_t}$ be a parameterization of $\overline{E_t}$ as in Proposition 2.6, so we have :

- $\phi(0)=\phi(\infty)=([0:1],[0:1])$;
- $\tilde{\iota}_1(s)=\frac{1}{s}$ and $\tilde{\iota}_2(s)=\frac{q}{s}$ for some $q\in\mathbb{R}\setminus\{\pm 1\}$;
- $\tilde{\sigma}(s)=qs$.

Recall the functional equation (1.1):

$$K(x,y,t)Q(x,y,t)=xy-F^1(x,t)-F^2(y,t)+td_{-1,-1}Q(0,0,t).$$

This equation is a formal identity but for $|x|<1$ and $|y|<1$, the series $Q(x,y,t)$, $F^1(x,t)$ and $F^2(y,t)$ are convergent. Using our parameterization of $\overline{E_t}$, we will show how we can pull back these convergent series and analytically continue them to meromorphic functions on $\mathbb{C}$ so that these latter functions satisfy simple q-difference equations.

The set $V = \{([x:1],[y:1]) \in \overline{E_t} \mid |x|,|y|<1\}$ is an open neighborhood of $([0:1],[0:1])$ in $\overline{E_t}$ for the analytic topology, and, for all $(x,y) \in V$, we have

$$0 = xy - F^1(x,t) - F^2(y,t) + td_{-1,-1}Q(0,0,t). \tag{3.1}$$

Since $\phi(0) = \phi(\infty) = ([0:1],[0:1])$, there exists $U \subset \mathbb{P}^1(\mathbb{C})$ which is the union of two small open discs centered at 0 and ∞ such that $\phi(U) \subset V$.

For any $s \in U$, we set $\breve{F}^1(s) = F^1(x(s),t)$ and $\breve{F}^2(s) = F^2(y(s),t)$. Then, $\breve{F}^1$ and $\breve{F}^2$ are meromorphic functions over U and (3.1) yields, for all $s \in U$,

$$0 = x(s)y(s) - \breve{F}^1(s) - \breve{F}^2(s) + td_{-1,-1}Q(0,0,t). \tag{3.2}$$

Replacing s by $\tilde{\iota}_2(s)$ in (3.2), we obtain, for all s close to 0 or ∞, (in what follows, we use $x(\tilde{\iota}_1(s)) = x(s)$, $y(\tilde{\iota}_2(s)) = y(s)$, $\breve{F}^1(\tilde{\iota}_1(s)) = \breve{F}^1(s)$ and $\breve{F}^2(\tilde{\iota}_2(s)) = \breve{F}^2(s)$)

$$\begin{aligned} 0 &= x(\tilde{\iota}_2(s))y(\tilde{\iota}_2(s)) - \breve{F}^1(\tilde{\iota}_2(s)) - \breve{F}^2(\tilde{\iota}_2(s)) + td_{-1,-1}Q(0,0,t) \\ &= x(\tilde{\iota}_1(\tilde{\iota}_2(s)))y(s) - \breve{F}^1(\tilde{\iota}_1(\tilde{\iota}_2(s))) - \breve{F}^2(s) + td_{-1,-1}Q(0,0,t) \\ &= x(q^{-1}s)y(s) - \breve{F}^1(q^{-1}s) - \breve{F}^2(s) + td_{-1,-1}Q(0,0,t). \end{aligned} \tag{3.3}$$

Subtracting (3.2) from (3.3), and then replacing s by qs, we obtain, for all s close to 0 or ∞,

$$\breve{F}^1(qs) - \breve{F}^1(s) = (x(qs) - x(s))y(qs). \tag{3.4}$$

Similarly, replacing s by $\tilde{\iota}_1(s)$ in (3.2), we obtain, for all s close to 0 or ∞,

$$\begin{aligned} 0 &= x(\tilde{\iota}_1(s))y(\tilde{\iota}_1(s)) - \breve{F}^1(\tilde{\iota}_1(s)) - \breve{F}^2(\tilde{\iota}_1(s)) + td_{-1,-1}Q(0,0,t) \\ &= x(s)y(\tilde{\iota}_2(\tilde{\iota}_1(s))) - \breve{F}^1(s) - \breve{F}^2(\tilde{\iota}_2(\tilde{\iota}_1(s))) + td_{-1,-1}Q(0,0,t) \\ &= x(s)y(qs) - \breve{F}^1(s) - \breve{F}^2(qs) + td_{-1,-1}Q(0,0,t). \end{aligned} \tag{3.5}$$

Subtracting (3.2) from (3.5), we obtain, for all s close to 0 or ∞,

$$\breve{F}^2(qs) - \breve{F}^2(s) = x(s)(y(qs) - y(s)). \tag{3.6}$$

We let $\widetilde{F}^1$ and $\widetilde{F}^2$ be the restrictions of $\breve{F}^1$ and $\breve{F}^2$ to a small disc around 0. They satisfy the functional equations (3.4) and (3.6) for s close to 0. This implies that $\widetilde{F}^1$and $\widetilde{F}^2$ can be continued to a meromorphic function on $\mathbb{C}$ with (3.4) satisfied for all $s \in \mathbb{C}$. Note that there is a priori no reason why, in the neighborhood of ∞, these functions should coincide with $\breve{F}^1$ and $\breve{F}^2$.

3.2. **Application to differential transcendence.**

In this subsection, we derive differential transcendency criteria for $x \mapsto Q(x,0,t)$ and $y \mapsto Q(0,y,t)$. It is based on the fact that the related functions $\widetilde{F}^1$ and $\widetilde{F}^2$ satisfy difference equations.

Definition 3.1. *Let $(E,\delta) \subset (F,\delta)$ be differential fields. We say that $f \in F$ is* differentially algebraic *over E if it satisfies a non trivial algebraic differential equation with coefficients in E, i.e., if for some m there exists a nonzero polynomial $P(y_0,\dots,y_m) \in E[y_0,\dots,y_m]$ such that*

$$P(f,\delta(f),\dots,\delta^m(f)) = 0.$$

We say that f is holonomic *over E if in addition, the equation is linear. We say that f is* differentially transcendental *over E if it is not differentially algebraic.*

Other terms have been used for the above concepts: hypotranscendental or hyperalgebraic or δ-algebraic for differentially algebraic and hypertranscendental or transcendentally transcendental for differentially transcendental.

Proposition 3.2. *The series $x \mapsto Q(x,0,t)$ is differentially algebraic over $\mathbb{C}(x)$ if and only if $\widetilde{F}^1$ is differentially algebraic over $\mathbb{C}(s)$. The series $y \mapsto Q(0,y,t)$ is differentially algebraic over $\mathbb{C}(y)$ if and only if $\widetilde{F}^2$ is differentially algebraic over $\mathbb{C}(s)$.*

Proof. This follows from Lemmas 6.3 and 6.4 of [DHRS17a]. □

Consequently, we only need to study $\widetilde{F}^1$ and $\widetilde{F}^2$. Recall that they belong to $\mathcal{M}er(\mathbb{C})$ the field of meromorphic functions on $\mathbb{C}$.

Using a result due to Ishizaki [Ish98, Theorem 1.2] (see also [HS08, Proposition 3.5], where a Galoisian proof of Ishizaki's result is given), we get, for any $i \in \{1,2\}$, the following dichotomy[‡] :

- either $\widetilde{F}^i \in \mathbb{C}(s)$, or
- $\widetilde{F}^i$ is differentially transcendental.

So, we need to understand when $\widetilde{F}^i \in \mathbb{C}(s)$. We set

$$\widetilde{b}_1(s) = y(qs)(x(qs) - x(s)) \text{ and } \widetilde{b}_2(s) = x(s)(y(qs) - y(s)),$$

so that the functional equations (3.4) and (3.6) can be restated as

$$\widetilde{F}^1(qs) - \widetilde{F}^1(s) = \widetilde{b}_1(s) \text{ and } \widetilde{F}^2(qs) - \widetilde{F}^2(s) = \widetilde{b}_2(s) \tag{3.7}$$

for $s \in \mathbb{C}$.

Lemma 3.3. *For any $i \in \{1,2\}$, the following facts are equivalent:*

- *$\widetilde{F}^i \in \mathbb{C}(s)$;*
- *there exists $f_i \in \mathbb{C}(s)$ such that $\widetilde{b}_i(s) = f_i(qs) - f_i(s)$.*

Proof. If $\widetilde{F}^i \in \mathbb{C}(s)$ then (3.7) shows that $\widetilde{b}_i(s) = f_i(qs) - f_i(s)$ with $f_i = \widetilde{F}^i \in \mathbb{C}(s)$. Conversely, assume that there exists $f_i \in \mathbb{C}(s)$ such that $\widetilde{b}_i(s) = f_i(qs) - f_i(s)$. Using (3.7) again, we find that $(\widetilde{F}^i - f_i)(s) = (\widetilde{F}^i - f_i)(qs)$. Therefore, since the function $\widetilde{F}^i - f_i$ is meromorphic over $\mathbb{C}$, its Taylor expansion yields that it is necessarily constant, and this ensures that $\widetilde{F}^i \in \mathbb{C}(s)$. □

Lemma 3.4. *The following properties are equivalent:*

- *$\widetilde{F}^1 \in \mathbb{C}(s)$;*
- *$\widetilde{F}^2 \in \mathbb{C}(s)$.*

Proof. Assume that $\widetilde{F}^1 \in \mathbb{C}(s)$. With Lemma 3.3, there exists $f_1 \in \mathbb{C}(s)$ such that $\widetilde{b}_1(s) = f_1(qs) - f_1(s)$. Note that $\widetilde{b}_1(s) + \widetilde{b}_2(s) = (xy)(qs) - (xy)(s)$, so that we have $\widetilde{b}_2(s) = f_2(qs) - f_2(s)$, with $xy(s) - f_1(s) = f_2(s) \in \mathbb{C}(s)$. With Lemma 3.3, we obtain $\widetilde{F}^2 \in \mathbb{C}(s)$. The converse may be proved in the same way. □

Theorem 3.5. *The following properties are equivalent:*

1. *The function $x \mapsto Q(x,0,t)$ is differentially algebraic over $\mathbb{C}(x)$.*
2. *The function $x \mapsto Q(x,0,t)$ is algebraic over $\mathbb{C}(x)$.*
3. *The function $y \mapsto Q(0,y,t)$ is differentially algebraic over $\mathbb{C}(y)$.*
4. *The function $y \mapsto Q(0,y,t)$ is algebraic over $\mathbb{C}(y)$.*
5. *There exists $f_1 \in \mathbb{C}(s)$ such that $\widetilde{b}_1(s) = f_1(qs) - f_1(s)$.*
6. *There exists $f_2 \in \mathbb{C}(s)$ such that $\widetilde{b}_2(s) = f_2(qs) - f_2(s)$.*

[‡]A slightly weaker result, in the spirit of the considerations of [DHRS17a], would suffice to establish this dichotomy, see [HS08, Corollary 3.2, Proposition 6.4] or [Har08].

Proof. Assume that (1) holds true. Proposition 3.2 implies that $\widetilde{F}^1$ is differentially algebraic over $\mathbb{C}(s)$. Ishizaki's Theorem ensures that $\widetilde{F}^1 \in \mathbb{C}(s)$. But $x : \mathbb{P}^1(\mathbb{C}) \to \mathbb{P}^1(\mathbb{C})$ is locally (for the analytic topology) invertible at all but finitely many points of $\mathbb{P}^1(\mathbb{C})$ and the corresponding local inverses are algebraic over $\mathbb{C}(x)$. It follows that $F^1(\cdot, t)$ can be expressed as a rational expression, with coefficients in $\mathbb{C}$, of an algebraic function, and, hence, is algebraic over $\mathbb{C}(x)$. Hence (2) is satisfied.

The fact that (2) implies (1) is obvious.

The fact that (3) is equivalent to (4) can be shown similarly.

The fact that (1) to (4) are equivalent now follows from Lemma 3.4 combined with [Ish98, Theorem 1.2].

The remaining equivalences follow from Lemma 3.3. □

So, to decide whether $Q(x,0,t)$, $Q(0,y,t)$ are differentially transcendental, we are led to the following problem:

Given $b \in \mathbb{C}(s)$, decide whether there exists $f \in \mathbb{C}(s)$ such that $b(s) = f(qs) - f(s)$.

Such a problem is known as a q-summation problem and has been solved by Abramov [Abr95]. This procedure was recast in [CS12] in terms of the so-called q-residues of b (see [CS12, Definition 2.7]) and depends on the criteria given in [CS12, Lemma 2.10]:

A function $b \in \mathbb{C}(s)$ is q-summable, i.e., there exists $f \in \mathbb{C}(s)$ such that $b(s) = f(qs) - f(s)$ if and only if the q-residues of b at non zero poles are zero and the partial fraction decomposition of b has no constant term.

Roughly, the q-residues are weighted sums of coefficients appearing in the principal part of the power series expansion of a rational function $b(s)$ at a pole s_0 and at poles that appear in the q-orbit $q^{\mathbb{Z}^*}s_0$ of s_0. We will not describe this concept in detail but only note that if s_0 is a pole of order $N \geq 1$ of b and if b has no other pole of order $\geq N$ in $q^{\mathbb{Z}^*}s_0$, then some q-residue is not zero. Using this fact, we may deduce the following from [CS12, Lemma 2.10]:

Lemma 3.6. *If $s_0 \in \mathbb{C}^*$ is a pole of $b \in \mathbb{C}(x)$ of order $N \geq 1$ if b has no other pole of order $\geq N$ in $q^{\mathbb{Z}^*}s_0$ then $b(s)$ is not q-summable, that is there is no $f(s) \in \mathbb{C}(s)$ such that $b(s) = f(qs) - f(s)$.*

Using the parameterization $\phi : \mathbb{P}^1(\mathbb{C}) \to \overline{E_t}$, we can translate this to give a criterion for the differential transcendence of $x \mapsto Q(x,0,t)$, $y \mapsto Q(0,y,t)$ over $\mathbb{C}(x)$ and $\mathbb{C}(y)$ respectively. We set (see Section 1 for notations)

$$b_1 = \iota_1(y)(\iota_2(x) - x) \text{ and } b_2 = x(\iota_1(y) - y),$$

so that we have

$$\widetilde{b}_1 = b_1 \circ \phi \text{ and } \widetilde{b}_2 = b_2 \circ \phi.$$

Proposition 3.7. *We suppose that Assumption 1.6 holds true and recall that $|q| \neq 1$. Let $b \in \mathbb{C}(x,y)$ represent a rational function on $\overline{E_t}$. Assume that $P \in \overline{E_t} \setminus \{\Omega\}$ is a pole of b of order $m \geq 1$ such that none of the $\sigma^i(P)$ with $i \in \mathbb{Z}\backslash\{0\}$ is a pole of b of order $\geq m$, then*

$$b = \sigma(g) - g$$

has no solution $g \in \mathbb{C}(x,y)$ which restricts to a rational function on $\overline{E_t}$.

In particular, if $b_2 = x(\iota_1(y) - y)$ satisfies this condition, then $x \mapsto Q(x,0,t)$, (resp. $y \mapsto Q(0,y,t)$) is differentially transcendental over $\mathbb{C}(x)$ (resp. differentially transcendental over $\mathbb{C}(y)$).

Proof. We know that the parameterization $\phi = (x, y) : \mathbb{P}^1(\mathbb{C}) \to \overline{E_t}$ that we have constructed induces an isomorphism between $\mathbb{P}^1(\mathbb{C}) \setminus \{0, \infty\}$ and $\overline{E_t} \setminus \{\Omega\}$. If $s_0 \in \mathbb{P}^1(\mathbb{C}) \setminus \{0, \infty\}$ is such that $\phi(s_0) = P$, then s_0 is a pole of order $m \geq 1$ of $b \circ \phi$ such that none of the $\tilde{\sigma}^i(s_0)$ with $i \in \mathbb{Z}\backslash\{0\}$ is a pole of $b \circ \phi$ of order $\geq m$. If $g \in \mathbb{C}(x, y)$ restricts to a rational function on $\overline{E_t}$ and satisfies $b = \sigma(g) - g$, then $f = g \circ \phi$ would satisfy $b(s) = f(qs) - f(s)$ contradicting Lemma 3.6.

If $b_2 = x(\iota_1(y) - y)$ satisfies the condition of the Proposition, then $b_2 = \sigma(g) - g$ has no solution g that is a rational function on $\overline{E_t}$. Pulling this back to $\mathbb{P}^1(\mathbb{C})$, we see that for $\widetilde{b}_2(s) = b_2 \circ \phi(s) = x(s)(y(1/s) - y(s))$, the equation $\widetilde{b}_2(s) = f(qs) - f(s)$ has no solution in $\mathbb{C}(s)$. Theorem 3.5 yields our conclusion. □

As we will see in Theorem 4.1, the criteria given by Proposition 3.7 is strong enough to ensure that $x \mapsto Q(x, 0, t)$ (resp. $y \mapsto Q(0, y, t)$) is differentially transcendental over $\mathbb{C}(x)$ (resp. over $\mathbb{C}(y)$).

4. Differential transcendence for genus zero walks

In this section, we will prove the main result of this paper :

Theorem 4.1. *We suppose that Assumption 1.6 is satisfied. Then, the functions $x \mapsto Q(x, 0, t)$ and $y \mapsto Q(0, y, t)$ are differentially transcendental over $\mathbb{C}(x)$ and $\mathbb{C}(y)$ respectively.*

Remark 4.2. *Walks in 3 dimension in the octant have been recently studied. In* [BBMKM16, DHW16]*, the authors study unweighted such walks having at most 6 steps. Among the non trivial 35548 models, 527 are equivalent to weighted walks in the quarter plane, and more precisely, Assumption 1.6 is satisfied for 69 such models, see* [DHW16, Section 3].

The proof of the above theorem will be given at the very end of this section. Our strategy will be to use Proposition 3.7. So, we begin by collecting information on the polar divisor of $b_2 = x(\iota_1(y) - y)$.

4.1. Preliminary results on the polar divisor of b_2.

We write

$$b_2 = x(\iota_1(y) - y)$$

in the projective coordinates with $x = \frac{x_0}{x_1}$ and $y = \frac{y_0}{y_1}$. We note that $\Omega = ([0:1], [0:1])$ is not a pole of b_2. Let us focus our attention on the points $([x_0 : x_1], [y_0 : y_1])$ of $\overline{E_t}$ corresponding to the equation $x_1 y_1 = 0$, namely:

$$P_1 = ([1:0], [\beta_0 : \beta_1]), \quad P_2 = \iota_1(P_1) = ([1:0], [\beta_0' : \beta_1']),$$
$$Q_1 = ([\alpha_0 : \alpha_1], [1:0]), \quad Q_2 = \iota_2(Q_1) = ([\alpha_0' : \alpha_1'], [1:0]).$$

Since $P_1, P_2 \in \overline{E_t}$, to compute $[\beta_0 : \beta_1]$ and $[\beta_0' : \beta_1']$, we have to solve $\overline{K}(1, 0, y_0, y_1, t) = 0$. Then we find that $[\beta_0 : \beta_1]$ and $[\beta_0' : \beta_1']$ are the roots in $\mathbb{P}^1(\mathbb{C})$ of the homogeneous polynomial in y_0 and y_1 given by

$$d_{1,-1} y_1^2 + d_{1,0} y_0 y_1 + d_{1,1} y_0^2 = 0.$$

Similarly, the x-coordinates $[\alpha_0 : \alpha_1]$ and $[\alpha_0' : \alpha_1']$ of Q_1 and Q_2 are the roots in $\mathbb{P}^1(\mathbb{C})$ of the homogeneous polynomial in x_0 and x_1 given by

$$d_{-1,1} x_1^2 + d_{0,1} x_0 x_1 + d_{1,1} x_0^2 = 0.$$

The following Lemma already appears in [DHRS17a, Lemma 4.11], we give its proof to be self contain.

Lemma 4.3. *The set of poles of* $b_1 = \iota_1(y)\,(\sigma(x) - x)$ *in* $\overline{E_t}$ *is contained in*

$$\mathcal{S}_1 = \{\iota_1(Q_1), \iota_1(Q_2), P_1, P_2, \sigma^{-1}(P_1), \sigma^{-1}(P_2)\}.$$

Similarly, the set of poles of $b_2 = x(\iota_1(y) - y)$ *in* $\overline{E_t}$ *is contained in*

$$\mathcal{S}_2 = \{P_1, P_2, Q_1, Q_2, \iota_1(Q_1), \iota_1(Q_2)\} = \{P_1, P_2, Q_1, Q_2, \sigma^{-1}(Q_1), \sigma^{-1}(Q_2)\}.$$

Moreover, we have

$$(b_2)^2 = \frac{x_0^2 \Delta^x_{[x_0:x_1]}}{x_1^2(\sum_{i=0}^2 x_0^i x_1^{2-i} t d_{i-1,1})^2}. \tag{4.1}$$

Proof. The proofs of the assertions about the location of the poles of b_1 and b_2 are straightforward. Let us prove (4.1). By definition, $\iota_1(\frac{y_0}{y_1})$ and $\frac{y_0}{y_1}$ are the two roots of the polynomial $y \mapsto \overline{K}(x_0, x_1, y, t)$. The square of their difference equals to the discriminant divided by the square of the leading term. Then, we have

$$\left(\iota_1(\frac{y_0}{y_1}) - \frac{y_0}{y_1}\right)^2 = \frac{\Delta^x_{[x_0:x_1]}}{(\sum_i x_0^i x_1^{2-i} t d_{i-1,1})^2}.$$

Therefore, we find

$$b_2\left(\frac{x_0}{x_1}, \frac{y_0}{y_1}\right)^2 = \frac{x_0^2 \Delta^x_{[x_0:x_1]}}{x_1^2(\sum_i x_0^i x_1^{2-i} t d_{i-1,1})^2}.$$

□

To apply Proposition 3.7 we now need to separate the orbits. Let us begin by P_1 and P_2 (resp. Q_1 and Q_2).

Proposition 4.4. *If* $P_1 \neq P_2$*, then one of the following properties holds true :*

- $P_1 \not\sim P_2$;
- $d_{0,1} = d_{1,1} = 0$.

If $Q_1 \neq Q_2$*, then one of the following properties holds true :*

- $Q_1 \not\sim Q_2$;
- $d_{1,0} = d_{1,1} = 0$.

Proof. We only prove the statement for the P_i, the proof for the Q_j is similar. Let $p_1, p_2 \in \mathbb{C}^*$ be such that $\phi(p_1) = P_1$ and $\phi(p_2) = P_2$. Recall that Lemma 1.7 ensures that

$$\begin{array}{rcl} \alpha_2 & = & 1 - 2td_{0,0} + t^2 d_{0,0}^2 - 4t^2 d_{-1,1} d_{1,-1} \\ \alpha_3 & = & 2t^2 d_{1,0} d_{0,0} - 2td_{1,0} - 4t^2 d_{0,1} d_{1,-1} \\ \alpha_4 & = & t^2(d_{1,0}^2 - 4d_{1,1} d_{1,-1}). \end{array}$$

and that, according to Proposition 2.7, one of the two complex numbers q or q^{-1} is equal to

$$\frac{-1 + d_{0,0}t - \sqrt{(1 - d_{0,0}t)^2 - 4d_{1,-1}d_{-1,1}t^2}}{-1 + d_{0,0}t + \sqrt{(1 - d_{0,0}t)^2 - 4d_{1,-1}d_{-1,1}t^2}}.$$

The explicit formula for ϕ given in Proposition 2.6 shows that p_1 and p_2 are the roots of

$$-\sqrt{\alpha_3^2 - 4\alpha_2\alpha_4}X^2 + 2\alpha_3 X - \sqrt{\alpha_3^2 - 4\alpha_2\alpha_4} = 0.$$

So, we have (for suitable choices of the square roots)

$$p_1 = \frac{-\alpha_3 - 2\sqrt{\alpha_2\alpha_4}}{-\sqrt{\alpha_3^2 - 4\alpha_2\alpha_4}} \text{ and } p_2 = \frac{-\alpha_3 + 2\sqrt{\alpha_2\alpha_4}}{-\sqrt{\alpha_3^2 - 4\alpha_2\alpha_4}}.$$

Assume that $P_1 \sim P_2$. Then, there exists $\ell \in \mathbb{Z}^*$ such that $\frac{p_1}{p_2} = q^\ell$ ($\ell \neq 0$ because $P_1 \neq P_2$). Using the above formulas for p_1,p_2 and q and replacing ℓ by $-\ell$ if necessary, this can be rewritten as :

$$\frac{-\alpha_3 - 2\sqrt{\alpha_2\alpha_4}}{-\alpha_3 + 2\sqrt{\alpha_2\alpha_4}} = \left(\frac{-1 + d_{0,0}t - \sqrt{(1-d_{0,0}t)^2 - 4d_{1,-1}d_{-1,1}t^2}}{-1 + d_{0,0}t + \sqrt{(1-d_{0,0}t)^2 - 4d_{1,-1}d_{-1,1}t^2}}\right)^\ell. \tag{4.2}$$

Recall that t is transcendental. We shall treat t as a variable and both sides of (4.2) as functions of the variable t, algebraic over $\mathbb{Q}(t)$. Formula (4.2) shows that these algebraic functions coincide at some transcendental number, therefore they are equal.

We now consider these algebraic functions near 0 (we choose an arbitrary branch) and will derive a contradiction by proving that they have different behaviors at 0.

If $d_{1,1} \neq 0$, then, considering the Taylor expansions at 0 in (4.2), we obtain, up to replacing ℓ by $-\ell$ if necessary:

$$\frac{d_{1,0} - \Delta_1}{d_{1,0} + \Delta_1} + O(t) = \left(\frac{1}{t^2}\left(\frac{1}{d_{1,-1}d_{-1,1}} + \mathrm{O(1/t)}\right)\right)^\ell$$

where Δ_1 is some square root of $d_{1,0}^2 - 4d_{1,1}d_{1,-1}$, and $d_{1,0} - \Delta_1$ and $d_{1,0} + \Delta_1$ are not 0 because $d_{1,1} \neq 0$ (note that, by Assumption 1.6, we have $d_{1,-1}d_{-1,1} \neq 0$). This equality is impossible.

If $d_{1,1} = 0$, then (4.2) gives

$$t\frac{d_{0,1}d_{1,-1}}{d_{1,0}} + O(t^2) = \left(\frac{1}{t^2}\left(\frac{1}{d_{1,-1}d_{-1,1}} + \mathrm{O(1/t)}\right)\right)^\ell$$

(note that we have $d_{1,0} \neq 0$ because $P_1 \neq P_2$). This implies $d_{0,1} = 0$ and conclude the proof. $\square$

Proposition 4.5. *Assume that $d_{1,1} \neq 0$. Then, for any $i, j \in \{1, 2\}$, we have $P_i \not\sim Q_j$.*

Proof. Let $p_i, q_j \in \mathbb{C}^*$ be such that $\phi(p_i) = P_i$ and $\phi(q_j) = Q_j$. As seen at the beginning of the proof of Proposition 4.4, we have (for suitable choices of the square roots)

$$p_i = \frac{-\alpha_3 - 2\sqrt{\alpha_2\alpha_4}}{-\sqrt{\alpha_3^2 - 4\alpha_2\alpha_4}}.$$

Similarly, we have (for suitable choices of the square roots)

$$q_j = \lambda\frac{-\beta_3 - 2\sqrt{\beta_2\beta_4}}{-\sqrt{\beta_3^2 - 4\beta_2\beta_4}}.$$

Suppose to the contrary that $P_i \sim Q_j$. The condition $d_{1,1} \neq 0$ yields that $P_i \neq Q_j$. Then, there exists $\ell \in \mathbb{Z}^*$ such that $\frac{p_i}{q_j} = q^\ell$. Using the above formulas for p_i and q_j, using Proposition 2.7 and replacing ℓ by $-\ell$ if necessary, this can be rewritten as:

$$\frac{\alpha_3 + 2\sqrt{\alpha_2\alpha_4}}{\sqrt{\alpha_3^2 - 4\alpha_2\alpha_4}}\frac{\sqrt{\beta_3^2 - 4\beta_2\beta_4}}{\beta_3 + 2\sqrt{\beta_2\beta_4}} = \left(\frac{-1 + d_{0,0}t - \sqrt{(1-d_{0,0}t)^2 - 4d_{1,-1}d_{-1,1}t^2}}{-1 + d_{0,0}t + \sqrt{(1-d_{0,0}t)^2 - 4d_{1,-1}d_{-1,1}t^2}}\right)^{\ell+\frac{1}{2}}. \tag{4.3}$$

As in the the proof of Proposition 4.4, on can treat t as a variable and both sides of (4.3) as functions of the variable t algebraic over $\mathbb{Q}(t)$, the above equality shows that they coincide, and we shall now consider these algebraic functions near 0 (we choose an arbitrary branch). Considering the Taylor expansions at 0 in (4.3), we obtain :

$$\frac{-d_{1,0} - \Delta_1}{\sqrt{d_{1,0}^2 - {\Delta_1}^2}}\frac{\sqrt{d_{0,1}^2 - {\Delta_2}^2}}{-d_{0,1} - \Delta_2} + O(t) = \left(\frac{1}{t^2}\left(\frac{1}{d_{1,-1}d_{-1,1}} + \mathrm{O(t)}\right)\right)^{\ell+\frac{1}{2}}$$

where Δ_1 and Δ_2 are suitable square roots of $d_{1,0}^2 - 4d_{1,1}d_{1,-1}$ and $d_{0,1}^2 - 4d_{1,1}d_{-1,1}$ respectively, and none of the numbers $-d_{1,0} - \Delta_1, \sqrt{d_{1,0}^2 - {\Delta_1}^2}, \sqrt{d_{0,1}^2 - {\Delta_2}^2}, -d_{0,1} - {\Delta_2}^2$ is zero because $d_{1,1} \neq 0$. This equality is impossible. □

4.2. **Proof of Theorem 4.1.** First we note that (4.1) and the fact that $\Delta^x_{[x_0:x_1]}$ seen as a function on $\mathbb{P}^1(\mathbb{C})$ has at most a simple zero at P_1 and P_2 (see Lemma 1.7) imply that P_1 and P_2 are poles of b_2.

If $d_{1,1} = d_{1,0} = 0$ (and $d_{0,1} \neq 0$ by Assumption 1.6), then a direct calculation shows that the polar divisor of b_2 on $\overline{E_t}$ is $3P_1 + Q_2 + \iota_1(Q_2)$ where

- $P_1 = P_2 = Q_1 = ([1:0],[1:0])$,
- $Q_2 = ([-d_{-1,1} : d_{0,1}],[1:0])$,
- $\iota_1(Q_2) = ([-d_{-1,1} : d_{0,1}],[-td_{1,-1}d_{-1,1} : d_{0,1}(1 - td_{0,0})]) \neq Q_2$.

The result is now a direct consequence of Proposition 3.7 because P_1 is a pole of order 3 of b_2, and all the other poles of b_2 have order 1.

The case $d_{1,1} = d_{0,1} = 0$ is similar.

Assume that $d_{1,1} = 0$ and $d_{1,0}d_{0,1} \neq 0$. In this case, we have

- $P_1 = Q_1 = ([1:0],[1:0])$,
- $P_2 = \iota_1(Q_1) = ([1:0],[-d_{1,-1} : d_{1,0}])$,
- $Q_2 = ([-d_{-1,1} : d_{0,1}],[1:0])$,
- $\iota_1(Q_2) = ([-d_{-1,1} : d_{0,1}],[-td_{1,-1}d_{-1,1} : d_{0,1}(1 - td_{0,0}) + td_{1,0}d_{-1,1}])$

Note that these four points are two by two distinct (since $d_{0,1} \neq 0$ and $t \notin \overline{\mathbb{Q}}$, the quantity $d_{0,1}(1 - td_{0,0}) + td_{1,0}d_{-1,1}$ does not vanish).
A direct computation shows that the polar divisor of b_2 on $\overline{E_t}$ is $2P_1 + 2P_2 + Q_2 + \iota_1(Q_2)$. Proposition 4.4 ensures that $P_1 \not\sim P_2$. So, $P = P_1$ or P_2 is such that such that none of the $\sigma^i(P)$ with $i \in \mathbb{Z}\backslash\{0\}$ is a pole of order ≥ 2 of b_2. The result is now a consequence of Proposition 3.7.

Last, assume that $d_{1,1} \neq 0$. Then, combining Proposition 4.4 and Proposition 4.5, and using the fact that the set of poles of b_2 is included in $\{P_1, P_2, Q_1, Q_2, \sigma^{-1}(Q_1), \sigma^{-1}(Q_2)\}$, we get that P_1 is such that none of the $\sigma^i(P_1)$ with $i \in \mathbb{Z}\backslash\{0\}$ is a pole of b_2. The result is now a consequence of Proposition 3.7.

References

[Abr95] S. A. Abramov, *Indefinite sums of rational functions*, Proceedings of the 1995 International Symposium on Symbolic and Algebraic Computation (1995), 303–308.

[BBMKM16] A. Bostan, M. Bousquet-Mélou, M. Kauers, and S. Melczer, *On 3-dimensional lattice walks confined to the positive octant*, Ann. Comb. **20** (2016), no. 4, 661–704. MR 3572381

[BBMR15] Olivier Bernadi, Mireille Bousquet-Mélou, and Kilian Raschel, *Counting quadrant walks via Tutte's invariant method*, An extended abstract to appeared in *Proceedings of FPSAC 2016*, Discrete Math. Theor. Comput. Sci. Proc., arXiv:1511.04298, 2015.

[BBMR17] ———, *Counting quadrant walks via Tutte's invariant method*, Preprint, arXiv:1708.08215, 2017.

[BMM10] Mireille Bousquet-Mélou and Marni Mishna, *Walks with small steps in the quarter plane*, Algorithmic probability and combinatorics, Contemp. Math., vol. 520, Amer. Math. Soc., Providence, RI, 2010, pp. 1–39.

[BRS14] Alin Bostan, Kilian Raschel, and Bruno Salvy, *Non-D-finite excursions in the quarter plane*, J. Combin. Theory Ser. A **121** (2014), 45–63. MR 3115331

[BvHK10] Alin Bostan, Mark van Hoeij, and Manuel Kauers, *The complete generating function for gessel walks is algebraic*, Proc. Amer. Math. Soc. **138** (2010), no. 9, 3063–3078.

[CS12] Shaoshi Chen and Michael F. Singer, *Residues and telescopers for bivariate rational functions*, Adv. in Appl. Math. **49** (2012), no. 2, 111–133.

[DHRS17a] Thomas Dreyfus, Charlotte Hardouin, Julien Roques, and Michael F. Singer, *On the nature of the generating series of walks in the quarter plane*, Preprint (2017).

[DHRS17b] ______, *Walks whose discriminant has a double zero*, 2017, This is a MAPLE worksheet which can be found at `http://www4.ncsu.edu/~singer/worksheet.html`.

[DHW16] D. Du, Q.-H. Hou, and R.-H. Wang, *Infinite orders and non-D-finite property of 3-dimensional lattice walks*, Electron. J. Combin. **23** (2016), no. 3, Paper 3.38, 15. MR 3558075

[FIM99] Guy Fayolle, Roudolf Iasnogorodski, and Vadim Malyshev, *Random walks in the quarter-plane*, Applications of Mathematics (New York), vol. 40, Springer-Verlag, Berlin, 1999, Algebraic methods, boundary value problems and applications. MR 1691900

[FR11] G. Fayolle and K. Raschel, *Random walks in the quarter-plane with zero drift: an explicit criterion for the finiteness of the associated group*, Markov Process. Related Fields **17** (2011), no. 4, 619–636.

[Har77] Robin Hartshorne, *Algebraic geometry*, Springer-Verlag, New York-Heidelberg, 1977, Graduate Texts in Mathematics, No. 52. MR 0463157

[Har08] Charlotte Hardouin, *Hypertranscendance des systèmes aux différences diagonaux*, Compos. Math. **144** (2008), no. 3, 565–581. MR 2422340

[HS08] Charlotte Hardouin and Michael F. Singer, *Differential Galois theory of linear difference equations*, Math. Ann. **342** (2008), no. 2, 333–377.

[Ish98] Katsuya Ishizaki, *Hypertranscendency of meromorphic solutions of a linear functional equation*, Aequationes Math. **56** (1998), no. 3, 271–283. MR 1639233

[KR12] Irina Kurkova and Kilian Raschel, *On the functions counting walks with small steps in the quarter plane*, Publ. Math. Inst. Hautes Études Sci. **116** (2012), 69–114. MR 3090255

[KY15] Manuel Kauers and Rika Yatchak, *Walks in the quarter plane with multiple steps*, Proceedings of FPSAC 2015, Discrete Math. Theor. Comput. Sci. Proc., Assoc. Discrete Math. Theor. Comput. Sci., Nancy, 2015, pp. 25–36.

[Mis09] Marni Mishna, *Classifying lattice walks restricted to the quarter plane*, J. Combin. Theory Ser. A **116** (2009), no. 2, 460–477. MR 2475028

[MM14] Stephen Melczer and Marni Mishna, *Singularity analysis via the iterated kernel method*, Combin. Probab. Comput. **23** (2014), no. 5, 861–888. MR 3249228

[MR09] Marni Mishna and Andrew Rechnitzer, *Two non-holonomic lattice walks in the quarter plane*, Theoret. Comput. Sci. **410** (2009), no. 38-40, 3616–3630. MR 2553316

[Ras12] Kilian Raschel, *Counting walks in a quadrant: a unified approach via boundary value problems*, J. Eur. Math. Soc. (JEMS) **14** (2012), no. 3, 749–777. MR 2911883

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