ELSEVIER Journal of Pure and Applied Algebra 110 (1996) 185–194

JOURNAL OF
PURE AND
APPLIED ALGEBRA

On Ramis's solution of the local inverse problem of differential Galois theory

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Communicated by M.F. Roy; received 10 June 1994; revised 15 April 1995

Abstract

Recently, Ramis gave necessary and sufficient conditions for a linear algebraic group to be the Galois group of a Picard–Vessiot extension of the field $\mathbb{C}\{x\}[x^{-1}]$ of germs of meromorphic functions at zero. In this paper, we give equivalent simple group theoretic conditions, and show how these generalize previous conditions of Kovacic in the solvable case.

1. Introduction

The general inverse problem in differential Galois theory can be stated as follows:

Let k denote a differential field of characteristic 0 and C the subfield of constants of k, which we assume to be algebraically closed. Characterize those linear algebraic groups G that are Galois groups of Picard–Vessiot extensions of k.

An early contribution to this problem is due to Bialynicki-Birula [1] who showed that if the transcendence degree of k over C is finite and nonzero then any connected nilpotent group is a Galois group over k. This result was generalized by Kovacic, who showed that the same is true for any connected solvable group.

When one considers specific fields, more is known. If $K = C(x)$, the field of rational functions over C, C. Tretkoff and M. Tretkoff [15] have shown that any linear algebraic group is a Galois group when $C = \mathbb{C}$, the field of complex numbers. For arbitrary C, Singer [14] showed that a large class of linear algebraic groups (including all connected

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SSDI 0022-4049(95)00104-2

groups) are Galois groups over $C(x)$. A different, purely algebraic proof of this result for connected linear algebraic groups can be found in [9]. If $K = \mathbb{C}\{x\}[x^{-1}]$, Kovacic [5] showed that a necessary and sufficient condition for a connected solvable group G to be a Galois group over K is that the unipotent radical of the center of $G/[R_u, R_u]$ have dimension at most 1, where R_u is the unipotent radical of G. In [12], Ramis showed that any connected semisimple group is a Galois group over K. Recently, Ramis extended this result to show that a necessary and sufficient condition for a linear algebraic group to be a Galois group over K is that it have a *local Galois structure* (cf. infra), a condition expressed in terms of the Lie algebra of the group.

In this paper, we give a simpler group theoretic condition that is equivalent to this latter condition and more in line with the condition of Kovacic. We can now state the solution of the inverse problem over $K = \mathbb{C}\{x\}[x^{-1}]$ as:

Theorem 1.1. *Let G be a linear algebraic group. The following statements are equivalent:*
(1) *G is the Galois group of some Picard–Vessiot extension of* $\mathbb{C}\{x\}[x^{-1}]$.
(2) *The following three conditions hold:*
 (a) *G/G^0 is cyclic,*
 (b) *the dimension of $R_u/[R_u, G^0]$ is at most* 1
 (c) *G/G^0 acts trivially on $R_u/[R_u, G^0]$.*

The rest of the paper is organized as follows. In section 2, we show the equivalence of Ramis's conditions with the above group theoretic criteria. In section 3, we give two illustrative examples.

2. Local Galois structures on linear algebraic groups

The following theorem of Ramis [11] solves the local inverse problem of differential Galois theory.

Theorem 2.1. *Let G be a complex algebraic group. Then G is the differential Galois group of some Picard–Vessiot extension of* $\mathbb{C}\{x\}[x^{-1}]$ *if and only if there is a* local Galois structure *on G.*

A local Galois structure on G, a linear algebraic group, was defined by Ramis as a triple $(T, a, \mathcal{N})$ such that:

1. T is a torus of $G, a \in N_G(T)$,
2. The image of a in G/G^0 generates this finite group,
3. $\mathcal{N}$ is an algebraic sub-Lie algebra of $\mathcal{G}$ (the Lie algebra of G) commuting with T and a (via the adjoint action) and $\dim(\mathcal{N}) \leq 1$,
4. Let $\mathcal{T}$ be the Lie algebra of T. We decompose $\mathcal{G} = \coprod_{\alpha\in\Phi} \mathcal{G}_\alpha$, where Φ is the set of weights of $\mathcal{T}$ and $\mathcal{G}_\alpha$ is the weight space corresponding to α. Then $\mathcal{G}$ is the Lie

algebra $\mathscr{T}+\mathscr{C}(\mathscr{T})+\mathscr{N}$, where $\mathscr{C}(\mathscr{T})$ is the Lie algebra generated by $\coprod_{\alpha\in\Phi^*}\mathscr{G}_\alpha$ and Φ^* is the set of roots (i.e., nonzero weights).

We note that the last condition is equivalent to the condition that $\mathscr{G}_0$ is the Lie algebra $\mathscr{T}+\coprod_{\alpha\in\Phi^*}[\mathscr{G}_\alpha,\mathscr{G}_{-\alpha}]+\mathscr{N}$. Furthermore, $\coprod_{\alpha\in\Phi^*}[\mathscr{G}_\alpha,\mathscr{G}_{-\alpha}]$ is an ideal of $\mathscr{G}_0$.

We now begin the proof of Theorem 1.1. The following result will be useful. We include the proof from [4] for the convenience of the reader.

Lemma 2.2. *Let $\mathscr{A}$ be a nilpotent Lie algebra such that* $\dim \mathscr{A}/[\mathscr{A},\mathscr{A}]\leq 1$. *Then $\mathscr{A}$ is commutative and so* $\dim\mathscr{A}\leq 1$.

Proof. Since $\mathscr{A}$ is nilpotent, we have $\mathscr{A}^n=0$ for some positive integer n, where $\mathscr{A}^0=\mathscr{A}$ and $\mathscr{A}^i=[\mathscr{A},\mathscr{A}^{i-1}]$. Therefore to show that $[\mathscr{A},\mathscr{A}]=0$ it is enough to show that $[\mathscr{A},\mathscr{A}]=\mathscr{A}^1=\mathscr{A}^2=[\mathscr{A},[\mathscr{A},\mathscr{A}]]$ (and so $\mathscr{A}^1=\mathscr{A}^i$ for all i). Clearly $[\mathscr{A},[\mathscr{A},\mathscr{A}]]\subset[\mathscr{A},\mathscr{A}]$. To show the reverse inclusion, it is enough to show that for any $u,v\in\mathscr{A}$, we have that $[u,v]\in[\mathscr{A},[\mathscr{A},\mathscr{A}]]$. By assumption, there exists an element $x\in\mathscr{A}$ such that $u=c_ux+u_1$, $v=c_vx+v_1$ where $c_u,c_v\in\mathscr{C}$ and $u_1,v_1\in[\mathscr{A},\mathscr{A}]$. We then have

$$\begin{aligned}[u,v]&=[c_ux+u_1,c_vx+v_1]\\&=[c_ux,c_vx]+[c_ux,v_1]+[u_1,c_vx]+[u_1,v_1]\\&=c_u[x,v_1]-c_v[x,u_1]+[u_1,v_1]\ \in\ [\mathscr{A},[\mathscr{A},\mathscr{A}]].\qquad\square\end{aligned}$$

Let G^0 be the identity component of G (see [2] as a general reference). We may write G^0 as a semidirect product $G^0=R_{\mathrm{u}}\rtimes P$, where R_{u} is the unipotent radical of G and P is a reductive subgroup (this is the *Levi decomposition*, [10]). This allows us to write the Lie algebra $\mathscr{G}$ of G as $\mathscr{G}=\mathscr{R}_{\mathrm{u}}+\mathscr{P}$, where $\mathscr{R}_{\mathrm{u}}=\mathrm{Lie}(R_{\mathrm{u}})$ and $\mathscr{P}=\mathrm{Lie}(P)$. Note that we can further decompose $\mathscr{P}=\mathscr{T}+\coprod_{\alpha\in\Phi^*}\mathscr{P}_\alpha$ where $\mathscr{T}$ is the Lie algebra of a maximal torus T of P, and the $\mathscr{P}_\alpha$ are the weight spaces corresponding to the set Φ^* of roots of T on $\mathscr{P}$. Note that $\mathscr{T}=\mathscr{P}_0$ since P is reductive and that T is also a maximal torus for G. Conversely, for any given maximal torus T of G there is a Levi decomposition $G=R_{\mathrm{u}}P$ such that $T\subset P$. Since R_{u} is normal in G, the adjoint action of T leaves $\mathscr{R}_{\mathrm{u}}$ invariant so we may decompose $\mathscr{R}_{\mathrm{u}}$ into weight spaces for T. We again denote the set of weights by Φ and denote the corresponding weight spaces by $(\mathscr{R}_{\mathrm{u}})_\alpha$. Using this notation, we have $\mathscr{G}_\alpha=(\mathscr{R}_{\mathrm{u}})_\alpha+\mathscr{P}_\alpha$. We also introduce $\mathscr{L}_0=\mathscr{T}+\coprod_{\alpha\in\Phi^*}[\mathscr{G}_\alpha,\mathscr{G}_{-\alpha}]$, which is an ideal of $\mathscr{G}_0$. We gather some simple facts in the following technical lemma.

Lemma 2.3. *With the notation as above, the following hold:*

(1) $\mathscr{T}+\coprod_{\alpha\in\Phi^*}[\mathscr{G}_\alpha,\mathscr{G}_{-\alpha}]=\mathscr{T}+\coprod_{\alpha\in\Phi^*}[(\mathscr{R}_{\mathrm{u}})_\alpha,(\mathscr{R}_{\mathrm{u}})_{-\alpha}]+\coprod_{\alpha\in\Phi^*}[\mathscr{P}_\alpha,(\mathscr{R}_{\mathrm{u}})_{-\alpha}]$.

(2) $[(\mathscr{R}_{\mathrm{u}})_\alpha,(\mathscr{R}_{\mathrm{u}})_{-\alpha}]$ *and* $[\mathscr{P}_\alpha,(\mathscr{R}_{\mathrm{u}})_{-\alpha}]$ *are in* $(\mathscr{R}_{\mathrm{u}})_0$. *Furthermore,* $(\mathscr{R}_{\mathrm{u}})_0\cap\coprod_{\alpha\in\Phi^*}[\mathscr{G}_\alpha,\mathscr{G}_{-\alpha}]=\coprod_{\alpha\in\Phi^*}[(\mathscr{R}_{\mathrm{u}})_\alpha,(\mathscr{R}_{\mathrm{u}})_{-\alpha}]+\coprod_{\alpha\in\Phi^*}[\mathscr{P}_\alpha,(\mathscr{R}_{\mathrm{u}})_{-\alpha}]$.

(3) $[\mathscr{R}_{\mathrm{u}},\mathscr{G}]$ *is invariant under the adjoint action of $\mathscr{T}$ and* $[\mathscr{R}_{\mathrm{u}},\mathscr{G}]=[(\mathscr{R}_{\mathrm{u}})_0,(\mathscr{R}_{\mathrm{u}})_0]+\coprod_{\alpha\in\Phi^*}[(\mathscr{R}_{\mathrm{u}})_\alpha,(\mathscr{R}_{\mathrm{u}})_{-\alpha}]+\coprod_{\alpha\in\Phi^*}[\mathscr{P}_\alpha,(\mathscr{R}_{\mathrm{u}})_{-\alpha}]+\coprod_{\alpha\in\Phi^*}(\mathscr{R}_{\mathrm{u}})_\alpha$.

(4) $\mathscr{R}_u/[\mathscr{R}_u,\mathscr{G}] = (\mathscr{R}_u)_0/([(\mathscr{R}_u)_0,(\mathscr{R}_u)_0] + \coprod_{\alpha\in\Phi^*}[(\mathscr{R}_u)_\alpha,(\mathscr{R}_u)_{-\alpha}] + \coprod_{\alpha\in\Phi^*}[\mathscr{P}_\alpha,(\mathscr{R}_u)_{-\alpha}])$.

(5) $[\mathscr{R}_u,\mathscr{R}_u]_0 = [(\mathscr{R}_u)_0,(\mathscr{R}_u)_0] + \coprod_{\alpha\in\Phi^*}[(\mathscr{R}_u)_\alpha,(\mathscr{R}_u)_{-\alpha}]$.

(6) *There is a natural isomorphism*

$$\mathscr{G}_0/\mathscr{L}_0 \longrightarrow (\mathscr{R}_u)_0\Big/\left(\coprod_{\alpha\in\Phi^*}[(\mathscr{R}_u)_\alpha,(\mathscr{R}_u)_{-\alpha}] + \coprod_{\alpha\in\Phi^*}[\mathscr{P}_\alpha,(\mathscr{R}_u)_{-\alpha}]\right).$$

Proof. The first claim follows by writing $\mathscr{G}_\alpha = (\mathscr{R}_u)_\alpha + \mathscr{P}_\alpha$ and noting that $[\mathscr{P}_\alpha,\mathscr{P}_{-\alpha}] \subset \mathscr{T}$. The second claim follows similarly. To verify the third claim, we write $\mathscr{G} = (\mathscr{R}_u)_0 + \coprod_{\alpha\in\Phi^*}(\mathscr{R}_u)_\alpha + \mathscr{T} + \coprod_{\alpha\in\Phi^*}\mathscr{P}_\alpha$ and $\mathscr{R}_u = (\mathscr{R}_u)_0 + \coprod_{\alpha\in\Phi^*}(\mathscr{R}_u)_\alpha$. Taking the brackets of each of the components separately and noting that $[(\mathscr{R}_u)_\alpha,\mathscr{T}] = (\mathscr{R}_u)_\alpha$ for $\alpha \in \Phi^*$ gives claim (3). Claim (4). follows from claim (3). Claim (5). is proved in a manner similar to claim (3). To prove claim (6). we note that $\mathscr{G}_0 = (\mathscr{R}_u)_0 + \mathscr{T}$ and that $(\mathscr{R}_u)_0 \cap \mathscr{L}_0 = \coprod_{\alpha\in\Phi^*}[(\mathscr{R}_u)_\alpha,(\mathscr{R}_u)_{-\alpha}] + \coprod_{\alpha\in\Phi^*}[\mathscr{P}_\alpha,(\mathscr{R}_u)_{-\alpha}]$. □

If we let $\mathscr{U} = \mathscr{R}_u/[\mathscr{R}_u,\mathscr{G}]$, $\mathscr{W} = \mathscr{G}_0/\mathscr{L}_0$, then Lemma 2.3 yields a surjective homomorphism π from $\mathscr{W}$ to $\mathscr{U}$. One can show that $\mathscr{U} = \mathscr{R}_u/[\mathscr{R}_u,\mathscr{G}]$ is the Lie algebra of $R_u/[R_u,G^0]$. There is a natural action (via conjugation) of G on $R_u/[R_u,G^0]$. Since G^0 acts trivially on this latter group, we have an action of G/G^0 on $R_u/[R_u,G^0]$ with its corresponding adjoint action on $\mathscr{U}$.

Lemma 2.4. *We have* dim $\mathscr{U} \le 1$ *if and only if* dim $\mathscr{W} \le 1$. *In this case π is an isomorphism, so G/G^0 acts on $\mathscr{W}$.*

Proof. Since the homomorphism π is surjective, dim $\mathscr{U} \le$ dim $\mathscr{W}$ always holds. Note that the kernel of π is $[\mathscr{W},\mathscr{W}]$. If dim $\mathscr{U} \le 1$ then by Lemma 2.2 the kernel of π is trivial and dim $\mathscr{W} \le 1$. □

Lemma 2.5. *If G has a local Galois structure, then G has a local Galois structure $(T,a,\mathscr{N})$ where T is a maximal torus of G and a is semisimple.*

Proof. Let $(T',a',\mathscr{N}')$ be a local Galois structure of G .Then T' is contained in a maximal torus T of G. Let us denote by $\mathcal{G}_\alpha$ (resp. $\mathcal{G}'_\alpha$) the weight spaces relative to T (resp. T'). Recall that $\mathscr{L}_0 = \mathcal{T} + \coprod_{\alpha\in\Phi^*}[\mathcal{G}_\alpha,\mathcal{G}_{-\alpha}]$ is an ideal of $\mathcal{G}_0$ and let $\mathscr{L}'_0 = \mathcal{T}' + \coprod_{\alpha\in\Psi^*}[\mathcal{G}'_\alpha,\mathcal{G}'_{-\alpha}]$, which is an ideal of $\mathcal{G}'_0$.

We have a decomposition $\mathcal{G}'_0 = \mathcal{G}_0 \oplus (\coprod_{\beta\in B}\mathscr{G}_\beta)$ for some subset of roots $B \subset \Phi^*$ and the local Galois structure gives $\mathcal{G}'_0 = \mathscr{L}'_0 + \mathscr{N}'$. Since clearly $\mathcal{L}'_0 \cap \mathcal{G}_0 \subset \mathcal{L}_0$ and $\mathscr{L}'_0 = (\mathscr{L}'_0 \cap \mathscr{G}_0) \oplus (\mathscr{L}'_0 \cap \coprod_{\beta\in B}\mathscr{G}_\beta)$ we get surjective homomorphisms

$$\mathscr{G}'_0/\mathscr{L}'_0 \longrightarrow \mathscr{G}_0/\mathscr{L}'_0 \cap \mathscr{G}_0 \longrightarrow \mathscr{G}_0/\mathscr{L}_0$$

so dim $\mathcal{G}'_0/\mathcal{L}'_0 \le 1$ implies dim $\mathcal{G}_0/\mathcal{L}_0 \le 1$.

Since $(a')^{-1}Ta'$ is again a maximal torus of G there exists $g \in G^0$ such that $(a')^{-1}Ta' = gTg^{-1}$. Therefore $a'g \in N_G(T)$. Let $a = (a'g)_s$ denote the semisimple part of $a'g$. Then we also have $a \in N_G(T)$ and $\mathrm{Ad}(a)$ permutes the weight spaces of T, leaving $\mathcal{L}_0$ invariant. Since a is semisimple, there exists a complement $\mathcal{N}$ of $\mathcal{L}_0$ in $\mathcal{G}_0$ which is $\mathrm{Ad}(a)$-invariant. Note that a and a' have the same image in G/G^0 and that the action of G/G^0 on $\mathscr{W} = \mathscr{G}_0/\mathscr{L}_0$ (described in Lemma 2.4) is induced by the adjoint action of a' on $\mathscr{G}_0'/\mathscr{L}_0'$ given by the local Galois structure. Since a' acts trivially on $\mathscr{N}'$, the action of a on $\mathscr{G}_0/\mathscr{L}_0$, hence on $\mathscr{N}$, is also trivial and $(T, a, \mathcal{N})$ provides a local Galois structure on G. $\square$

Lemma 2.6. *G has a local Galois structure if and only if for some maximal torus T and $a \in N_G(T)$, and with notations as before,*

(1) *a is semisimple and the image of a generates G/G^0,*

(2) $\dim(\mathscr{W}) \leq 1$,

(3) *the action of a on $\mathscr{W}$ is trivial.*

Proof. Let $(T, a, \mathscr{N})$ be a Galois structure. By Lemma 2.5, we may assume that T is a maximal torus and that a is semisimple. Note that $\mathscr{G}_0 = \mathscr{T} + (\mathscr{R}_u)_0$. Therefore $(\mathscr{R}_u)_0 = (\coprod_{\alpha\in\Phi^*}[\mathscr{G}_\alpha, \mathscr{G}_{-\alpha}] \cap (\mathscr{R}_u)_0) + \mathscr{N} = (\coprod_{\beta\in\Phi^*}[(\mathscr{R}_u)_\beta, (\mathscr{R}_u)_{-\beta}] + \coprod_{\beta\in\Phi^*}[\mathscr{P}_\beta, (\mathscr{R}_u)_{-\beta}]) + \mathscr{N}$. This implies that there is a natural surjective map

$$\phi: \mathscr{N} \to (\mathscr{R}_u)_0 \Big/ \Big(\coprod_{\beta\in\Phi^*}[(\mathscr{R}_u)_\beta, (\mathscr{R}_u)_{-\beta}] + \coprod_{\beta\in\Phi^*}[\mathscr{P}_\beta, (\mathscr{R}_u)_{-\beta}] \Big),$$

that is, from $\mathscr{N}$ to $\mathscr{W}$ and that this map commutes with the action of a. Since $\dim \mathscr{N} \leq 1$ and a commutes with $\mathscr{N}$, we get the conclusion of the lemma.

Now assume that there is a maximal torus T and $a \in N_G(T)$ satisfying (1), (2), and (3). As in the proof of Lemma 2.4, since a normalizes T, $\mathrm{Ad}(a)$ will normalize $\mathrm{Ad}(T)$. Therefore $\mathrm{Ad}(a)$ will permute the weight spaces of T, preserving the group structure of the weights. In particular, $\mathrm{Ad}(a)$ will preserve $\coprod_{\beta\in\Phi^*}[(\mathscr{R}_u)_\beta, (\mathscr{R}_u)_{-\beta}] + \coprod_{\beta\in\Phi^*}[\mathscr{P}_\beta, (\mathscr{R}_u)_{-\beta}]$. Since a is semisimple, there is a complementary $\mathrm{Ad}(a)$-invariant Lie algebra $\mathscr{N} \subset (\mathscr{R}_u)_0$ such that

$$\begin{aligned}(\mathscr{R}_u)_0 &= \Big(\coprod_{\alpha\in\Phi^*}[\mathscr{G}_\alpha, \mathscr{G}_{-\alpha}] \cap (\mathscr{R}_u)_0 \Big) + \mathscr{N} \\ &= \Big(\coprod_{\beta\in\Phi^*}[(\mathscr{R}_u)_\beta, (\mathscr{R}_u)_{-\beta}] + \coprod_{\beta\in\Phi^*}[\mathscr{P}_\beta, (\mathscr{R}_u)_{-\beta}] \Big) + \mathscr{N}.\end{aligned}$$

Therefore

$$\begin{aligned}\mathscr{G}_0 &= \mathscr{T} + (\mathscr{R}_u)_0 \\ &= \mathscr{T} + \Big(\coprod_{\beta\in\Phi^*}[(\mathscr{R}_u)_\beta, (\mathscr{R}_u)_{-\beta}] + \coprod_{\beta\in\Phi^*}[\mathscr{P}_\beta, (\mathscr{R}_u)_{-\beta}] \Big) + \mathscr{N} \\ &= \mathscr{T} + \coprod_{\alpha\in\Phi^*}[\mathscr{G}_\alpha, \mathscr{G}_{-\alpha}] + \mathscr{N}.\end{aligned}$$

Note that the action of a on $\mathcal{N}$ is trivial since a acts trivially on $\mathcal{W}$. This proves that $(T, a, \mathcal{N})$ is a local Galois structure. □

Proposition 2.7. *Let G be a linear algebraic group. Then G has a local Galois structure if and only if the following three conditions hold:*
(1) *G/G^0 is cyclic,*
(2) *the dimension of $R_u/[R_u, G^0]$ is at most* 1,
(3) *G/G^0 acts trivially on $R_u/[R_u, G^0]$.*

Proof. The proof follows directly from Lemma 2.4 and 2.6 and from the fact that $\mathcal{U}$ is the Lie algebra of $R_u/[R_u, G^0]$. □

This proposition, together with Ramis's Theorem, finishes the proof of Theorem 1.1.

Corollary 2.8. *Let G be a connected* nilpotent *group. The following statements are equivalent:*
(1) *G is the Galois group of some Picard-Vessiot extension of* $\mathbb{C}\{x\}[x^{-1}]$.
(2) *The dimension of R_u is at most* 1.

Proof. A connected nilpotent group may be written as a direct product $G = R_u T$, so the quotient $R_u/[R_u, G^0]$ is $R_u/[R_u, R_u]$. This has dimension at most one if and only if R_u has dimension at most one, by Lemma 2.2. □

Kovacic proves the following result in [5]. Kovacic's techniques allow him to reduce the inverse problem for an arbitrary connected group to the same problem for semisimple groups [6], but he readily admits that the techniques seem to take him no further.

Corollary 2.9. *Let G be a connected* solvable *group. The following statements are equivalent:*
(1) *G is the Galois group of some Picard–Vessiot extension of* $\mathbb{C}\{x\}[x^{-1}]$.
(2) *The dimension of the unipotent radical of the center of $G/[R_u, R_u]$ is at most one.*

Proof. Since G is solvable, it is a semi-direct product $G = R_u \rtimes T$ for some torus T. Therefore $G/[R_u, R_u] = R_u/[R_u, R_u] \rtimes T$. The quotient $R_u/[R_u, R_u]$ is a commutative unipotent group and so isomorphic to $\mathbb{C}^m$ for some m. Since T is reductive and acts on $R_u/[R_u, R_u]$ by conjugation, we may write $R_u/[R_u, R_u]$ as the sum of weight spaces for T (we write the group of weights *additively*). Therefore the unipotent radical of the center of $R_u/[R_u, R_u] \rtimes T$ is $(R_u/[R_u, R_u])_0$. We shall calculate the Lie algebra of this group. Clearly this Lie algebra is $(\mathcal{R}_u/[\mathcal{R}_u, \mathcal{R}_u])_0$. Since T is reductive, we may write $\mathcal{R}_u = \tilde{\mathcal{G}} + [\mathcal{R}_u, \mathcal{R}_u]$ for some T-invariant space $\tilde{\mathcal{G}}$. Therefore, $(\mathcal{R}_u)_0 = \tilde{\mathcal{G}}_0 + [\mathcal{R}_u, \mathcal{R}_u]_0$ so $\tilde{\mathcal{G}}_0 = (\mathcal{R}_u/[\mathcal{R}_u, \mathcal{R}_u])_0 = (\mathcal{R}_u)_0/[\mathcal{R}_u, \mathcal{R}_u]_0$. Lemma 2.2 implies that $(\mathcal{R}_u)_0/[\mathcal{R}_u, \mathcal{R}_u]_0 = (\mathcal{R}_u)_0/([(\mathcal{R}_u)_0, (\mathcal{R}_u)_0] + \coprod_{\alpha \in \Phi^*}[(\mathcal{R}_u)_\alpha, (\mathcal{R}_u)_{-\alpha}])$. Noting that $\mathcal{P}_\alpha = 0$ for all $\alpha \in \Phi^*$, Lemma 2.3 shows that this is the same as the Lie algebra of $R_u/[R_u, G^0]$. Therefore the unipotent radical of the center of $G/[R_u, R_u]$ and $R_u/[R_u, G^0]$ have the same dimension and the result now follows from the main theorem. □

Corollary 2.10. *Let G be a* reductive *group with G/G^0 cyclic. Then G is the Galois group of some Picard–Vessiot extension of* $\mathbb{C}\{x\}[x^{-1}]$.

Proof. R_u is trivial. □

3. Examples

We now give two examples that illustrate the above theorem.

Example 3.1. Let $k = \mathbb{C}\{x\}[x^{-1}]$ and consider the following linearly independent functions $\{\int e^{-1/\sqrt{x}}, \int e^{1/\sqrt{x}}, 1, \log x\}$. The first three satisfy a third order linear differential equation $L_1(y) = 0$ over $\mathbb{C}(x)$. The last two satisfy a second-order linear differential equation $L_2(y) = 0$. These two equations will have a one-dimensional solution space in common (the solution space generated by 1). Taking a least common multiple of these two equations, we will get a linear differential equation of order 4 with the above set as a basis for its solution space.

We will show that the Galois group of this equation over k is

$$\left\{ \begin{pmatrix} a & 0 & 0 & b \\ 0 & a^{-1} & 0 & c \\ 0 & 0 & 1 & d \\ 0 & 0 & 0 & 1 \end{pmatrix} \middle| a \in \mathbb{C}^*, b, c, d \in \mathbb{C} \right\} \cup \left\{ \begin{pmatrix} 0 & a & 0 & b \\ a^{-1} & 0 & 0 & c \\ 0 & 0 & 1 & d \\ 0 & 0 & 0 & 1 \end{pmatrix} \middle| a \in \mathbb{C}^*, b, c, d \in \mathbb{C} \right\}.$$

In group theoretic terms, this is the semidirect product $(\mathbb{C}^3 \rtimes \mathbb{C}^*) \rtimes \mathbb{Z}/2\mathbb{Z}$, where the action of $\mathbb{C}^*$ on $\mathbb{C}^3$ is given by the representation

$$a \mapsto \begin{pmatrix} a & 0 & 0 \\ 0 & a^{-1} & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

and the action of $\mathbb{Z}/2\mathbb{Z}$ is given by permuting the first two columns of any matrix. One sees that $R_u = \mathbb{C}^3$ and that $[R_u, G^0] = \mathbb{C}^2 \times \{0\}$. It then follows that the action of $\mathbb{Z}/2\mathbb{Z}$ on $R_u/[R_u, G^0]$ is trivial.

To verify that this is actually the Galois group, one checks that the Picard–Vessiot extension of k corresponding to $L(y) = 0$ is

$$k\left(\sqrt{x}, \log x, e^{1/\sqrt{x}}, \int e^{1/\sqrt{x}}, \int e^{-1/\sqrt{x}}\right).$$

Letting $k_0 = k(\sqrt{x})$, we have the following facts:

(1) $\log x$ *is transcendental over* k_0.

(2) $e^{1/\sqrt{x}}$ *is transcendental over* $k_0(\log x)$. This follows from the Kolchin–Ostrowski Theorem [4] which implies that if $e^{1/\sqrt{x}}$ is not transcendental over $k_0(\log x)$ then $(e^{1/\sqrt{x}})^n \in k_0$ for some nonzero $n \in \mathbb{Z}$. This is easily seen to be impossible by expanding power series.

(3) $\int e^{1/\sqrt{x}}$ *and* $\int e^{-1/\sqrt{x}}$ *are algebraically independent over* $k_0(\log x, e^{1/\sqrt{x}})$. If not, the Kolchin–Ostrowski Theorem implies that $c_1 \int e^{1/\sqrt{x}} + c_2 \int e^{-1/\sqrt{x}} \in k_0(\log x, e^{1/\sqrt{x}})$ for some constants c_1, c_2, not both zero. In particular $c_1 \int e^{1/\sqrt{x}} + c_2 \int e^{-1/\sqrt{x}}$ lies in an elementary extension of $k_0(e^{1/\sqrt{x}})$. We will now use the following result of Rosenlicht ([13, Theorem 2]):

Let $k \subset k(y_1, \dots, y_n)$ be an ordinary differential field of characteristic zero with the same field of constants. Assume that the field of constants is algebraically closed, that $y_i'/y_i \in k$ and assume that $y_i/y_j \notin k$ for $i \neq j$. If $y_1 + \cdots + y_n$ is the derivative of an element in some elementary differential extension of $k(y_1, \dots, y_n)$ having the same constants as k, then the same is true of each y_i. In this case, if y_i is not algebraic over k, then it is the derivative of ay_i for some $a_i \in k$, i.e., there is an $a_i \in k$ such that $a_i' + a_i y_i'/y_i = 1$.

We therefore need to show that neither

$$a' - \frac{1}{2}x^{-3/2}a = 1 \quad \text{nor} \quad a' + \frac{1}{2}x^{-3/2}a = 1$$

has a solution in k_0. This can be done by showing that any Laurent series solution (in powers of $x^{1/2}$) must be divergent. Note that the situation changes completely if one replaces k by the field of formal Laurent series.

Now one easily sees that any differential isomorphism σ must do one of the following: $\sigma(1) = 1, \sigma(\sqrt{x}) = \sqrt{x}, \sigma(\log x) = \log x + d, \sigma(\int e^{1/\sqrt{x}}) = a \int e^{1/\sqrt{x}} + b, \sigma(\int e^{-1/\sqrt{x}}) = a^{-1} \int e^{-1/\sqrt{x}} + c$ or $\sigma(1) = 1, \sigma(\sqrt{x}) = -\sqrt{x}, \sigma(\log x) = \log x + d, \sigma(\int e^{1/\sqrt{x}}) = a^{-1} \int e^{-1/\sqrt{x}} + b, \sigma(\int e^{-1/\sqrt{x}}) = a \int e^{1/\sqrt{x}} + c$ for some constants $a \in \mathbb{C}^*, b, c, d \in \mathbb{C}$.

Example 3.2. cf. [3, 7].

Lemma 3.3. *The local Galois group at infinity of any confluent generalized hypergeometric equation*

$$D_{qp} = (-1)^{q-p} x \left(\prod_{j=1}^{p} \partial + \mu_j \right) - \left(\prod_{j=1}^{q} \partial + \nu_j - 1 \right),$$

$\partial = x\frac{\mathrm{d}}{\mathrm{d}x}$, $\mu_i - \mu_j \notin \mathbb{Z}$ *for* $i \neq j$, *satisfies*

$$R_u = [R_u, G^0] .$$

Proof. The *formal monodromy* at ∞ (cf. [8]) is

$$\hat{M} = \begin{pmatrix} D & 0 \\ 0 & R \end{pmatrix}$$

where $D = \mathrm{diag}(\mathrm{e}^{-2\mathrm{i}\pi\mu_1}, \ldots, \mathrm{e}^{-2\mathrm{i}\pi\mu_p})$, and $R = \mathrm{e}^{2\mathrm{i}\pi\lambda/(q-p)}P$,

$$P = \begin{pmatrix} 0 & \ldots & \ldots & \ldots & 1 \\ 1 & 0 & \ldots & \ldots & 0 \\ & \ddots & \ddots & \ddots & \\ 0 & \ldots & 0 & 1 & 0 \end{pmatrix},$$

$\lambda = \frac{1}{2}(q-p+1) - \sum_{j=1}^{p} \mu_j - \sum_{j=1}^{q} \nu_j$. Therefore $\hat{M}$ is semisimple.

Following Ramis, we know that G is endowed with a local Galois structure $(\mathscr{T}, a, \mathscr{N})$, where $\mathscr{N}$ is generated by the logarithm of $\hat{M}_{\mathrm{u}}$, the unipotent part of $\hat{M}$. Considering the form of $\hat{M}$ above, we see that $\mathscr{N} = 0$. By Lemma 2.5 we may moreover suppose that T is a maximal torus. From the proof of Lemma 2.6, we get a surjective homorphism from $\mathscr{N}$ to $\mathscr{U}$, hence dim $\mathscr{U} = 0$, where $\mathscr{U}$ is the Lie algebra of $R_{\mathrm{u}}/[R_{\mathrm{u}}, G^0]$. $\square$

As an example, for $\partial = x\frac{\mathrm{d}}{\mathrm{d}x}$, the equation

$$\begin{aligned} D_{31}y &= \partial^3 y - x\partial y + \frac{1}{2}y \\ &= x^2 y''' + 3xy'' + (1-x)y' + \frac{1}{2}y \\ &= 0 \end{aligned}$$

has Galois group

$$G = (\mathrm{PSL}(2,\mathbb{C}) \rtimes \mathbb{C}^*) \rtimes \mathbb{Z}/2\mathbb{Z}$$

and a local Galois structure is

$$(\mathbb{C}^*, a, 0)$$

where the maximal torus $\mathbb{C}^*$ is the *exponential torus* (cf. [8]) $T = \{\mathrm{diag}(1, \lambda, \lambda^{-1}) \mid \lambda \in \mathbb{C}^*\}$ and a is the formal monodromy

$$\begin{pmatrix} -1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & -1 & 0 \end{pmatrix}$$

generating $G/G^0 \simeq \mathbb{Z}/2\mathbb{Z}$.

Acknowledgements

The authors would like to thank J.P. Ramis for helpful discussions and for bringing to our attention a gap in a previous version of this paper.

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