REMARKS ON ANALYTIC CONTINUATION

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The collection $\mathcal{M}$ of multi-valued analytic functions with three singular points $z = a, b, c$ on the Riemann sphere $\hat{\mathbb{C}}$ includes such classical examples as the hypergeometric functions and the inverse of Legendre's modular function $\lambda(\tau)$. In the present note we show that this class also contains $2^{\aleph_0}$ transcendentally transcendental functions with essentially distinct branching behaviour. Here, we say that two functions with the same set of singular points have "essentially distinct branching behaviour" if their monodromy groups are not isomorphic qua permutation groups to conjugate subgroups of the group of all permutations of a countable set.

A function $w(z)$ is called *transcendentally transcendental* if it does not satisfy any algebraic differential equation; thus there is no polynomial $P(z, w_0, \ldots, w_n) \not\equiv 0$ with complex coefficients which vanishes identically when w_0 is replaced by $w(z)$ and the w_j are replaced by $d^j w/dz^j$. According to a theorem of Hölder, the gamma function has this property, but $\Gamma(z)$ is single-valued and has poles at the negative integers.

Our result is obtained by supplementing the reasoning employed in [**4**] with that of Ritt and Gourin [**3**]. Moreover, replacing the Golod-Shafarevitch group utilized in [**4**] by interesting two-generator groups permits us to restrict our attention to the collection of functions with three singular points and to make a few additional observations about the branching behaviour of some of its members. Finally, we prove a theorem about the symmetric group on a countable set which yields yet another interesting function in our class $\mathcal{M}$.

Throughout this paper we replace the phrase "homogeneous linear differential equation with single-valued analytic coefficients with singularities at $z = a_i$, $i = 1, \ldots, r+1$, r a positive integer", by the term "linear differential equation".

Now, suppose that G is an infinite group generated by r elements; of course, G is countable. Selecting $r+1$ arbitrary points $z = a_i$, $i = 1, \ldots, r+1$ on $\hat{\mathbb{C}}$, the construction given in [**4**] establishes the existence of transcendental functions $T(z)$ on $\hat{\mathbb{C}}$ with singular points $z = a_i$, $i = 1, \ldots, r+1$ and monodromy group isomorphic to G. The analytic key to this result is the Mittag-Leffler Anschmiegungssatz for non-compact Riemann surfaces; it permits us to prescribe arbitrary polynomials

$$R(\sigma) = \sum_{j=0}^{n(\sigma)} a_j(\sigma)(z-z_0)^j, \quad n(\sigma) \geqslant 0, \sigma \in G, a_0(\sigma) \neq a_0(\tau) \quad \text{when} \quad \sigma \neq \tau,$$

as the initial terms of the Taylor expansions of the branches of $T(z)$ at a base point $z_0 \neq a_i, i = 1, \ldots, r+1$.

Adapting the beautiful argument of Ritt and Gourin [**3**], we find that the $a_j(\sigma)$ may be selected so that $T(z)$ is transcendentally transcendental. Namely, we have:

THEOREM 1. *If $z = a_i, i = 1, \ldots, r+1$ are arbitrary points on $\hat{\mathbb{C}}$, and if the infinite group G is generated by r elements, then there are transcendentally transcendental functions on $\hat{\mathbb{C}}$ whose only singularities occur at $z = a_i, i = 1, \ldots, r+1$ and whose monodromy groups are isomorphic to G.*

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Proof. Since the collection $\{P_\sigma(z, w_0, \ldots, w_{n(\sigma)})\}$ of polynomials in a finite number of variables with integral coefficients is countable, we may suppose that the indices σ belong to G. Now, select complex numbers $b_0(\sigma), \ldots, b_n(\sigma), n = n(\sigma)$, such that $b_0(\sigma) \neq b_0(\tau)$ when $\sigma \neq \tau$ and $P_\sigma(z_0, b_0(\sigma), \ldots, b_n(\sigma)) \neq 0$. Setting $a_j(\sigma)j! = b_j(\sigma)$, we see that the branch of $T(z)$ beginning with $R(\sigma)$ cannot satisfy the differential equation defined by $P_\sigma(z, w_0, \ldots, w_n)$.

However, if $T(z)$ satisfies the algebraic differential equation defined by $P(z, w_0, \ldots, w_m)$, it must also satisfy an algebraic differential equation defined by a polynomial with integral coefficients. Indeed, viewing the monomials appearing in $P(z, w_0, \ldots, w_m)$ as linearly dependent analytic functions of z, we see that their Wronskian vanishes and furnishes the required polynomial $P_\sigma(z, w_0, \ldots, w_n)$, which can be seen to be not identically zero. We can therefore conclude that $T(z)$ must in fact be transcendentally transcendental and our proof is complete.

Of course, not every function with an infinite monodromy group is transcendentally transcendental; the transcendental hypergeometric functions are obvious counter examples. In fact, Theorem 1 shows that we cannot find a sufficient condition for a function to satisfy an algebraic differential equation which is expressed only in terms of branching behaviour. However, since the totality of branches at $z = z_0$ of a solution of a linear differential equation spans a finite dimensional complex vector space V, the monodromy group of such a function has a faithful representation as a group of non-singular linear transformations. We call groups with this property *linear*, and recall some theorems about them (cf. **[5]**, pages 52, 71, 112).

THEOREM. *Let G be a finitely generated linear group.*

A. *If every element has finite order, G is finite* (Burnside).

B. *If G is simple, G is finite* (Malcev).

C. *G has soluble word problem* (Rabin).

R. Camm **[2]** exhibited $2^{\aleph_0}$ non-isomorphic two-generator infinite simple groups. It follows from B that these are not linear, so if we take G in Theorem 1 to be one of these groups we obtain:

THEOREM 2. *There are $2^{\aleph_0}$ transcendentally transcendental functions on $\hat{\mathbb{C}}$ with singular points $z = a, b, c$. No two of these functions have essentially the same branching behaviour. Moreover, none of these functions has the branching behaviour of a solution of a linear differential equation.*

There are two-generator groups with unsolvable word problem (see, for example, Adyan **[1]**). Also, Adyan proved that there are two-generator infinite groups in which each element has order dividing a certain positive integer N. In view of (A), (C) and Theorem 1, we have the following results:

THEOREM 3. *There is a transcendentally transcendental function $T(z)$ with singular points $z = a, b, c$ whose branching behaviour is the same as that of an algebraic function. More precisely, there is a positive integer N such that continuing any branch of $T(z)$ along any loop avoiding $z = a, b, c$ yields the original branch if repeated N times.*

Theorem 3 generalizes the result in **[4]**.

THEOREM 4. *If a function $f(z)$ in $\mathscr{M}$ has a monodromy group with unsolvable word problem, then it cannot satisfy a linear differential equation.*

The unsolvability of the word problem suggests the following interpretation: "There is no algorithm to decide whether analytic continuation of a branch of $f(z)$ along a loop avoiding the singularities of $f(z)$ yields the chosen branch." This statement can be made rigorous if we know that the power series expansion of a branch of $f(z)$ is computable; that is, there is a computer program that gives the mth term of the decimal expansions of the real and imaginary parts of the nth coefficient, for all non-negative integers m and n. In turn, this leads to the questions: (i) Can the Anschmiegungssatz be strengthened to assert the existence of computable functions and (ii) If a function in $\mathscr{M}$ has computable coefficients, must its monodromy group have a solvable word problem?

Finally we show:

THEOREM 5. *There is a function whose monodromy group is free of rank two and whose Riemann surface has a single "infinite spiral ramp" over two of its three singular points.*

By way of contrast, the inverse of Legendre's modular function also has a free group of rank two as its monodromy group, but its Riemann surface has "countably many infinite spiral ramps" over each singularity. Invoking the existence of functions with prescribed monodromy group, we see that Theorem 5 is a purely algebraic statement:

LEMMA. *There are two infinite cycles which permute the integers and generate a free group of rank two.*

Proof. Recall that a permutation of the integers is called an infinite cycle if it is conjugate to the permutation given by $\tau(n) = n+1$, n an integer.

We begin by partitioning the integers into a family of disjoint finite subsets S_j, j an integer, of consecutive integers. The cardinality of S_j is supposed to exceed by one the length of the corresponding member $w_j(x, y)$ of a list of all reduced words on the symbols x and y. Recall that $w_j(x, y)$ is an expression of the form $x_1^{e_1} \dots x_t^{e_t}$, where each x_i is x or y, $e_i = \pm 1$ and $e_i e_{i+1} > 0$ if $x_i = x_{i+1}$; its length is t. Now, for simplicity, suppose $S_j = \{1, \dots, t+1\}$, and if $x_i = x$ write $\sigma_j(i) = i+1$ when $e_i = 1$ and $\sigma_j(i+1) = i$ when $e_i = -1$. Thus, σ_j maps a proper subset of S_j injectively into S_j. Selecting an integer P_j not in the domain of σ_j and an integer Q_j not in its image, we see that there is a permutation $\hat{\sigma}_j$ of S_j which extends σ_j, maps P_j to Q_j, and is a cycle of length $t+1$. Replacing $x_i = x$ by $x_i = y$ in this construction, we obtain a cycle $\hat{\tau}_j$ of length $t+1$ carrying M_j to N_j. Substituting $\hat{\sigma}_j$ and $\hat{\tau}_j$ for x and y in $w_j(x, y)$, we find that the permutation $w_j(\hat{\sigma}_j, \hat{\tau}_j)$ maps 1 to $t+1$, so it is not the identity. Finally, we construct an infinite cycle σ permuting the integers by setting $\sigma(P_j) = Q_{j+1}$ and $\sigma(R) = \hat{\sigma}_j(R)$ for all other $R \in S_j$, $j = 1, 2, \dots$.

Replacing $\hat{\sigma}_j$, P_j and Q_j by $\hat{\tau}_j$, M_j and N_j respectively, this construction yields a second infinite cycle τ. Moreover, $w_j(\sigma, \tau)$ is not the identity, $j = 1, 2, \dots$, so σ and τ generate a free group of rank two as desired.

Noting that $\sigma^* = \mu^{-1} \sigma \mu$ and $\tau^* = \mu^{-1} \tau \mu$ generate a free group of rank two for all permutations μ of the integers, we see that one of the two infinite cycles we have constructed may be prescribed arbitrarily. In fact, with obvious modifications of the construction of the collection of S_j we have:

THEOREM 6. *If τ is a permutation of the integers that has infinite order, then there is a permutation σ of the integers such that τ and σ generate a free group of rank two.*

COROLLARY. *If τ is a permutation of the integers that has infinite order, then there is a function in $\mathscr{M}$ whose monodromy group is free of rank two and is generated by τ and another permutation, σ.*

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