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Telescopers for differential forms with one parameter

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Abstract

Telescopers for a function are linear differential (resp. difference) operators annihilating the definite integral (resp. definite sum) of this function. They play a key role in Wilf–Zeilberger theory and algorithms for computing them have been extensively studied in the past 30 years. In this paper, we introduce the notion of telescopers for differential forms with D-finite function coefficients. These telescopers appear in several areas of mathematics, for instance parametrized differential Galois theory and mirror symmetry. We give a sufficient and necessary condition for the existence of telescopers for a differential form and describe a method to compute them if they exist. Algorithms for verifying this condition are also given.

Keywords Telescopers · Differential forms · D-finite elements · Parametrized Poincaré's lemma

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1 Introduction

In the Wilf–Zeilberger theory, telescopers usually refer to the operators in the output of the method of creative telescoping, which are linear differential (resp. difference)

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operators annihilating the definite integrals (resp. the definite sums) of the input functions. The telescopers have emerged at least from the work of Euler [22] and have found many applications in the various areas of mathematics such as combinatorics, number theory, knot theory as well as others (see Sect. 7 of [26] for details or [28] for applications in Feyman integrals). In particular, telescopers for a function are often used to prove the identities involving this function or even obtain a simpler expression for the definite integral or sum of this function. As a clever and algorithmic process for constructing telescopers, creative telescoping firstly appeared as a term in the essay of van der Poorten on Apréy's proof of the irrationality of $\zeta(3)$ [39]. However, it was Zeilberger and his collaborators [3, 36, 44, 45, 48] in the early 1990s who equipped creative telescoping with a concrete meaning and formulated it as an algorithmic tool. Since then, algorithms for creative telescoping have been extensively studied. Based on the techniques used in the algorithms, the existing algorithms are divided into four generations, see [15] for the details. Most recent algorithms are called reduction-based algorithms which were first introduced by Bostan et al. [7] and further developed, for example, in [8, 9, 16, 20]. The termination of these algorithms relies on the existence of telescopers. The question for which input functions the algorithms will terminate has been answered in [1, 2, 11, 21, 46] for several classes of functions such as rational functions and hypergeometric functions as well as others. The algorithmic framework for creative telescoping is now called the Wilf–Zeilberger theory.

Most algorithms for creative telescoping focus on the case of one bivariate function as input. There are only a few algorithms which deal with multivariate case (see, for example, [10, 12, 14, 27]). It is still a challenge to develop the multivariate analogue of the existing algorithms (see Sect. 5 of [15]). In the language of differential forms (with m variables and one parameter), the results in [12, 27] dealt with the cases of differential 1-forms and differential m-forms respectively. On the other hand, in the applications to other domains such as mirror symmetry (see [30, 33, 34]), one needs to deal with the case of differential p-forms with $1 \leq p \leq m$. Below is an example.

Example 1 Consider the following one-parameter family of the quintic polynomials

$$W(t) = \frac{1}{5}\left(x_1^5 + x_2^5 + x_3^5 + x_4^5 + x_5^5\right) - t x_1 x_2 x_3 x_4 x_5$$

where t is a parameter. Set

$$\omega = \sum_{i=1}^{5} \frac{(-1)^{i-1} x_i}{W(t)} \mathrm{d}x_1 \wedge \cdots \wedge \widehat{\mathrm{d}x_i} \wedge \cdots \wedge \mathrm{d}x_5.$$

To obtain the Picard–Fuchs equation for the mirror quintic, the geometers want to compute a fourth order linear differential operator L in t and ∂_t such that $L(\omega) = \mathrm{d}\eta$ for some differential 3-form η. Here one has that

$$L = (1 - t^5)\frac{\partial^4}{\partial t^4} - 10t^4 \frac{\partial^3}{\partial t^3} - 25t^3 \frac{\partial^2}{\partial t^2} - 15t^2 \frac{\partial}{\partial t} - 1.$$

Set $\theta_t = t\partial/\partial t$. Then

$$\tilde{L} = -\frac{1}{5^4}L\frac{1}{t} = \theta_t^4 - 5t(5\theta_t + 1)(5\theta_t + 2)(5\theta_t + 3)(5\theta_t + 4)$$

and the equation $\tilde{L}(y) = 0$ is the required Picard–Fuchs equation.

We call the operator L appearing in the above example a telescoper for the differential form ω (see Definition 4). In this paper, we study the telescopers for differential forms with D-finite function coefficients. Instead of the geometric method used in [30, 33, 34], we provide an algebraic treatment. We give a sufficient and necessary condition guaranteeing the existence of telescopers and describe a method to compute them if they exist. In addition, we also present algorithms to verify this condition.

The rest of this paper is organized as follows. In Sect. 2, we recall differential forms with D-finite function coefficients and introduce the notion of telescopers for differential forms. In Sect. 3, we give a sufficient and necessary condition for the existence of telescopers, which can be considered as a parametrized version of Poincaré's lemma on differential manifolds. In Sect. 4, we give two algorithms for verifying the condition presented in Sect. 3.

Throughout this paper, we assume the following notations:

- $\partial_t = \frac{\partial}{\partial t}$, the usual derivation with respect to t;
- $\partial_{x_i} = \frac{\partial}{\partial x_i}$, the usual derivation with respect to x_i;
- $\mathbf{x} = \{x_1, \ldots, x_n\}$;
- $\boldsymbol{\partial}_{\mathbf{x}} = \{\partial_{x_1}, \ldots, \partial_{x_n}\}$.

The following formulas will also be frequently used:

$$\partial_x^\mu x^\nu = \begin{cases} \nu(\nu-1)\cdots(\nu-\mu+1)x^{\nu-\mu} + P\partial_x, & \nu \geq \mu \\ P\partial_x, & \nu < \mu \end{cases} \tag{1}$$

$$x^\mu \partial_x^\nu = \begin{cases} (-1)^\nu \mu(\mu-1)\cdots(\mu-\nu+1)x^{\mu-\nu} + \partial_x P, & \mu \geq \nu \\ \partial_x P, & \mu < \nu \end{cases} \tag{2}$$

where $P \in k\langle x, \partial_x\rangle$.

2 *D*-finite elements and differential forms

Throughout this paper, let k be an algebraically closed field of characteristic zero and let K be the differential field $k(t, x_1, \ldots, x_n)$ with the derivations $\partial_t, \partial_{x_1}, \ldots, \partial_{x_n}$. Let $\mathfrak{D} = K\langle\partial_t, \boldsymbol{\partial}_{\mathbf{x}}\rangle$ be the ring of linear differential operators with coefficients in K. For $S \subset \{t, \mathbf{x}, \partial_t, \boldsymbol{\partial}_{\mathbf{x}}\}$, denote by $k\langle S\rangle$ the subalgebra over k of $\mathfrak{D}$ generated by S. For brevity, we denote $k\langle t, \mathbf{x}, \partial_t, \boldsymbol{\partial}_{\mathbf{x}}\rangle$ by $\mathfrak{W}$. Let $\mathcal{U}$ be the universal differential extension of K in which every algebraic differential equation having a solution in an extension of $\mathcal{U}$ has a solution in $\mathcal{U}$ (see page 133 of [24] for more precise description).

Definition 2 An element $f \in \mathcal{U}$ is said to be D-finite over K if for every $\delta \in \{\partial_t, \partial_{x_1}, \ldots, \partial_{x_n}\}$, there is a nonzero operator $L_\delta \in K\langle\delta\rangle$ such that $L_\delta(f) = 0$.

Denote by R the ring of D-finite elements over K, and by $\mathcal{M}$ a free R-module of rank n with base $\{\mathfrak{a}_1, \ldots, \mathfrak{a}_n\}$. Define a map $\mathfrak{D} \times \mathcal{M} \to \mathcal{M}$ given by

$$\left(L, \sum_{i=1}^{n} f_i \mathfrak{a}_i\right) \longrightarrow L\left(\sum_{i=1}^{n} f_i \mathfrak{a}_i\right) := \sum_{i=1}^{n} L(f_i)\mathfrak{a}_i.$$

This map endows $\mathcal{M}$ with a left $\mathfrak{D}$-module structure. Let

$$\bigwedge(\mathcal{M}) = \bigoplus_{i=0}^{n} \bigwedge^i(\mathcal{M})$$

be the exterior algebra of $\mathcal{M}$, where $\bigwedge^i(\mathcal{M})$ denotes the i-th homogeneous part of $\bigwedge(\mathcal{M})$ as a graded R-algebra. We call an element in $\bigwedge^i(\mathcal{M})$ an i-form. Note that $\bigwedge(\mathcal{M})$ inherites a left $\mathfrak{D}$-module structure from $\mathcal{M}$. In fact, for $L \in \mathfrak{D}$ and $\omega = \sum f_{s_1,\ldots,s_i} \mathfrak{a}_{s_1} \wedge \cdots \wedge \mathfrak{a}_{s_i} \in \bigwedge^i(\mathcal{M})$, one can define

$$L(\omega) = \sum L(f_{s_1,\ldots,s_i})\mathfrak{a}_{s_1} \wedge \cdots \wedge \mathfrak{a}_{s_i}$$

and for $\omega = \sum_i \omega_i$ with $\omega_i \in \bigwedge^i(\mathcal{M})$, define $L(\omega) = \sum_i L(\omega_i)$. Let $\mathrm{d} : R \to \mathcal{M}$ be a map defined as

$$\mathrm{d}f = \partial_{x_1}(f)\mathfrak{a}_1 + \cdots + \partial_{x_n}(f)\mathfrak{a}_n$$

for any $f \in R$. Then d is a derivation over k. Note that for each $i = 1, \ldots, n$ one has that $\mathrm{d}x_i = \mathfrak{a}_i$. Hence in the rest of this paper we shall use $\{\mathrm{d}x_1, \ldots, \mathrm{d}x_n\}$ instead of $\{\mathfrak{a}_1, \ldots, \mathfrak{a}_n\}$. The map d can be extended to a derivation on $\bigwedge(\mathcal{M})$ which is defined recursively as

$$\mathrm{d}(\omega_1 \wedge \omega_2) = \mathrm{d}\omega_1 \wedge \omega_2 + (-1)^i \omega_1 \wedge \mathrm{d}\omega_2$$

for any $\omega_1 \in \bigwedge^i(\mathcal{M})$ and $\omega_2 \in \bigwedge^j(\mathcal{M})$. For detailed definitions on exterior algebra and differential forms, we refer the readers to Chapter 19 of [29] and Chapter 1 of [43] respectively. As the usual differential forms, we introduce the following definition.

Definition 3 Let $\omega \in \bigwedge(\mathcal{M})$ be a form.

(1) ω is said to be closed if $\mathrm{d}\omega = 0$, and exact if there is $\eta \in \bigwedge(\mathcal{M})$ such that $\omega = \mathrm{d}\eta$.
(2) ω is said to be ∂_t-closed (resp. ∂_t-exact) if there is a nonzero $L \in k(t)\langle\partial_t\rangle$ such that $L(\omega)$ is closed (resp. exact).

Definition 4 Assume that $\omega \in \bigwedge(\mathcal{M})$. A nonzero $L \in k(t)\langle\partial_t\rangle$ is called a telescoper for ω if $L(\omega)$ is exact.

3 Parametrized Poincaré's lemma

The famous Poincaré's lemma states that if B is an open ball in $\mathbb{R}^n$, any smooth closed i-form ω defined on B is exact, for any integer i with $1 \leq i \leq n$. In this section, we shall prove the following lemma which can be viewed as a parametrized analogue of Poincaré's lemma for $\bigwedge(\mathcal{M})$.

Lemma 5 (Parameterized Poincaré's lemma) *Let $\omega \in \bigwedge^p(\mathcal{M})$. If ω is ∂_t-closed then it is ∂_t-exact.*

To prove the above lemma, we need some lemmas. Note that the annihilated ideal of a D-finite element in the Weyl algebra $k\langle t, x_1, \ldots, x_n, \partial_t, \partial_{x_1}, \ldots, \partial_{x_n}\rangle$ is holonomic, as demonstrated in [19] (or refer to [5, 23] for the proofs). Using a dimension argument, the holonomic property implies the existence of a specific operator L in the annihilated ideal. However, the proofs of two lemmas below are constructive, providing algorithms for computing L.

Lemma 6 (Lipshitz's lemma, see Lemma 3 of [32]) *Assume that f is a D-finite element over $k(\mathbf{x})$. For each $i = 1, 3, 4, \ldots, n$, there is a nonzero operator $L \in k(x_1, x_3, \ldots, x_n)\langle\partial_{x_2}, \partial_{x_i}\rangle$ such that $L(f) = 0$.*

The following lemma is a generalization of Lipshitz's lemma.

Lemma 7 *Assume that $f_1, \ldots, f_m$ are D-finite elements over $k(\mathbf{x}, t)$ and*

$$S \subset \{t, x_1, \ldots, x_n, \partial_t, \partial_{x_1}, \ldots, \partial_{x_n}\}$$

with $|S| > n + 1$. Then one can compute a nonzero operator L in $k\langle S\rangle$ such that $L(f_i) = 0$ for all $i = 1, \ldots, m$.

Proof For each $\delta \in \{\partial_t, \partial_{x_1}, \ldots, \partial_{x_n}\}$ and $i = 1, \ldots, m$, let $T_{i,\delta}$ be a nonzero operator in $K\langle\delta\rangle$ such that $T_{i,\delta}(f_i) = 0$. Set T_δ to be the least common left multiple of $T_{1,\delta}, \ldots, T_{m,\delta}$. Then $T_\delta(f_i) = 0$ for all $i = 1, \ldots, m$ and $\delta \in \{\partial_t, \partial_{x_1}, \ldots, \partial_{x_n}\}$. The lemma then follows from an argument similar to that in the proof of Lipshitz's lemma. □

Remark 8 Lemma 7 originally appears in [47] (see Lemma 4.1), where Zeilberger proves the existence of the operator L in the setting of Weyl algebra and gives an algorithm to compute L in the case of two variables. Furthermore, there is a Mathematica package called HolonomicFunctions developed by Koutschan which allows one to compute L (see [25]).

Lemma 9 *Assume that $f_1, \ldots, f_m$ are D-finite over $k(\mathbf{x}, t)$, $I, J \subset \{1, \ldots, n\}$ and $I \cap J = \emptyset$. Assume further that $V \subset \{x_i, \partial_{x_i} | i \in \{1, \ldots, n\}\backslash(I \cup J)\}$ with $|V| = n - |I| - |J|$. Then one can compute an operator P of the form*

$$L + \sum_{i\in I} \partial_{x_i} M_i + \sum_{j\in J} N_j \partial_{x_j}$$

such that $P(f_l)=0$ *for all* $l=1,\ldots,m$*, where* L *is a nonzero operator in* $k\langle\{t,\partial_t\}\cup V\}\rangle$, $M_i, N_j\in\mathfrak{W}$ *and* N_j *is free of* x_i *for all* $i\in I$ *and* $j\in J$.

Proof Without loss of generality, we assume that $I=\{1,\ldots,r\}$ and $J=\{r+1,\ldots,r+s\}$ where $r=|I|$ and $s=|J|$. Let

$$S=\{t,\partial_t\}\cup\{\partial_{x_i}|i\in I\}\cup\{x_j|j=r+1,\ldots,r+s\}\cup V.$$

Then $|S|=n+2>n+1$. By Lemma 7, one can compute a $T\in k\langle S\rangle\backslash\{0\}$ such that $T(f_l)=0$ for all $l=1,\ldots,m$. Write

$$T=\sum_{\mathbf{d}=(d_1,\ldots,d_r)\in\Gamma_1}\partial_{x_1}^{d_1}\cdots\partial_{x_r}^{d_r}T_{\mathbf{d}}$$

where $T_{\mathbf{d}}\in k\langle\{t,\partial_t,x_{r+1},\ldots,x_{r+s}\}\cup V\}\rangle\backslash\{0\}$ and Γ_1 is a finite subset of $\mathbb{Z}^r$. Let $\bar{\mathbf{d}}=(\bar{d}_1,\ldots,\bar{d}_r)$ be the minimal element of Γ_1 with respect to the lex order on $\mathbb{Z}^r$. Multiplying T by $\prod_{i=1}^r x_i^{\bar{d}_i}$ on the left and using the formula (2) yield that

$$\left(\prod_{i=1}^r x_i^{\bar{d}_i}\right)T=\alpha T_{\bar{\mathbf{d}}}+\sum_{i=1}^r\partial_{x_i}\tilde{T}_i \tag{3}$$

where α is a nonzero integer and $\tilde{T}_i\in k\langle S\cup\{x_i|i\in I\}\rangle$. Write

$$T_{\bar{\mathbf{d}}}=\sum_{\mathbf{e}=(e_1,\ldots,e_s)\in\Gamma_2}L_{\mathbf{e}}x_{r+1}^{e_1}\cdots x_{r+s}^{e_s}$$

where $L_{\mathbf{e}}\in k\langle\{t,\partial_t\}\cup V\rangle\backslash\{0\}$ and Γ_2 is a finite subset of $\mathbb{Z}^s$. Let $\bar{\mathbf{e}}=(\bar{e}_1,\ldots,\bar{e}_s)$ be the maximal element of Γ_2 with respect to the lex order on $\mathbb{Z}^s$. Multiplying $T_{\bar{\mathbf{d}}}$ by $\prod_{i=1}^s\partial_{x_{r+i}}^{\bar{e}_i}$ on the left and using the formula (1) yield that

$$\left(\prod_{i=1}^s\partial_{x_{r+i}}^{\bar{e}_i}\right)T_{\bar{\mathbf{d}}}=\beta L_{\bar{\mathbf{e}}}+\sum_{j\in J}\tilde{L}_j\partial_{x_j} \tag{4}$$

where $\tilde{L}_i\in k\langle\{t,\partial_t,x_{r+1},\ldots,x_{r+s},\partial_{x_{r+1}},\ldots,\partial_{x_{r+s}}\}\cup V\rangle$ and β is a nonzero integer. Combining (3) with (4) yields the required operator P. $\square$

Corollary 10 *Assume that* $f_1,\ldots,f_m$ *are D-finite over* $k(\mathbf{x},t)$*,* J *is a subset of* $\{1,\ldots,n\}$ *and* $V\subset\{x_i,\partial_{x_i}|i\in\{1,\ldots,n\}\backslash J\}$ *with* $|V|=n-|J|$*. Assume further that* $\partial_{x_j}(f_l)=0$ *for all* $j\in J$ *and* $l=1,\ldots,m$*. Then one can compute a nonzero* $L\in k\langle\{t,\partial_t\}\cup V\rangle$ *such that* $L(f_l)=0$ *for all* $l=1,\ldots,m$.

Proof In Lemma 9, set $I=\emptyset$. $\square$

The main result of this section is the following theorem which can be viewed as a generalization of Corollary 10 to differential forms. To describe and prove this theorem, let us recall some notation from the first chapter of [43]. For any $f \in R$, we define $\mathrm{d}_0(f) = 0$ and

$$\mathrm{d}_s(f) = \partial_{x_1}(f)\mathrm{d}x_1 + \cdots + \partial_{x_s}(f)\mathrm{d}x_s$$

for $s \in \{1, 2, \ldots, n\}$. We can extend d_s to the module $\bigwedge(\mathcal{M})$ in a natural way. Precisely, let $\omega = \sum_{i=1}^m f_i \mathfrak{m}_i$ where $\mathfrak{m}_i$ is a monomial in $\mathrm{d}x_1, \ldots, \mathrm{d}x_n$. Then $\mathrm{d}_0(\omega) = 0$ and

$$\mathrm{d}_s(\omega) = \sum_{i=1}^m \sum_{j=1}^s \partial_{x_j}(f_i)\mathrm{d}x_j \wedge \mathfrak{m}_i = \sum_{j=1}^s \mathrm{d}x_j \wedge \partial_{x_j}(\omega).$$

By definition, one sees that

$$\mathrm{d}_s(u \wedge \mathrm{d}x_s) = \mathrm{d}_{s-1}(u) \wedge \mathrm{d}x_s \quad \text{and} \quad \mathrm{d}_s(u) = \mathrm{d}_{s-1}(u) + \mathrm{d}x_s \wedge \partial_{x_s}(u).$$

Theorem 11 *Assume that $0 \le s \le n$, $V \subset \{x_{s+1}, \ldots, x_n, \partial_{x_{s+1}}, \ldots, \partial_{x_n}\}$ with $|V| = n-s$ and $\omega \in \bigwedge^p(\mathcal{M})$. If $\mathrm{d}_s\omega = 0$, then one can compute a nonzero $L \in k\langle\{t, \partial_t\} \cup V\rangle$ and $\mu \in \bigwedge^{p-1}(\mathcal{M})$ such that $L(\omega) = \mathrm{d}_s\mu$.*

Remark 12 1. If $p = 0$, then $\omega = f \in R$ and $\mathrm{d}_s f = 0$ if and only if $s = 0$ or $\partial_{x_i}(f) = 0$ for all $1 \le i \le s$ if $s > 0$. Therefore Corollary 10 is a special case of Theorem 11.
2. Note that the parametrized Poincaré's lemma is just the special case of Theorem 11 when $s = n$.

Proof We proceed by induction on s. Assume that $s = 0$ and write

$$\omega = \sum_{i=1}^m f_i \mathfrak{m}_i$$

where $\mathfrak{m}_i$ a monomial in $\mathrm{d}x_1, \mathrm{d}x_2, \ldots, \mathrm{d}x_n$ and $f_i \in R$. By Corollary 10 with $I = \emptyset$, one can compute a nonzero $L \in k\langle\{t, \partial_t\} \cup V\rangle$ such that $L(f_i) = 0$ for all $i = 1, \ldots, m$. Then one has that

$$L(\omega) = \sum_{i=1}^m L(f_i)\mathfrak{m}_i = 0.$$

This proves the base case. Now assume that the theorem holds for $s < \ell$ and consider the case $s = \ell$. Write

$$\omega = u \wedge \mathrm{d}x_\ell + v$$

where both u and v do not involve $\mathrm{d}x_\ell$. Then the assumption $\mathrm{d}_\ell \omega = 0$ implies that

$$\mathrm{d}_{\ell-1}u \wedge \mathrm{d}x_\ell + \mathrm{d}_\ell v = \mathrm{d}_{\ell-1}u \wedge \mathrm{d}x_\ell + \mathrm{d}_{\ell-1}v + \mathrm{d}x_\ell \wedge \partial_{x_l}(v) = 0.$$

Since all of $\mathrm{d}_{\ell-1}u, \mathrm{d}_{\ell-1}v, \partial_{x_\ell}(v)$ do not involve $\mathrm{d}x_\ell$, one has that $\mathrm{d}_{\ell-1}v = 0$ and $\mathrm{d}_{\ell-1}(u) - \partial_{x_\ell}(v) = 0$. By the induction hypothesis, one can compute a nonzero $\tilde{L} \in k\langle \{t, x_\ell, \partial_t\} \cup V\rangle$ and $\tilde{\mu} \in \bigwedge^{p-1}(\mathcal{M})$ such that

$$\tilde{L}(v) = \mathrm{d}_{\ell-1}(\tilde{\mu}). \tag{5}$$

We claim that $\tilde{L}$ can be chosen to be free of x_ℓ. Write

$$\tilde{L} = \sum_{j=0}^{d} N_j x_\ell^d$$

where $N_j \in k\langle \{t, \partial_t\} \cup V\rangle$ and $N_d \neq 0$. Multiplying $\tilde{L}$ by $\partial_{x_\ell}^d$ on the left and using the formula (2) yield that

$$\partial_{x_\ell}^d \tilde{L} = \sum_{j=0}^{d} N_j \partial_{x_\ell}^d x_\ell^j = \alpha N_d + \tilde{N}\partial_{x_\ell} \tag{6}$$

where α is a nonzero integer and $\tilde{N} \in k\langle \{t, x_\ell, \partial_t, \partial_{x_\ell}\} \cup V\rangle$. The equalities (5) and (6) together with $\partial_{x_\ell}(v) = \mathrm{d}_{\ell-1}(\tilde{u})$ yield that $N_d(v) = \mathrm{d}_{\ell-1}(\pi)$ for some $\pi \in \bigwedge^{p-1}(\mathcal{M})$. This proves the claim. Now one has that

$$\tilde{L}(\omega) = \tilde{L}(u) \wedge \mathrm{d}x_\ell + \mathrm{d}_{\ell-1}(\tilde{\mu}) = \tilde{L}(u) \wedge \mathrm{d}x_\ell + \mathrm{d}x_\ell \wedge \partial_{x_\ell}(\tilde{\mu}) + \mathrm{d}_\ell(\tilde{\mu}).$$

Since $\tilde{L}$ is free of $x_1, \ldots, x_\ell$, $\tilde{L}\mathrm{d}_\ell = \mathrm{d}_\ell \tilde{L}$. This implies that

$$\begin{aligned} 0 = \tilde{L}(\mathrm{d}_\ell(\omega)) = \mathrm{d}_\ell(\tilde{L}(\omega)) &= \mathrm{d}_{\ell-1}(\tilde{L}(u)) \wedge \mathrm{d}x_\ell + \mathrm{d}x_\ell \wedge \mathrm{d}_{\ell-1}(\partial_{x_\ell}(\tilde{\mu})) \\ &= \mathrm{d}_{\ell-1}\left(\tilde{L}(u) - \partial_{x_\ell}(\tilde{\mu})\right) \wedge \mathrm{d}x_\ell. \end{aligned}$$

Note that $\tilde{\mu}$ can always be chosen to be free of $\mathrm{d}x_\ell$. Hence one has that $\mathrm{d}_{\ell-1}(\tilde{L}(u) - \partial_{x_\ell}(\tilde{\mu})) = 0$. By the induction hypothesis, one can compute a nonzero $\bar{L} \in k\langle \{t, \partial_{x_\ell}, \partial_t\} \cup V\rangle$ and $\bar{\mu} \in \bigwedge^{p-1}(\mathcal{M})$ such that

$$\bar{L}\left(\tilde{L}(u) - \partial_{x_\ell}(\tilde{\mu})\right) = \mathrm{d}_{\ell-1}(\bar{\mu}). \tag{7}$$

Write

$$\bar{L} = \sum_{j=e_1}^{e_2} \partial_{x_\ell}^j M_j$$

where $M_j \in k\langle\{t, \partial_t\} \cup V\rangle$ and $M_{e_1} \neq 0$. Multiplying $\bar{L}$ by $x_\ell^{e_1}$ on the left and using the formula (2) yield that

$$x_\ell^{e_1}\bar{L} = \beta M_{e_1} + \partial_{x_\ell}\tilde{M}$$

where β is a nonzero integer and $\tilde{M} \in k\langle\{t, \partial_t, \partial_{x_\ell}, x_\ell\} \cup V\rangle$. Hence applying $x_\ell^{e_1}$ to the equality (7), one gets that

$$\beta M_{e_1}\left(\tilde{L}(u) - \partial_{x_\ell}(\tilde{\mu})\right) = \mathrm{d}_{\ell-1}(x_\ell^{e_1}\bar{\mu}) + \partial_{x_\ell}\left(\tilde{M}\left(\tilde{L}(u) - \partial_{x_\ell}(\tilde{\mu})\right)\right).$$

Set $L = \beta M_{e_1}\tilde{L}$. Then one has that

$$\begin{aligned} L(\omega) &= \beta M_{e_1}\left((\tilde{L}(u) - \partial_{x_\ell}(\tilde{\mu})) \wedge \mathrm{d}x_\ell + \mathrm{d}_\ell(\tilde{\mu})\right) \\ &= \left(\beta M_{e_1}\left(\tilde{L}(u) - \partial_{x_\ell}(\tilde{\mu})\right)\right) \wedge \mathrm{d}x_\ell + \mathrm{d}_\ell(\beta M_{e_1}(\tilde{\mu})) \\ &= \mathrm{d}_{\ell-1}(x_\ell^{e_1}\bar{\mu}) \wedge \mathrm{d}x_\ell + \partial_{x_\ell}\tilde{M}\left(\tilde{L}(u) - \partial_{x_\ell}(\tilde{\mu})\right) \wedge \mathrm{d}x_\ell + \mathrm{d}_\ell(\beta M_{e_1}(\tilde{\mu})) \\ &= \mathrm{d}_\ell\left(x_\ell^{e_1}\bar{\mu} + \tilde{M}\left(\tilde{L}(u) - \partial_{x_\ell}(\tilde{\mu})\right) + \beta M_{e_1}(\tilde{\mu})\right). \end{aligned}$$

The last equality holds because

$$\mathrm{d}_{\ell-1}\left(\tilde{M}\left(\tilde{L}(u) - \partial_{x_\ell}(\tilde{\mu})\right)\right) = \tilde{M}\mathrm{d}_{\ell-1}\left(\tilde{L}(u) - \partial_{x_\ell}(\tilde{\mu})\right) = 0.$$

□

Remark 13 Lemma 5 can be derived from the finiteness of the de Rham cohomology groups of D-modules in the Bernstein class. To see this, let ω be a differential s-form with coefficients in R and let M be the D-module generated by all coefficients of ω and all derivatives of these coefficients with respect to ∂_t. By Proposition 5.2 on page 12 of [6], M is a D-module in the Bernstein class. Assume that ω is closed. Then $\partial_t^j(\omega) \in H_{DR}^s(M)$, the s-th de Rham cohomology group of M, for all nonnegative integers j. By Theorem 6.1 on page 16 of [6], $H_{DR}^s(M)$ is of finite dimension over $k(t)$. This implies that there are $a_0, \ldots, a_m \in k(t)$ such that $\sum_{j=0}^m a_j\partial_t^j(\omega) = 0$ in $H_{DR}^s(M)$, i.e., $\sum_{j=0}^m a_j\partial_t^j(\omega)$ is exact. This proves the existence of telescopers for the ∂_t-closed differential forms. However the proof of Theorem 11 is constructive and it provides a method to compute a telescoper if it exists.

The proof of Theorem 11 can be summarized as the following algorithm.

Algorithm 14 Input: $\omega \in \bigwedge^p(\mathcal{M})$ and $V \subset \{x_i, \partial_{x_i} | i = s+1, \ldots, n\}$ satisfying that $\mathrm{d}_s(\omega) = 0$ and $|V| = n - s$.

Output: a nonzero $L \in k\langle\{t, \partial_t\} \cup V\rangle$ such that $L(\omega) = \mathrm{d}_s(\mu)$.

1. If $\omega \in R$, then by Corollary 10, compute a nonzero $L \in k\langle\{t, \partial_t\} \cup V\rangle$ such that $L(\omega) = 0$. Return L.

2. Write $\omega = u \wedge \mathrm{d}x_s + v$ with u, v not involving $\mathrm{d}x_s$.
3. Call Algorithm 14 with v and $V \cup \{x_s\}$ as inputs and let $\tilde{L}$ be the output.
   (a) Write $\tilde{L} = \sum_{j=0}^{d} N_j x_s^j$ with $N_j \in k\langle\{t, \partial_t\} \cup V\rangle$ and $N_d \neq 0$.
   (b) Compute a $\tilde{\mu} \in \bigwedge^{p-1}(\mathcal{M})$ such that $N_d(v) = \mathrm{d}_{s-1}(\tilde{\mu})$.
4. Write $N_d(\omega) = (N_d(u) - \partial_{x_s}(\tilde{\mu})) \wedge \mathrm{d}x_s + \mathrm{d}_s(\tilde{\mu})$.
5. Call Algorithm 14 with $N_d(u) - \partial_{x_s}(\tilde{\mu})$ and $V \cup \{\partial_{x_s}\}$ as inputs and let $\bar{L}$ be the output.
6. Write $\bar{L} = \sum_{j=e_1}^{e_2} \partial_{x_s}^j M_j$ with $M_j \in k\langle\{t, \partial_t\} \cup V\rangle$ and $M_{e_1} \neq 0$.
7. Return $M_{e_1} N_d$.

4 The existence of telescopers

It is easy to see that if a differential form is ∂_t-exact then it is ∂_t-closed. Therefore Lemma 5 implies that given an $\omega \in \bigwedge^p(\mathcal{M})$, to decide whether it has a telescoper, it suffices to decide whether there is a nonzero $L \in k(t)\langle\partial_t\rangle$ such that $L(\mathrm{d}\omega) = 0$. Suppose that

$$\mathrm{d}\omega = \sum_{1 \leq i_1 < \cdots < i_{p+1} \leq n} a_{i_1,\ldots,i_{p+1}} \mathrm{d}x_{i_1} \cdots \mathrm{d}x_{p+1}, \quad a_{i_1,\ldots,a_{p+1}} \in \mathcal{U}.$$

Then $L(\mathrm{d}\omega) = 0$ if and only if $L(a_{i_1,\ldots,i_{p+1}}) = 0$ for all $1 \leq i_1 < \cdots < i_{p+1} \leq n$. So the existence problem of telescopers can be reduced to the following problem.

Problem 15 *Given an element $f \in R$ and its minimal annihilating operator $P \in K\langle\partial_t\rangle$, decide whether there exists a nonzero $L \in k(t)\langle\partial_t\rangle$ such that $L(f) = 0$.*

Example 16 Let $W(t)$ be as in Example 1. Then $W(t) \in R$ since it is rational in $x_1, \ldots, x_5, t$. Its minimal annihilating operator in $K\langle\partial_t\rangle$ is

$$P = \partial_t + \frac{x_1x_2x_3x_4x_5}{\frac{1}{5}(x_1^5 + x_2^5 + x_3^5 + x_4^5 + x_5^5) - tx_1x_2x_3x_4x_5}.$$

Set $L = \partial_t^2$. Then L is a nonzero operator in $k(t)\langle\partial_t\rangle$ such that $L(W(t)) = 0$.

Note that f is annihilated by a nonzero $L \in k(t)\langle\partial_t\rangle$ if and only if P is a right-hand factor of L, i.e. $L = QP$ for some $Q \in K\langle\partial_t\rangle$. For convenience, we introduce the following definition.

Definition 17 An operator $P \in K\langle\partial_t\rangle$ is called $(\mathbf{x}, t)$-separable if there is a nonzero $L \in k(t)\langle\partial_t\rangle$ such that $L = QP$ for some $Q \in K\langle\partial_t\rangle$.

Problem 15 then is reduced to the following one.

Problem 18 *Given a $P \in K\langle\partial_t\rangle\backslash\{0\}$, decide whether P is $(\mathbf{x}, t)$-separable.*

The above problem was called *the separability problem* in [13] that investigates the possibility of eliminating the parameters $\mathbf{x}$ (not t) by left-multiplying the operator P by a specific operator. Many special cases of the separability problem had been studied in [13] and we will address the general D-finite case in this section. A similar idea has been successfully applied in the desingularization of linear differential operators. In this process, multiplying P by an operator on the left enables the removal of factors of the leading coefficient of P that correspond to the removable singularities, see for example [17, 18]. It is important to note that desingularization focuses on removing factors (in t) within the leading coefficient, whereas in our case, the objective is to eliminate the parameters $\mathbf{x}$ present in all coefficients. The rest of this paper is aimed at developing an algorithm to solve the above problem. Let us first investigate the solutions of $(\mathbf{x}, t)$-separable operators.

Notation 19

$$C_t := \{c \in \mathcal{U} \mid \partial_t(c) = 0\}, \quad C_{\mathbf{x}} := \{c \in \mathcal{U} \mid \forall x \in \mathbf{x}, \partial_x(c) = 0\}.$$

Assume that $L \in k(t)\langle\partial_t\rangle \setminus \{0\}$. By Corollary 1.2.12 of [38], the solution space of $L(y) = 0$ in $\mathcal{U}$ is a C_t-vector space of dimension $\mathrm{ord}(L)$. Moreover we have the following lemma.

Lemma 20 *If $L \in k(t)\langle\partial_t\rangle \setminus \{0\}$, then the solution space of $L(y) = 0$ in $\mathcal{U}$ has a basis in $C_{\mathbf{x}}$.*

Proof Let A_0 be the companion matrix of $L(y) = 0$, i.e.

$$A_0 = \begin{pmatrix} 0 & 1 & & & \\ & 0 & 1 & & \\ & & \ddots & \ddots & \\ & & & 0 & 1 \\ -a_0 & -a_1 & -a_2 & \dots & -a_{m-1} \end{pmatrix}$$

where $m = \mathrm{ord}(L)$ and $L = \partial_t^m + a_{m-1}\partial_t^{m-1} + \cdots + a_0$. Set $A_i = 0$ for all $i = 1, \dots, n$. Let $\partial_0 = \partial_t$, $\partial_i = \partial_{x_i}$ for $i = 1, \dots, n$. Then the system

$$\partial_0(Y) = A_0 Y, \quad \partial_1(Y) = A_1 Y, \dots, \quad \partial_n(Y) = A_n Y$$

satisfies the integrability conditions:

$$\partial_i(A_j) - \partial_j(A_i) = A_i A_j - A_j A_i$$

for all $0 \le i < j \le n$. Therefore there is a solution V in $\mathrm{GL}_m(\mathcal{U})$. Let $\mathbf{v}$ be the first row of V. Note that $\det(V)$ is the Wronskian determinant of $\mathbf{v}$ and $\det(V) \neq 0$. These imply that $\mathbf{v}$ is a basis of the solution space of $L(y) = 0$. Since $\partial_i(\mathbf{v}) = 0$ for all $1 \le i \le n$, $\mathbf{v}$ has entries in $C_{\mathbf{x}}$. □

As a consequence, we have the following corollary.

Corollary 21 *Assume that $P \in K\langle \partial_t \rangle \setminus \{0\}$. Then P is $(\mathbf{x}, t)$-separable if and only if the solutions of $P(y) = 0$ in $\mathcal{U}$ are of the form*

$$\sum_{i=1}^{s} g_i h_i, \quad g_i \in C_t, \quad h_i \in C_{\mathbf{x}} \cap \{f \in \mathcal{U} \mid Q(f) = 0\} \tag{8}$$

for some $Q \in K\langle \partial_t \rangle$.

Proof The “only if" part is a direct consequence of Lemma 20. For the “if" part, one only need to prove that if $h \in C_{\mathbf{x}} \cap \{f \in \mathcal{U} \mid Q(f) = 0\}$ then h is annihilated by a nonzero operator in $k(t)\langle \partial_t \rangle$. Suppose that $h \in C_{\mathbf{x}} \cap \{f \in \mathcal{U} \mid Q(f) = 0\}$. Let L be the monic operator in $K\langle \partial_t \rangle \setminus \{0\}$ which annihilates h and is of minimal order. Write

$$L = \partial_t^{\ell} + \sum_{i=0}^{\ell-1} a_i \partial_t^i, \quad a_i \in K.$$

Then for every $j \in \{1, \ldots, n\}$

$$0 = \partial_{x_j}(L(h)) = \sum_{i=0}^{\ell-1} \partial_{x_j}(a_i)\partial_t^i(h) + L(\partial_{x_j}(h)) = \sum_{i=0}^{\ell-1} \partial_{x_j}(a_i)\partial_t^i(h).$$

The last equality holds because $h \in C_{\mathbf{x}}$. By the minimality of L, one sees that $\partial_{x_j}(a_i) = 0$ for all $i = 0, \ldots, \ell - 1$ and all $j = 1, \ldots, n$. Hence $a_i \in k(t)$ for all i. In other words, $L \in k(t)\langle \partial_t \rangle$. □

For convention, we introduce the following definition.

Definition 22 (1) We say $f \in \mathcal{U}$ is split if it can be written in the form $f = gh$ where $g \in C_t$ and $h \in C_{\mathbf{x}}$, and say f is semisplit if it is the sum of finitely many split elements.
(2) We say a nonzero operator $P \in K\langle \partial_t \rangle$ is semisplit if it is monic and all its coefficients are semisplit.

The semisplit operators have the following property.

Lemma 23 *Assume that $P = Q_1 Q_2$ where P, Q_1, Q_2 are monic operators in $K\langle \partial_t \rangle$. Assume further that $Q_2 \in k(t)[\mathbf{x}, 1/r]\langle \partial_t \rangle$ where $r \in k[\mathbf{x}, t]$. Then $P \in k(t)[\mathbf{x}, 1/r]\langle \partial_t \rangle$ if and only if Q_1 is also.*

Proof Comparing the coefficients on both sides of $P = Q_1 Q_2$ concludes the proof. □

As a direct consequence, we have the following corollary.

Corollary 24 *Assume that $P = Q_1 Q_2$ where P, Q_1, Q_2 are monic operators in $K\langle \partial_t \rangle$. Assume further that Q_2 is semisplit. Then P is semisplit if and only if Q_1 is also.*

4.1 The completely reducible case

In Proposition 10 of [12], we show that given a hyperexponential function h over K, $\operatorname{ann}(h) \cap k(t)\langle \partial_t \rangle \neq \{0\}$ if and only if there is a nonzero $p \in k(\mathbf{x})[t]$ and $r \in k(t)$ such that

$$a = \frac{\partial_t(p)}{p} + r,$$

where $a = \partial_t(h)/h$. Remark that a, p, r with $p \neq 0$ satisfy the above equality if and only if $\frac{1}{p}(\partial_t - a) = (\partial_t - r)\frac{1}{p}$. Under the notion of $(\mathbf{x}, t)$-separable and the language of differential operators, Proposition 10 of [12] states that $\partial_t - a$ is $(\mathbf{x}, t)$-separable if and only if it is similar to a first order operator in $k(t)\langle \partial_t \rangle$ by some $1/p$ with p being a nonzero polynomial in t. In this section, we shall generalize Proposition 10 of [12] to the case of completely reducible operators. We shall use $\operatorname{lclm}(Q_1, Q_2)$ to denote the monic operator of minimal order which is divisible by both Q_1 and Q_2 on the right. We shall prove that if P is $(\mathbf{x}, t)$-separable and completely reducible then there is a nonzero $L \in k(t)\langle \partial_t \rangle$ such that P is the transformation of L by some Q with semisplit coefficients. To this end, we need to introduce some notations from [35].

Definition 25 Assume that $P, Q \in K\langle \partial_t \rangle \setminus \{0\}$.

1. We say $\tilde{P}$ is the transformation of P by Q if $\tilde{P}$ is the monic operator satisfying that $\tilde{P}Q = \lambda \operatorname{lclm}(P, Q)$ for some $\lambda \in K$.
2. We say $\tilde{P}$ is similar to P (by Q) if there is an operator Q with $\operatorname{gcrd}(P, Q) = 1$ such that $\tilde{P}$ is the transformation of P by Q, where $\operatorname{gcrd}(P, Q)$ denotes the greatest common right-hand factor of P and Q.

Definition 26 1. We say $P \in K\langle \partial_t \rangle$ is completely reducible if it is the lclm of a family of irreducible operators in $K\langle \partial_t \rangle$.
2. We say $Q \in K\langle \partial_t \rangle$ is the maximal completely reducible right-hand factor of $P \in K\langle \partial_t \rangle$ if Q is the lclm of all irreducible right-hand factors of P.

Given a $P \in K\langle \partial_t \rangle$, Theorem 7 of [35] or Theorem 1.1 on page 4 of [37] implies that P has the following unique decomposition called Loewy decomposition,

$$P = \lambda H_r H_{r-1} \dots H_1$$

where $\lambda \in K$ and H_i is the maximal completely reducible right-hand factor of $H_r \dots H_i$. For an $L \in k(t)\langle \partial_t \rangle$, it has two Loewy decompositions viewed as an operator in $k(t)\langle \partial_t \rangle$ and an operator in $K\langle \partial_t \rangle$ respectively. In the following, we shall prove that these two decompositions coincide. For convenience, we shall denote by $P_{x_i = c_i}$ the operator obtained by replacing x_i by $c_i \in k$ in P.

Lemma 27 *Assume that P, L are two monic operators in $K\langle \partial_t \rangle$. Assume further that $P \in k(t)[\mathbf{x}, 1/r]\langle \partial_t \rangle$ with $r \in k[\mathbf{x}, t]$, and $L \in k(t)\langle \partial_t \rangle$. Let $\mathbf{c} \in k^n$ be such that $r(\mathbf{c}) \neq 0$.*

1. *If* $\operatorname{gcrd}(P_{\mathbf{x}=\mathbf{c}}, L) = 1$ *then* $\operatorname{gcrd}(P, L) = 1$.

2. *If* $\mathrm{gcrd}(P, L) = 1$ *then there is* $\mathbf{a} \in k^n$ *such that* $r(\mathbf{a}) \neq 0$ *and* $\mathrm{gcrd}(P_{\mathbf{x}=\mathbf{a}}, L) = 1$.

Proof 1. We shall prove the lemma by induction on $n = |\mathbf{x}|$. Assume that $n = 1$, and $\mathrm{gcrd}(P, L) \neq 1$. Then there are $M, N \in k(t)[x_1]\langle\partial_t\rangle$ with $\mathrm{ord}(M) < \mathrm{ord}(L)$ such that $MP + NL = 0$. Write

$$M = \sum_{i=0}^{m-1} a_i \partial_t^i, \quad N = \sum_{i=0}^{s} b_i \partial_t^i$$

where $m = \mathrm{ord}(L)$. If the a_i's have a common factor c in $k(t_1)[x_1]$, then one sees that c is a common factor of the b_i's. Thus we can cancel this factor c. So without loss of generality, we may assume that the a_i's have no common factor. This implies that $M_{x_1=c_1} \neq 0$ and $M_{x_1=c_1} P_{x_1=c_1} + N_{x_1=c_1} L = 0$. Since $\mathrm{ord}(M_{x_1=c_1}) < \mathrm{ord}(L)$, $\mathrm{gcrd}(P_{x_1=c_1}, L) \neq 1$, a contradiction. For the general case, set $Q = P_{x_1=c_1}$. Then $Q_{x_2=c_2,\dots,x_n=c_n} = P_{\mathbf{x}=\mathbf{c}}$. This implies that $\mathrm{gcrd}(Q_{x_2=c_2,\dots,x_n=c_n}, L) = 1$. By the induction hypothesis, $\mathrm{gcrd}(Q, L) = 1$. Finally, regarding P and L as operators with coefficients in $k(t, x_2, \dots, x_n)[x_1, 1/r]$ and by the induction hypothesis again, we get $\mathrm{gcrd}(P, L) = 1$.

2. Since $\mathrm{gcrd}(P, L) = 1$, there are $M, N \in K\langle\partial_t\rangle$ such that $MP + NL = 1$. Let $\mathbf{a} \in k^n$ be such that $r(\mathbf{a}) \neq 0$ and both $M_{\mathbf{x}=\mathbf{a}}$ and $N_{\mathbf{x}=\mathbf{a}}$ are well-defined. For such $\mathbf{a}$, one has that $M_{\mathbf{x}=\mathbf{a}} P_{\mathbf{x}=\mathbf{a}} + N_{\mathbf{x}=\mathbf{a}} L = 1$ and then $\mathrm{gcrd}(P_{\mathbf{x}=\mathbf{a}}, L) = 1$. □

Lemma 28 *Let* $L \in k(t)\langle\partial_t\rangle$. *The Loewy decompositions of* L *viewed as an operator in* $k(t)\langle\partial_t\rangle$ *and an operator in* $K\langle\partial_t\rangle$ *respectively coincide.*

Proof We first claim that an irreducible operator of $k(t)\langle\partial_t\rangle$ is irreducible in $K\langle\partial_t\rangle$. Let P be a monic irreducible operator in $k(t)\langle\partial_t\rangle$ and assume that Q is a monic right-hand factor of P in $K\langle\partial_t\rangle$ with $1 \leq \mathrm{ord}(Q) < \mathrm{ord}(P)$. Then $P = \tilde{Q}Q$ for some $\tilde{Q} \in K\langle\partial_t\rangle$. Suppose that $Q \in k(t)[\mathbf{x}, 1/r]\langle\partial_t\rangle$. By Lemma 23, $\tilde{Q}$ belongs to $k(t)[\mathbf{x}, 1/r]\langle\partial_t\rangle$. Let $\mathbf{c} \in k^n$ be such that $r(\mathbf{c}) \neq 0$. Then $P = \tilde{Q}_{\mathbf{x}=\mathbf{c}} Q_{\mathbf{x}=\mathbf{c}}$ and $1 \leq \mathrm{ord}(Q_{\mathbf{x}=\mathbf{c}}) < \mathrm{ord}(P)$. These imply that P is reducible in $k(t)\langle\partial_t\rangle$, a contradiction. So P is irreducible in $K\langle\partial_t\rangle$ and thus the claim holds. Let $L = \lambda H_r H_{r-1} \cdots H_1$ be the Loewy decomposition in $k(t)\langle\partial_t\rangle$. The above claim implies that H_1 viewed as an operator in $K\langle\partial_t\rangle$ is completely reducible. Assume that H_1 is not the maximal completely reducible right-hand factor of L in $K\langle\partial_t\rangle$. Let $M \in K\langle\partial_t\rangle \setminus K$ be a monic irreducible right-hand factor of L satisfying that $\mathrm{gcrd}(M, H_1) = 1$. Due to Lemma 27, there is $\mathbf{a} \in k^n$ satisfying that $\mathrm{gcrd}(M_{\mathbf{x}=\mathbf{a}}, H_1) = 1$. Note that $M_{\mathbf{x}=\mathbf{a}}$ is a right-hand factor of L. Therefore $M_{\mathbf{x}=\mathbf{a}}$ has some irreducible right-hand factor of L as a right-hand factor. Such irreducible factor must be a right-hand factor of H_1 and thus $\mathrm{gcrd}(M_{\mathbf{x}=\mathbf{a}}, H_1) \neq 1$, a contradiction. Therefore H_1 is the maximal completely reducible right-hand factor of L in $K\langle\partial_t\rangle$. Using the induction on the order, one sees that $\lambda H_r H_{r-1} \cdots H_1$ is the Loewy decomposition of L in $K\langle\partial_t\rangle$. □

Lemma 29 *Assume that* P *is monic,* $(\mathbf{x}, t)$*-separable and completely reducible. Assume further that* $P \in k(t)[\mathbf{x}, 1/r]\langle\partial_t\rangle$ *with* $r \in k[\mathbf{x}, t]$. *Let* $\mathbf{c} \in k^n$ *be such that* $r(\mathbf{c}) \neq 0$. *Then* $P_{\mathbf{x}=\mathbf{c}}$ *is similar to* P.

Proof Let $\tilde{L}$ be a nonzero monic operator in $k(t)\langle\partial_t\rangle$ with P as a right-hand factor. Since P is completely reducible, by Theorem 8 of [35], P is a right-hand factor of the maximal completely reducible right-hand factor of $\tilde{L}$. By Lemma 28, the maximal completely reducible right-hand factor of $\tilde{L}$ is in $k(t)\langle\partial_t\rangle$. Hence we may assume that $\tilde{L}$ is completely reducible after replacing $\tilde{L}$ by its maximal completely reducible right-hand factor. Assume that $\tilde{L} = QP$ for some $Q \in K\langle\partial_t\rangle$. By Lemma 23, $Q \in k(t)[\mathbf{x}, 1/r]\langle\partial_t\rangle$. Then $\tilde{L} = Q_{\mathbf{x}=\mathbf{c}}P_{\mathbf{x}=\mathbf{c}}$, i.e. $P_{\mathbf{x}=\mathbf{c}}$ is a right-hand factor of $\tilde{L}$. We claim that for a right-hand factor T of $\tilde{L}$, there is a right-hand factor L of $\tilde{L}$ satisfying that $\mathrm{gcrd}(T, L) = 1$ and $\mathrm{lclm}(T, L) = \tilde{L}$. We prove this claim by induction on $s = \mathrm{ord}(\tilde{L})-\mathrm{ord}(T)$. When $s = 0$, there is nothing to prove. Assume that $s > 0$. Then since $\tilde{L}$ is completely reducible, there is an irreducible right-hand factor L_1 of $\tilde{L}$ such that $\mathrm{gcrd}(T, L_1) = 1$. Let $N = \mathrm{lclm}(T, L_1)$. We have that $\mathrm{ord}(N) = \mathrm{ord}(T) + \mathrm{ord}(L_1)$. Therefore $\mathrm{ord}(\tilde{L}) - \mathrm{ord}(N) < s$. By induction hypothesis, there is a right-hand factor L_2 of $\tilde{L}$ such that $\mathrm{gcrd}(N, L_2) = 1$ and $\mathrm{lclm}(N, L_2) = \tilde{L}$. Let $L = \mathrm{lclm}(L_1, L_2)$. Then

$$\tilde{L} = \mathrm{lclm}(N, L_2) = \mathrm{lclm}(T, L_1, L_2) = \mathrm{lclm}(T, L).$$

Taking the order of the operators in the above equality yields that

$$\begin{aligned}\mathrm{ord}(\mathrm{lclm}(T, L)) &= \mathrm{ord}(\mathrm{lclm}(N, L_2)) = \mathrm{ord}(N) + \mathrm{ord}(L_2)\\ &= \mathrm{ord}(T) + \mathrm{ord}(L_1) + \mathrm{ord}(L_2).\end{aligned}$$

On the other hand, we have

$$\mathrm{ord}(\mathrm{lclm}(T, L)) \le \mathrm{ord}(T) + \mathrm{ord}(L) \le \mathrm{ord}(T) + \mathrm{ord}(L_1) + \mathrm{ord}(L_2).$$

This implies that

$$\mathrm{ord}(\mathrm{lclm}(T, L)) = \mathrm{ord}(T) + \mathrm{ord}(L).$$

So $\mathrm{gcrd}(T, L) = 1$ and then L is a required operator. This proves the claim. Now let $L_{\mathbf{c}}$ be a right-hand factor of $\tilde{L}$ satisfying that $\mathrm{gcrd}(P_{\mathbf{x}=\mathbf{c}}, L_{\mathbf{c}}) = 1$ and $\mathrm{lclm}(P_{\mathbf{x}=\mathbf{c}}, L_{\mathbf{c}}) = \tilde{L}$. Let $M \in k(t)\langle\partial_t\rangle$ be such that $\tilde{L} = ML_{\mathbf{c}}$. Then $P_{\mathbf{x}=\mathbf{c}}$ is similar to M. It remains to show that P is also similar to M. Due to Lemma 27, $\mathrm{gcrd}(P, L_{\mathbf{c}}) = 1$. Then

$$\mathrm{ord}(\mathrm{lclm}(P, L_{\mathbf{c}})) = \mathrm{ord}(P) + \mathrm{ord}(L_{\mathbf{c}}) = \mathrm{ord}(P_{\mathbf{x}=\mathbf{c}}) + \mathrm{ord}(L_{\mathbf{c}}) = \mathrm{ord}(\tilde{L}).$$

Note that $\mathrm{lclm}(P, L_{\mathbf{c}})$ is a right-hand factor of $\tilde{L}$. Hence $\mathrm{lclm}(P, L_{\mathbf{c}}) = \tilde{L}$ and thus P is similar to M. □

For the general case, the above lemma is not true anymore as shown in the following example.

Example 30 Let $y = x_1 \log(t+1) + x_2 \log(t-1)$ and

$$P = \partial_t^2 + \frac{(t-1)^2 x_1 + (t+1)^2 x_2}{(t^2-1)((t-1)x_1 + (t+1)x_2)} \partial_t.$$

Then P is (x, t)-separable since $\{1, y\}$ is a basis of the solution space of $P = 0$ in $\mathcal{U}$. We claim that P is not similar to $P_{\mathbf{x}=\mathbf{c}}$ for any $\mathbf{c} \in k^2 \backslash \{(0, 0)\}$. Suppose on the contrary that P is similar to $P_{\mathbf{x}=\mathbf{c}}$ for some $\mathbf{c} = (c_1, c_2) \in k^2 \backslash \{(0, 0)\}$, i.e. there are $a, b \in k(\mathbf{x}, t)$, not all zero, such that $\operatorname{gcrd}(a\partial_t + b, P_{\mathbf{x}=\mathbf{c}}) = 1$ and P is the transformation of $P_{\mathbf{x}=\mathbf{c}}$ by $a\partial_t + b$. Denote $Q = a\partial_t + b$. As $\{1, y_{\mathbf{x}=\mathbf{c}}\}$ is a basis of the solution space of $P_{\mathbf{x}=\mathbf{c}}$, $\{Q(1), Q(y_{\mathbf{x}=\mathbf{c}})\}$ is a basis of the solution space of $P = 0$. In other words, there is $C \in \mathrm{GL}_2(C_t)$ such that

$$\left(b, a\left(\frac{c_1}{t+1} + \frac{c_2}{t-1}\right) + b y_{\mathbf{x}=\mathbf{c}}\right) = (1, y)C.$$

Note that $\log(t+1), \log(t-1), 1$ are linearly independent over $k(x_1, x_2, t)$. We have that $b \in C_t \backslash \{0\}$ and $bc_1 = \tilde{c}x_1, bc_2 = \tilde{c}x_2$ for some $\tilde{c} \in C_t$. This implies that $x_1/x_2 = c_1/c_2 \in k$, a contradiction.

When the given two operators are of length two, i.e. they are the products of two irreducible operators, a criterion for the similarity is presented in [31]. For the general case, suppose that P is similar to $P_{\mathbf{x}=\mathbf{c}}$ by Q. Then the operator Q is a solution of the following mixed differential equation

$$Pz \equiv 0 \mod P_{\mathbf{x}=\mathbf{c}}. \tag{9}$$

An algorithm for computing all solutions of the above mixed differential equation is developed in [41]. In the following, we shall show that if P is $(\mathbf{x}, t)$-separable then Q is an operator with semisplit coefficients. Note that Q can be chosen to be of order less than $\operatorname{ord}(P_{\mathbf{x}=\mathbf{c}})$ and all solutions of the mixed differential equation with order less than $\operatorname{ord}(P_{\mathbf{x}=\mathbf{c}})$ form a vector space over $k(\mathbf{x})$ of finite dimension. Furthermore Q induces an isomorphism from the solution space of $P_{\mathbf{x}=\mathbf{c}}(y) = 0$ to that of $P(y) = 0$.

Proposition 31 *Assume that P is monic and completely reducible. Assume further that $P \in k(t)[\mathbf{x}, 1/r]\langle\partial_t\rangle$ with $r \in k[\mathbf{x}, t]$. Let $\mathbf{c} \in k^n$ be such that $r(\mathbf{c}) \neq 0$. Then P is $(\mathbf{x}, t)$-separable if and only if P is similar to $P_{\mathbf{x}=\mathbf{c}}$ by an operator Q with semisplit coefficients.*

Proof Denote $m = \operatorname{ord}(P_{\mathbf{x}=\mathbf{c}}) = \operatorname{ord}(P)$. Assume that $\{\alpha_1, \ldots, \alpha_m\}$ is a basis of the solution space of $P_{\mathbf{x}=\mathbf{c}}(y) = 0$ in $C_{\mathbf{x}}$ and P is similar to $P_{\mathbf{x}=\mathbf{c}}$ by Q. Write $Q = \sum_{i=0}^{m-1} a_i \partial_t^i$ where $a_i \in K$. Then

$$(Q(\alpha_1), \ldots, Q(\alpha_m)) = (a_0, \ldots, a_{m-1}) \begin{pmatrix} \alpha_1 & \alpha_2 & \cdots & \alpha_m \\ \alpha_1' & \alpha_2' & \cdots & \alpha_m' \\ \vdots & \vdots & & \vdots \\ \alpha_1^{(m-1)} & \alpha_2^{(m-1)} & \cdots & \alpha_m^{(m-1)} \end{pmatrix}$$

and $Q(\alpha_1), \ldots, Q(\alpha_m)$ form a basis of the solution space of $P(y) = 0$.

Now suppose that P is $(\mathbf{x}, t)$-separable. Due to Lemma 29, P is similar to $P_{\mathbf{x}=\mathbf{c}}$ by Q. By Corollary 21, the $Q(\alpha_i)$ are semisplit. The above equalities then imply that the a_i are semisplit. Conversely, assume that P is similar to $P_{\mathbf{x}=\mathbf{c}}$ by Q and the a_i are semisplit. It is easy to see the $Q(\alpha_i)$ are semisplit. By Corollary 21 again, P is $(\mathbf{x}, t)$-separable. □

Using the algorithm developed in [41], we can compute a basis of the solution space over $k(\mathbf{x})$ of the Eq. (9). It is clear that the solutions with semisplit entries form a subspace. We can compute a basis for this subspace as follows. Suppose that $\{Q_1, \ldots, Q_\ell\}$ is a basis of the solution space of the Eq. (9) consisting of solutions with order less than $\operatorname{ord}(P_{\mathbf{x}=\mathbf{c}})$. We may identify Q_i with a vector $\mathbf{g}_i \in K^m$ under the basis $1, \partial_t, \ldots, \partial_t^{m-1}$. Let $q \in k(\mathbf{x})[t]$ be a common denominator of all entries of the $\mathbf{g}_i$. Write $\mathbf{g}_i = \mathbf{p}_i/q$ for each $i = 1, \ldots, \ell$, where $\mathbf{p}_i \in k(\mathbf{x})[t]^m$. Write $q = q_1 q_2$ where q_2 is split but q_1 is not. Note that a rational function in t with coefficients in $k(\mathbf{x})$ is semisplit if and only if its denominator is split. For $c_1, \ldots, c_\ell \in k(\mathbf{x})$, $\sum_{i=1}^{\ell} c_i \mathbf{g}_i$ is semisplit if and only if all entries of $\sum_{i=1}^{\ell} c_i \mathbf{p}_i$ are divided by q_1. For $i = 1, \ldots, \ell$, let $\mathbf{h}_i$ be the vector whose entries are the remainders of the corresponding entries of $\mathbf{p}_i$ by q_1. Then all entries of $\sum_{i=1}^{\ell} c_i \mathbf{p}_i$ are divided by q_1 if and only if $\sum_{i=1}^{\ell} c_i \mathbf{h}_i = 0$. Let $\mathbf{c}_1, \ldots, \mathbf{c}_s$ be a basis of the solution space of $\sum_{i=1}^{\ell} z_i \mathbf{h}_i = 0$. Then $\{(Q_1, \ldots, Q_\ell)\mathbf{c}_i \mid i = 1, \ldots, s\}$ is the required basis. Consequently, the required basis can be computed by solving the system of linear equations $\sum_{i=1}^{\ell} z_i \mathbf{h}_i = 0$.

In the following, for the sake of notations, we assume that $\{Q_1, \ldots, Q_\ell\}$ is a basis of the solution space of the Eq. (9) consisting of solutions with semi-split coefficients. By Proposition 31 and the definition of similarity, P is $(\mathbf{x}, t)$-separable if and only if there is a nonzero $\tilde{Q}$ in the space spanned by $Q_1, \ldots, Q_\ell$ such that $\operatorname{gcrd}(P_{\mathbf{x}=\mathbf{c}}, \tilde{Q}) = 1$. Note that $\tilde{Q}$ induces a homomorphism from the solutions space of $P_{\mathbf{x}=\mathbf{c}}(y) = 0$ to that of $P(y) = 0$. Moreover, one can easily see that $\operatorname{gcrd}(P_{\mathbf{x}=\mathbf{c}}, \tilde{Q}) = 1$ if and only if $\tilde{Q}$ is an isomorphism i.e. $\tilde{Q}(\alpha_1), \ldots, \tilde{Q}(\alpha_m)$ form a basis of the solution space of $P(y) = 0$ where $\{\alpha_1, \ldots, \alpha_m\}$ is a basis of the solution space of $P_{\mathbf{x}=\mathbf{c}}(y) = 0$. Assume that $\tilde{Q} = \sum_{i=0}^{m-1} a_{0,i} \partial_t^i$ with $a_{0,i} \in K$. Using the relation $P_{\mathbf{x}=\mathbf{c}}(\alpha_j) = 0$ with $j = 1, \ldots, m$, one has that for all $j = 1, \ldots, m$

$$\tilde{Q}(\alpha_j)' = \left(\sum_{i=0}^{m-1} a_{0,i} \alpha_j^{(i)}\right)' = \sum_{i=0}^{m-1} a_{1,i} \alpha_j^{(i)}$$

for some $a_{1,i} \in K$. Repeating this process, we can compute $a_{l,i} \in K$ such that for all $j = 1, \ldots, m$ and $l = 1, \ldots, m-1$,

$$\tilde{Q}(\alpha_j)^{(l)} = \sum_{i=0}^{m-1} a_{l,i} \alpha_j^{(i)}.$$

Now suppose that $\tilde{Q} = \sum_{i=1}^{\ell} z_i Q_i$ with $z_i \in k(\mathbf{x})$. One sees that the $a_{l,i}$ are linear in $z_1, \ldots, z_\ell$. Set $A(\mathbf{z}) = (a_{i,j})_{0 \le i,j \le m-1}$ with $\mathbf{z} = (z_1, \ldots, z_\ell)$. Then one has that

$$A(\mathbf{z}) \begin{pmatrix} \alpha_1 & \cdots & \alpha_m \\ \vdots & & \vdots \\ \alpha^{(m-1)} & \cdots & \alpha_m^{(m-1)} \end{pmatrix} = \begin{pmatrix} \tilde{Q}(\alpha_1) & \cdots & \tilde{Q}(\alpha_m) \\ \vdots & & \vdots \\ \tilde{Q}(\alpha_1)^{(m-1)} & \cdots & \tilde{Q}(\alpha_m)^{(m-1)} \end{pmatrix}. \quad (10)$$

It is well-known that $\tilde{Q}(\alpha_1), \ldots, \tilde{Q}(\alpha_m)$ form a basis if and only if the right-hand side of the above equality is a nonsingular matrix and thus if and only if $A(\mathbf{z})$ is nonsingular. In the sequel, one can reduce the problem of the existence of $\tilde{Q}$ satisfying $\operatorname{gcrd}(\tilde{Q}, P_{\mathbf{x}=\mathbf{c}}) = 1$ to the problem of the existence of $\mathbf{a} \in k(\mathbf{x})^\ell$ in $k(\mathbf{x})$ such that $\det(A(\mathbf{a})) \neq 0$.

Suppose now we already have an operator Q with semisplit coefficients such that P is similar to $P_{\mathbf{x}=\mathbf{c}}$ by Q. Write $Q = \sum_{i=0}^{m-1} b_i \partial_t^i$ where $b_i \in K$ is semisplit. Write further $b_i = \sum_{j=1}^{s} h_{i,j} \beta_j$ where $h_{i,j} \in k(\mathbf{x})$ and $\beta_j \in k(t) \setminus \{0\}$. Let $L_0 = P_{\mathbf{x}=\mathbf{c}}$ and let L_i be the transformation of L_{i-1} by ∂_t for $i = 1, \ldots, m-1$. Then L_i annihilates $\alpha_j^{(i)}$ for all $j = 1, \ldots, m$ and $L_i \frac{1}{\beta_l}$ annihilates $\beta_l \alpha_j^{(i)}$ for all $l = 1, \ldots, s$ and $j = 1, \ldots, m$. Set

$$L = \operatorname{lclm}\left(\left\{L_i \frac{1}{\beta_l} \mid i = 0, \ldots, m-1, l = 1, \ldots, s\right\}\right).$$

Then L annihilates all $\tilde{Q}(\alpha_i)$ and thus has P as a right-hand factor. We summarize the previous discussion as the following algorithm.

Algorithm 32 Input: $P \in K\langle \partial_t \rangle$ that is monic and completely reducible.
Output: a nonzero $L \in k(t)\langle \partial_t \rangle$ which is divided by P on the right if it exists, otherwise 0.

1. Write
$$P = \partial_t^m + \sum_{i=0}^{m-1} \frac{a_i}{r} \partial_t^i$$
where $a_i \in k(t)[\mathbf{x}]$, $r \in k[\mathbf{x}, t]$.
2. Pick $\mathbf{c} \in k^n$ such that $r(\mathbf{c}) \neq 0$. By the algorithm in [41], compute a basis of the solution space V of the Eq. (9).
3. Compute a basis of the subspace of V consisting of operators with semisplit coefficients, say $Q_1, \ldots, Q_\ell$.
4. Set $\tilde{Q} = \sum_{i=1}^{\ell} z_i Q_i$ and using $\tilde{Q}$, compute the matrix $A(\mathbf{z})$ as in (10).
5. If $\det(A(\mathbf{z})) = 0$ then return 0 and the algorithm terminates. Otherwise compute $\mathbf{a} = (a_1, \ldots, a_\ell) \in k^\ell$ such that $\det(A(\mathbf{a})) \neq 0$.
6. Set b_i to be the coefficient of ∂_t^i in $\sum_{j=1}^{\ell} a_j Q_j$ and write $b_i = \sum_{j=1}^{s} h_{i,j} \beta_j$ where $h_{i,j} \in k(\mathbf{x})$ and $\beta_j \in k(t)$. Let $L_0 = P_{\mathbf{x}=\mathbf{c}}$ and for each $i = 1, \ldots, m-1$ compute L_i, the transformation of L_{i-1} by ∂_t.

7. Return $\operatorname{lclm}\left(\left\{L_i \frac{1}{\beta_j} \mid i = 0, \ldots, m-1, j = 1, \ldots, s\right\}\right)$.

4.2 The general case

Assume that P is $(\mathbf{x}, t)$-separable and $P = Q_1 Q_2$ where $Q_1, Q_2 \in K\langle\partial_t\rangle$. It is clear that Q_2 is also $(\mathbf{x}, t)$-separable. One may wonder whether Q_1 is also $(\mathbf{x}, t)$-separable. The following example shows that Q_1 may not be $(\mathbf{x}, t)$-separable.

Example 33 Let $K = k(x, t)$ and let $P = \partial_t^2$. Then P is $(\mathbf{x}, t)$-separable and

$$\partial_t^2 = \left(\partial_t + \frac{x}{xt+1}\right)\left(\partial_t - \frac{x}{xt+1}\right).$$

The operator $\partial_t + x/(xt+1)$ is not $(\mathbf{x}, t)$-separable, because $1/(xt+1)$ is one of its solutions and it is not semisplit.

On the other hand, the lemma below shows that if Q_2 is semisplit then Q_1 is also $(\mathbf{x}, t)$-separable.

Lemma 34 (1) *Assume that $Q_1, Q_2 \in K\langle\partial_t\rangle\backslash\{0\}$, and Q_2 is semisplit. Then $Q_1 Q_2$ is $(\mathbf{x}, t)$-separable if and only if both Q_1 and Q_2 are $(\mathbf{x}, t)$-separable.*

(2) *Assume that $P \in K\langle\partial_t\rangle\backslash\{0\}$ and L is a nonzero monic operator in $k(t)\langle\partial_t\rangle$. Then P is $(\mathbf{x}, t)$-separable if and only if the transformation of P by L is also.*

Proof Note that the solution space of $\operatorname{lclm}(P_1, P_2)(y) = 0$ is spanned by those of $P_1(y) = 0$ and $P_2(y) = 0$. Hence $\operatorname{lclm}(P_1, P_2)$ is $(\mathbf{x}, t)$-separable if and only if so are both P_1 and P_2.

(1) For the “only if" part, one only need to prove that Q_1 is $(\mathbf{x}, t)$-separable. Assume that g is a solution of $Q_1(y) = 0$ in $\mathcal{U}$. Let f be a solution of $Q_2(y) = g$ in $\mathcal{U}$. Such f exists because $\mathcal{U}$ is the universal differential extension of K. Then f is a solution of $Q_1 Q_2(y) = 0$ in $\mathcal{U}$. By Corollary 21, f is semisplit. Since Q_2 is semisplit, one sees that $g = Q_2(f)$ is semisplit. By Corollary 21 again, Q_1 is $(\mathbf{x}, t)$-separable.

Now assume that both Q_1 and Q_2 are $(\mathbf{x}, t)$-separable. Let $\tilde{Q} \in K\langle\partial_t\rangle$ be such that $\tilde{Q} Q_2 = L$ where $L \in k(t)\langle\partial_t\rangle$ is monic. By Corollary 24 and the “only if" part, $\tilde{Q}$ is semisplit and $(\mathbf{x}, t)$-separable. Thus $\operatorname{lclm}(Q_1, \tilde{Q})$ is $(\mathbf{x}, t)$-separable. Assume that $\operatorname{lclm}(Q_1, \tilde{Q}) = N\tilde{Q}$ with $N \in K\langle\partial_t\rangle$. Since $\tilde{Q}$ is semisplit, by the “only if" part again, N is $(\mathbf{x}, t)$-separable. Let $M \in K\langle\partial_t\rangle$ be such that MN is a nonzero operator in $k(t)\langle\partial_t\rangle$. We have that

$$M\operatorname{lclm}(Q_1, \tilde{Q})Q_2 = MN\tilde{Q}Q_2 = MNL \in k(t)\langle\partial_t\rangle.$$

On the other hand, $M\operatorname{lclm}(Q_1, \tilde{Q})Q_2 = M\tilde{M}Q_1Q_2$ for some $\tilde{M} \in K\langle\partial_t\rangle$. Hence $P = Q_1 Q_2$ is $(\mathbf{x}, t)$-separable.

(2) Since L is $(\mathbf{x}, t)$-separable, we have that P is $(\mathbf{x}, t)$-separable if and only if $\operatorname{lclm}(P, L)$ is also. Let $\tilde{P}$ be the transformation of P by L. Then $\tilde{P}L = \operatorname{lclm}(P, L)$. As L is semisplit, the assertion then follows from (1). $\square$

Assume that P is a nonzero operator in $K\langle \partial_t \rangle$. Let P_0 be an irreducible right-hand factor of P. By Algorithm 32, we can decide whether P_0 is $(\mathbf{x}, t)$-separable or not. Now assume that P_0 is $(\mathbf{x}, t)$-separable. Then we can compute a nonzero monic operator $L_0 \in k(t)\langle \partial_t \rangle$ having P_0 as a right-hand factor. Let P_1 be the transformation of P by L_0. Lemma 34 implies that P is $(\mathbf{x}, t)$-separable if and only if P_1 is also. Note that

$$\begin{aligned} \mathrm{ord}(P_1) &= \mathrm{ord}(\mathrm{lclm}(P, L_0)) - \mathrm{ord}(L_0) \\ &\leq \mathrm{ord}(P) + \mathrm{ord}(L_0) - \mathrm{ord}(P_0) - \mathrm{ord}(L_0) = \mathrm{ord}(P) - \mathrm{ord}(P_0). \end{aligned}$$

In other words, $\mathrm{ord}(P_1) < \mathrm{ord}(P)$. Replacing P by P_1 and repeating the above process yield an algorithm to decide whether P is $(\mathbf{x}, t)$-separable.

Algorithm 35 Input: a nonzero monic $P \in K\langle \partial_t \rangle$.
Output: a nonzero $L \in k(t)\langle \partial_t \rangle$ which is divided by P on the right if it exists, otherwise 0.

1. If $P = 1$ then return 1 and the algorithm terminates.
2. Compute an irreducible right-hand factor P_0 of P by algorithms developed in [4, 40, 42].
3. Apply Algorithm 32 to P_0 and let L_0 be the output.
4. If $L_0 = 0$ then return 0 and the algorithm terminates. Otherwise compute the transformation of P by L_0, denoted by P_1.
5. Apply Algorithm 35 to P_1 and let L_1 be the output.
6. Return $L_1 L_0$.

The termination of the algorithm is obvious. Assume that $L_1 \neq 0$. Then $L_1 = Q_1 P_1$ for some $Q_1 \in K\langle \partial_t \rangle$. We have that $P_1 L_0 = \mathrm{lclm}(P, L_0)$. Therefore

$$L_1 L_0 = Q_1 P_1 L_0 = Q_1 \mathrm{lclm}(P, L_0) = Q_1 Q_0 P$$

for some $Q_0 \in K\langle \partial_t \rangle$. This proves the correctness of the algorithm.

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