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Elementary and Liouvillian Solutions of Linear Differential Equations

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Let $L(y)=b$ be a linear differential equation with coefficients in a differential field k, of characteristic 0. We show that if $L(y)=b$ has a non-zero solution Liouvillian over k, then either $L(y)=0$ has a non-zero solution u such that u'/u is algebraic over k, or $L(y)=b$ has a solution in k. If $L(y)=b$ has a non-zero solution elementary over k, then either $L(y)=0$ has a non-zero solution algebraic over k, or $L(y)=b$ has a solution in k. This latter fact is a consequence of the fact that if $L(y)=b$ has a solution elementary over k, then it has a solution of the form $P(\log u_1, \ldots, \log u_m)$, where P is a polynomial with coefficients algebraic over k whose degree is at most equal to the order of $L(y)$, and the u_i are algebraic over k. Algorithmic considerations are also discussed.

1. Introduction

In this paper, we shall present some results concerning Liouvillian and elementary solutions of linear differential equations. We start by giving some definitions. Let $k \subset K$ be ordinary differential fields with derivation $'$. The constant subfield $C(K)$ of K is defined to be the set of c in K such that $c'=0$. We say K is a *Liouvillian extension of* k if (1) $C(K)=C(k)$ and (2) there is a tower of fields $k=K_0 \subset K_1 \subset \ldots \subset K_r=K$ such that for each $i=1, \ldots, r$, $K_i=K_{i-1}(\theta_i)$ where either (a) $\theta_i' \in K_{i-1}$ or (b) $\theta_i'/\theta_i \in K_{i-1}$ or (c) θ_i is algebraic over K_{i-1}. K is said to be an *elementary extension of* k if (1) $C(K)=C(k)$ and (2) there is a tower of fields $k=K_0 \subset K_1 \subset \ldots \subset K_r=K$ such that for each $i=1, \ldots, r$, $K_i=K_{i-1}(\theta_i)$ where either (a) for some $u \neq 0$ in K_{i-1}, $\theta_i'=u'/u$ or (b) for some u in K_{i-1}, $\theta_i'=u'\theta_i$ or (c) θ_i is algebraic over K_{i-1}. We say w is Liouvillian (elementary) over k if w belongs to a Liouvillian (elementary) extension of k. Our definitions differ from the standard definitions in that we assume that the extensions have the same field of constants as the base fields. If $C(k)$ is algebraically closed, then a differential equation, with coefficients in k, has a non-zero solution in a Liouvillian (elementary) extension of k possibly containing new constants, if and only if it has a solution in an extension of the same type with no new constants. Therefore we could remove this restriction and restate our results to allow for the possible introduction of new algebraic constants.

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Section 2 is devoted to proving:

THEOREM 1. *Let k be a differential field of characteristic 0 with $a_{n-1}, \ldots, a_0, b$ in k. If*

$$L(y) = y^{(n)} + a_{n-1}y^{(n-1)} + \ldots + a_0 y = b$$

has a non-zero solution Liouvillian over k, then either:

(*i*) *$L(y) = 0$ has a non-zero solution u such that u'/u is algebraic over k, or*
(*ii*) *$L(y) = b$ has a solution in k.*

This theorem generalises Theorem 2 of Davenport (1986) where the case $n = 2$ was treated. In Section 2 we also use Theorem 1 to show, when k is an algebraic extension of $Q(x)$, where $x' = 1$ and Q denotes the rational numbers, that one can decide if $L(y) = b$ has a Liouvillian solution, and if so find one.

Section 3 is devoted to proving the following generalisation of Liouville's theorem on integration in terms of elementary functions:

THEOREM 2. *Let k be a differential field of characteristic 0 and let $a_{n-1}, \ldots, a_0, b$ be elements of k. If*

$$L(y) = y^{(n)} + a_{n-1}y^{(n-1)} + \ldots + a_o y = b$$

has a solution elementary over k, then $L(y) = b$ has a solution of the form $P(t_1, \ldots, t_m)$ where P is a polynomial in $t_1, \ldots, t_m$ of degree at most n with coefficients algebraic over k and each t_i is transcendental over k and satisfies $t_i' = u_i'/u_i$ for some non-zero u_i algebraic over k. Furthermore, if $P_n(t_1, \ldots, t_m)$ is the homogeneous part of P of degree n and $a_0 = 0$, then the coefficients in $P_n(t_1, \ldots, t_m)$ are constants.

In Section 3, we shall discuss the relation between this theorem and Liouville's theorem and give an example to show that the coefficients of P and the u_i's do not necessarily lie in k. In the process of proving Theorem 2, we shall also generalise work of Ostrowski (1946) concerning the integration of elementary functions that are built up using only logarithms (and no exponentials), Lemma 7, as well as showing that the vector space of elementary solutions of $L(s) = 0$ has a basis of a very particular form, Lemma 8. As a corollary to Theorem 2, we shall also show the following "elementary" version of Theorem 1:

THEOREM 3. *Let k be a differential field of characteristic 0 with $a_{n-1}, \ldots, a_0, b$ in k. If*

$$L(y) = y^{(n)} + a_{n-1}y^{(n-1)} + \ldots + a_0 y = b$$

has a non-zero solution elementary over k, then either:

(*i*) *$L(y) = 0$ has a non-zero solution algebraic over k, or*
(*ii*) *$L(y) = b$ has a solution in k.*

This theorem generalises Theorems 1 and 4 of Davenport (1986) where the cases $n = 1$ and 2 are treated. We end the section with a discussion of the problem of deciding if $L(y) = b$ has an elementary solution. Although we are not able to completely solve this, we are able to reduce the problem to finding an effective procedure for the following: given $y_1, \ldots, y_n$, algebraic over $Q(x)$, find a system **L** of linear equations with constant coefficients such that $c_1 y_1 + \ldots + c_n y_n$ has an elementary anti-derivative if and only if $(c_1, \ldots, c_n)$ satisfies **L**. This problem can further be reduced (although we do not do so here) to a problem in algebraic geometry: given divisors $D_1, \ldots, D_n$ on a curve C, find a basis for the vectors of rational numbers c_i such that $\Sigma c_i D_i$ is a torsion divisor.

Section 4 concludes the paper with a discussion of open problems.

2. Liouvillian Solutions of Linear Differential Equations

Theorem 1 follows easily from parts (a) and (b) of Lemma 1 below. The proof of this lemma involves several facts from the theory of Puiseux expansions. Let K be a function field of one variable of characteristic 0, that is, K is a finite algebraic extension of a field of the form $k(t)$, where t is transcendental over k. It is known that for some m, we can embed K in $\bar{k}((t^{1/m}))$, the field of formal power series in $t^{1/m}$ with coefficients in $\bar{k}$, the algebraic closure of k, via a map that is the identity on $k(t)$ (Chevalley, 1951, p. 64). Furthermore, if D is a derivation of K that maps k into itself, then D extends to a derivation of $\bar{k}((t^{1/m}))$ in such a way that for any element

$$u = \sum_{i \geqslant r} \alpha_i t^{\frac{i}{m}}$$

in $\bar{k}((t^{1/m}))$ we have

$$Du = \sum_{i \geqslant r} (D\alpha_i) t^{\frac{i}{m}} + \Big(\sum_{i \geqslant r} (i/m) \alpha_i t^{\frac{i}{m}-1}\Big) Dt$$

(Chevalley, 1951, p. 114). We shall be interested in derivations D such that either $Dt \in k$ or $Dt/t \in k$, in which case this fact may be verified directly. Note that part (a) of Lemma 1 is proved by Rosenlicht (1973) and parts (b) and (c) follow from Corollary 3 of Singer (1976). We give a simplified presentation for the convenience of the reader.

LEMMA 1. *Let k be a differential field of characteristic 0 and let $a_{n-1}, \ldots, a_0, b$ be in k. Let*

$$L(y) = y^{(n)} + a_{n-1} y^{(n-1)} + \ldots + a_0 y.$$

(*a*) *If $L(y) = 0$ has a non-zero solution Liouvillian over k, then $L(y) = 0$ has a non-zero solution u such that u'/u is algebraic over k.*

(*b*) *If $L(y) = 0$ has no non-zero Liouvillian solutions and $L(y) = b$ has a non-zero Liouvillian solution z, then z is in k.*

(*c*) *If $L(y) = 0$ has no non-zero elementary solutions and $L(y) = b$ has a non-zero elementary solution z, then z is in k.*

PROOF. We first recall some facts about Ricatti equations. Let y be a non-zero solution of $L(y) = 0$ and let $w = y'/y$. We see that $y' = wy$, $y'' = w'y + w^2 y$, $y''' = w''y + 3w'wy + w^3 y$, etc. Substituting these expressions in $L(y) = 0$ and dividing by y, we see that w satisfies a non-linear differential equation of order $n-1$ of the form

$$R(w) = w^n + f(w, w', \ldots, w^{(n-1)}) = 0,$$

where f is a polynomial of degree less than n. Conversely, if w satisfies $R(w) = 0$, then $y = \exp(\int w)$ satisfies $L(y) = 0$. $R(w) = 0$ is called the Ricatti equation associated with $L(y) = 0$. Furthermore, the constant term of $f(w, w', \ldots, w^{(n-1)})$ (and therefore of $R(w)$) is a_0. Therefore $R(w)$ has no constant term if and only if $L(1) = 0$, i.e. $L(y)$ has no term of order 0. The Lemma is trivial if $L(1) = 0$ so we may assume $L(1) \neq 0$.

(a) From the preceding discussion, we see that it is sufficient to prove the following: Let $R(w) = 0$ be a Ricatti equation and let K be a Liouvillian extension of k. If $R(w) = 0$ has a solution in K, then $R(w) = 0$ has a solution algebraic over k. Replacing k by its algebraic closure and proceeding by induction on the transcendence degree of K over k, it suffices to prove this when K is an algebraic extension of $k(t)$ where $t' \in k$ or $t'/t \in k$. We can expand any $u \in K$ in fractional powers of t^{-1} as

$$u = \sum_{i \geq p} \alpha_i t^{-\frac{i}{m}}$$

where $\alpha_p \neq 0$. The integer p is called the order of u and is denoted by ord u. If $t' \in k$ we have

(1) $u' = \alpha_p' t^{\frac{-p}{m}} + \text{terms involving higher powers of } t^{\frac{-1}{m}}$.

If $t'/t \in k$,

(2) $u' = (\alpha_p' - (p/m)\alpha_p(t'/t))t^{\frac{-p}{m}} + \text{terms involving higher powers of } t^{\frac{-1}{m}}$.

In either case, we have ord $u' \geq$ ord u, so ord $u^{(i)} \geq$ ord u for all $i \geq 0$. We now claim that if w satisfies $R(w) = 0$, then ord $w = 0$. If ord $w < 0$, then ord $(w^n) = n$ ord w while ord $(f(w, w', \ldots, w^{(n-1)}) \geq (n-1)$ ord w contradicting the fact that

$$R(w) = w^n + f(w, w', \ldots, w^{(n-1)}) = 0.$$

If ord $w > 0$, then, recalling the fact that we are assuming f has a non-zero constant term, ord $f(w, w', \ldots, w^{(n-1)}) = 0$. Since ord $(w^n) > 0$, we get a contradiction from $w^n + f(w, w', \ldots, w^{(n-1)}) = 0$. Therefore ord $w = 0$. Setting $w = \alpha_0 + \alpha_1 t^{\frac{1}{m}} + \ldots$ and referring back to (1) and (2), we have that $w^{(i)} = \alpha_0^{(i)} + \text{terms involving positive powers of } t^{-\frac{1}{m}}$. Comparing coefficients of $(t^{-\frac{1}{m}})^0$ in $R(w) = 0$, we see that $R(\alpha_0) = 0$. So $R(w) = 0$ has a solution in k.

(b) Let K be a Liouvillian extension of k containing a solution u of $L(y) = b$, $b \neq 0$. We shall first show that u is algebraic over k. Using induction on the transcendence degree of K over k, we may assume that K is algebraic over $k_0(t)$, where k_0 is the algebraic closure of k and $t' \in k_0$ or $t'/t \in k_0$. Expanding u in fractional powers of t^{-1}, we have

$$u = \sum_{i \geq p} \alpha_i t^{-\frac{i}{m}},$$

where each $\alpha_i \in k_0$ and $\alpha_p \neq 0$. If $t' \in k$, then as before, we have $u' = \alpha_p' t^{-\frac{p}{m}} +$ higher powers of $t^{-\frac{1}{m}}$, $u'' = \alpha_p'' t^{-\frac{p}{m}} +$higher powers of $t^{-\frac{1}{m}}$, etc. Furthermore, if $p \neq 0$, we can compare coefficients of $t^{-\frac{p}{m}}$ in $L(u) = b$ and deduce that $L(\alpha_p) = 0$, contradicting the fact that $L(y) = 0$ has no non-zero Liouvillian solutions. If $t'/t \in k$, then

$$\begin{aligned} u' &= (\alpha_p' - (p/m)\alpha_p(t'/t))t^{-\frac{p}{m}} + \\ &\qquad \text{terms involving higher powers of } t^{-\frac{1}{m}} \\ &= (\alpha_p t^{-\frac{p}{m}})' + \text{terms involving higher powers of } t^{-\frac{1}{m}}. \end{aligned}$$

Similarly, $u'' = (\alpha_p t^{-\frac{p}{m}})''$ terms involving higher powers of $t^{-\frac{1}{m}}$. Therefore, the $t^{-\frac{p}{m}}$ term of $L(u)$ is of the form $L(\alpha_p t^{-\frac{p}{m}})$. If $p \neq 0$, we would have a Liouvillian solution of $L(y) = 0$, a contradiction. Therefore, in both cases, we have $p = 0$ and $L(\alpha_0) = b$. This implies that $L(\alpha_0 - u) = 0$, so $u = \alpha_0 \in k_0$. We now have that u is algebraic over k and wish to show that u is in k. Let Tr denote the trace function from $k(u)$ to k. We then have that $mb = \text{Tr}(L(u)) = L(\text{Tr}(u))$ for some integer m. Therefore $u - (1/m)\text{Tr}(u)$ is a Liouvillian solution of $L(y) = 0$ and so $u = (1/m)\text{Tr}(u) \in k$.

(c) The proof is identical to that of part (b) once one replaces all occurrences of the word Liouvillian with the word elementary. □

The following example, from Davenport (1986), shows that Theorem 1 cannot be improved.

EXAMPLE 1. Consider the equation $y'+y=1/x$. This has a solution,

$$y=\exp(-x)\left(\int(1/x)\exp x\right),$$

that is Liouvillian over $C(x)$, C being the complex numbers. By degree arguments, one can show that neither $y'+y=1/x$ nor $y'+y=0$ have non-zero solutions in $C(x)$ (or, by elementary galois theory, do not have non-zero solutions algebraic over $C(x)$) but, of course, $y'+y=0$ has a solution $u=\exp(-x)$ such that $u'/u\in C(x)$.

We now turn to the problem of deciding if $L(y)=b$ has a Liouvillian solution and finding such a solution if it exists. When k is a finite algebraic extension of $Q(x)$ and $L(y)$ has coefficients in k, the second author showed how to find, in a finite number of steps, a basis for the vector space of Liouvillian solutions of $L(y)=0$ (Singer, 1980). If, in addition, b is in k, this algorithm can be used to decide if $L(y)=b$ has a Liouvillian solution. Since we shall present a better algorithm below, we will only give an outline. Define a new homogeneous linear differential equation $L_1(y)=L_0(L(y))$ where $L_0(y)=y'-(b'/b)y$. Any solution of $L(y)=b$ will be a solution of $L_1(y)=0$. Using the algorithm of Singer (1980) we construct a basis $y_1,\ldots,y_m$ for the vector space of Liouvillian solutions of $L_1(y)=0$. Let $z_1=L(y_1)$, $z_2=L(y_2),\ldots,z_m=L(y_m)$. We must now decide if b is a constant linear combination of $z_1,\ldots,z_m$. First, we find a maximal linearly independent subset of $z_1,\ldots,z_m$. To do this we need only look at various Wronskian determinants $W_r(z_{i_1},\ldots,z_{i_r})$ and decide which are 0 and which are not. Say $\{z_1,\ldots,z_s\}$ form a maximal linearly independent set. We then decide if $Wr(b,z_1,\ldots,z_s)$ is zero or not. If it is not zero, then $L(y)=b$ does not have a Liouvillian solution. If $Wr(b,z_1,\ldots,z_s)=0$, then $L(y)=b$ will have a solution of the form $c_1y_1+\ldots+c_ry_r$ where the c_i are constants that can be determined from the minors of the Wronskian matrix of $Wr(b,z_1,\ldots,z_s)$.

This approach to the problem has several drawbacks. The first is that although we start with an nth order differential equation, we are immediately forced to find the Liouvillian solutions of an $(n+1)$st order differential equation. Since the algorithm for finding such solutions increases greatly in difficulty as n increases, and, in fact, has not been implemented beyond the case $n=2$, we would hope to avoid this. Second, in this algorithm, we are forced to perform calculations in a (possibly complicated) Liouvillian extension of k. This again could make the calculations very difficult. The above considerations allow us to give an algorithm that avoids these pitfalls. We need the following two lemmas before we can give the algorithm.

LEMMA 2. *Let k be a finite algebraic extension of $Q(x)$ and let $k[D]$ be the ring of linear differential operators with coefficients in k. Let* $\mathbf{A}$ *be an $n\times n$ matrix with entries in $k[D]$ and* $\mathbf{B}$ *an $n\times 1$ matrix with entries in k. Then one can decide in a finite number of steps if* $\mathbf{AY}=\mathbf{B}$ *has a solution* $\mathbf{Y}$ *that is an $n\times 1$ vector with entries in k and if so, find such a* $\mathbf{Y}$.

PROOF. We may write $k=Q(x,\alpha)$ where α is algebraic over $Q(x)$ of degree N. $1,\alpha,\ldots,\alpha^{N-1}$ will form a $Q(x)$ basis of k over $Q(x)$. Define new vector valued variables

$\mathbf{Y}_i$, $i = 0, \ldots, N-1$ and set $\mathbf{Y} = \mathbf{Y}_0 + \alpha\mathbf{Y}_1 + \ldots + \alpha^{N-1}\mathbf{Y}_{N-1}$. We may also write $\mathbf{A} = \mathbf{A}_0 + \alpha\mathbf{A}_1 + \ldots + \alpha^{N-1}\mathbf{A}_{N-1}$ and $\mathbf{B} = \mathbf{B}_0 + \alpha\mathbf{B}_1 + \ldots + \alpha^{N-1}B_{N-1}$, where the $\mathbf{A}_i$ have entries in $Q(x)[D]$ and the $\mathbf{B}_i$ have entries in $Q(x)$. Substituting these expressions into $\mathbf{AY} = \mathbf{B}$, replacing derivatives of α by their known expressions in terms of $1, \alpha, \ldots, \alpha^{N-1}$, multiplying out and again rewriting in terms of $1, \alpha, \ldots, \alpha^{N-1}$ we get

$$\bar{\mathbf{A}}_0\mathbf{Y}_0 + \alpha\bar{\mathbf{A}}_1\mathbf{Y}_1 + \ldots + \alpha^{N-1}\bar{\mathbf{A}}_{N-1}\mathbf{Y}_{N-1} = \mathbf{B}_0 + \alpha\mathbf{B}_1 + \ldots + \alpha^{N-1}\mathbf{B}_{N-1},$$

where the $\bar{\mathbf{A}}_i$ have entries in $Q(x)[D]$. Comparing powers of α, this latter equation is equivalent to $\bar{\mathbf{A}}\bar{\mathbf{Y}} = \bar{\mathbf{B}}$, where $\bar{\mathbf{A}}$ is an $nN \times nN$ matrix and $\bar{\mathbf{Y}}$ and $\bar{\mathbf{B}}$ are $nN \times 1$ matrix given by

$$\bar{\mathbf{A}} = \begin{pmatrix} \bar{\mathbf{A}}_0 & & & \\ & \bar{\mathbf{A}}_1 & \mathbf{0} & \\ & \mathbf{0} & \ddots & \\ & & & \bar{\mathbf{A}}_{N-1} \end{pmatrix}$$

$$\bar{\mathbf{Y}} = \begin{pmatrix} \mathbf{Y}_0 \\ \vdots \\ \mathbf{Y}_{N-1} \end{pmatrix}$$

$$\bar{\mathbf{B}} = \begin{pmatrix} \mathbf{B}_0 \\ \vdots \\ \mathbf{B}_N \end{pmatrix}.$$

Solving $\mathbf{AY} = \mathbf{B}$ over k is equivalent to solving $\bar{\mathbf{A}}\bar{\mathbf{Y}} = \bar{\mathbf{B}}$ over $Q(x)$. We therefore have reduced the problem to proving the lemma when $k = Q(x)$.

We now make a further reduction. As noted by Poole (1960, pp. 33–41) we can find $n \times n$ matrices $\mathbf{U}$ and $\mathbf{V}$ with entries in $Q(x)[D]$ such that $\mathbf{C} = \mathbf{UAV}$ is a diagonal matrix. Furthermore, U will have a left inverse $\mathbf{U}^{-1}$ and V will have a right inverse $\mathbf{V}^{-1}$. $\mathbf{Y}$ is then a solution of $\mathbf{AY} = \mathbf{B}$ if and only if $\mathbf{W} = \mathbf{V}^{-1}\mathbf{Y}$ is a solution of $\mathbf{CW} = \mathbf{UB}$. If we write $\mathbf{W} = (w_1, \ldots, w_n)^{\mathrm{T}}$ and $\mathbf{UB} = (f_1, \ldots, f_n)^{\mathrm{T}}$, then $\mathbf{CW} = \mathbf{UB}$ can be written as

$$\begin{aligned} L_1(w_1) &= f_1 \\ L_2(w_2) &= f_2 \\ &\vdots \\ L_n(w_n) &= f_n, \end{aligned}$$

where the L_i are linear differential operators with coefficients in $Q(x)$. Therefore, to prove the lemma, it is enough to be able to decide if $L(y) = f$ has a solution y in $Q(x)$, where L is a differential operator with coefficients in $Q(x)$ and f is in $Q(x)$. An algorithm for this is given by Singer (1981, p. 667). □

The following is an example of the method described in Lemma 2.

EXAMPLE 2. Let $k = Q(x, \sqrt{x})$ and $n = 1$. We wish to decide if

$$y' + \sqrt{x}y = (D + \sqrt{x})y = 1 + x\sqrt{x}$$

has a solution in k and, if so, find such a solution. Letting $y = Y_0 + Y_1\sqrt{x}$, we have

$$(DY_0 + xY_1) + (Y_0 + (D + (1/2x))Y_1) = x + x\sqrt{x}$$

or in matrix notation

$$\begin{pmatrix} D & x \\ 1 & D + \frac{1}{2x} \end{pmatrix} \begin{pmatrix} Y_0 \\ Y_1 \end{pmatrix} \begin{pmatrix} 1 \\ x \end{pmatrix}.$$

This is the equation $\overline{\mathbf{A}}\overline{\mathbf{Y}} = \overline{\mathbf{B}}$. We now perform elementary row column operations on $\bar{\mathbf{A}}$ to reduce it to a diagonal matrix. These are done in the following order: (1) Subtract $(D + (1/2x))$ times the first column from the second column (this corresponds to multiplying on the right by $\mathbf{V}_1$ below). (2) Subtract D times the second row from the first (this corresponds to multiplying by $\mathbf{U}_1$ on the left). (3) Interchange the first and second row (this corresponds to multiplying by $\mathbf{U}_2$ on the left). We then have

$$\mathbf{C} = \mathbf{U}_2\mathbf{U}_1\mathbf{A}\mathbf{V}_1 = \begin{pmatrix} 1 & 0 \\ 0 & -D^2 - \frac{1}{2x}D + \frac{1}{2x^2} + x \end{pmatrix},$$

where

$$\mathbf{U}_1 = \begin{pmatrix} 1 & -D \\ 0 & 1 \end{pmatrix}, \quad \mathbf{U}_2 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \mathbf{V}_1 = \begin{pmatrix} 1 & -\left(D + \frac{1}{2x}\right) \\ 0 & 1 \end{pmatrix}.$$

We denote $\mathbf{U}_2\mathbf{U}_1$ by $\mathbf{U}$ and $\mathbf{V}_1$ by $\mathbf{V}$. We must now solve $\mathbf{CW} = \mathbf{UB}$, that is

$$\begin{pmatrix} 1 & 0 \\ 0 & -D^2 - \frac{1}{2x}D + \frac{1}{2x^2} + x \end{pmatrix} \begin{pmatrix} W_0 \\ W_1 \end{pmatrix} = \begin{pmatrix} x \\ 0 \end{pmatrix}.$$

This corresponds to the two linear equations $W_0 = x$ and

$$W_1'' + (1/2x)W_1' - (1/2x^2)W_1 - xW_1 = 0.$$

The only solution in $Q(x)$ of the second equation is 0 as can be seen by expanding W_1 in partial fractions and comparing degrees. Computing $\bar{\mathbf{Y}} = \mathbf{VW}$, we see that $Y_0 = x$, $Y_1 = 0$. So $y = x$ is a solution of $y' + \sqrt{x}y = 1 + x\sqrt{x}$.

Lemma 3. *Let $E \subset F$ be differential fields with $C(E) = C(F)$ and let $a_{n-1}, \ldots, a_0$ and b be elements of E. Let $\theta \in F$ be transcendental over E and satisfy $\theta'/\theta \in E$. If*

$$L(y) = y^{(n)} + a_{n-1}y^{(n-1)} + \ldots + a_0 y = b\theta$$

has a solution in $E(\theta)$, then it has a solution of the form $B\theta$ with $B \in E$. Furthermore, B will satisfy a linear differential equation $L_0(y) = b$ where the coefficients of $L_0(y)$ lie in E and can be determined from $a_{n-1}, \ldots, a_0$ and θ'/θ and any solution B of $L_0(y) = b$ will determine a solution $B\theta$ of $L(y) = b\theta$.

Proof. Let $u \in E(\theta)$ be a solution of $L(y) = b\theta$. We claim that the only irreducible polynomial p in θ that could possibly divide the denominator of u is $p = \theta$. To see this assume that $p \neq \theta$ is an irreducible polynomial dividing the denominator of u. Since p' does not divide p, we can apply L to the partial fraction decomposition of u and conclude that p divides the denominator of $L(u)$. Since p does not divide the denominator of $b\theta$, we have a contradiction. Therefore,

$$u = B_{-r}\theta^{-r} + \ldots + B_0 + B_1\theta + \ldots + B_s\theta^s.$$

For each i, $L(B_i\theta^i)$ is a monomial in θ of degree i. Therefore, comparing powers of θ in $L(u) = b\theta$, we have $L(B_1\theta) = b\theta$. Since $\theta'/\theta \in E$, we can write $L(B_1\theta) = L_0(B_1)\theta$ where L_0 is a linear operator whose coefficients lie in E and can be determined from $a_{n-1}, \ldots, a_0$ and θ'/θ. □

We now present our algorithm based on Theorem 1. In fact we shall prove a stronger result below.

PROPOSITION 1. *Let k be a finite algebraic extension of $Q(x)$ and let $a_{n-1}, \ldots, a_0, b$ and u be in k. Let η satisfy $\eta'/\eta = u$ and assume $C(k(\eta)) = C(k)$. Furthermore, if η is algebraic over k, assume that we are given the minimum polynomial of η over k. One can then decide, in a finite number of steps, if*

$$L(y) = y^{(n)} + a_{n-1}y^{(n-1)} + \ldots + a_0 y = b\eta$$

has a Liouvillian solution, and if it does, produce such a solution.

PROOF. We shall proceed by induction on n. For $n = 1$, $L(y) = y' + a_0 y = b\eta$ always has a Liouvillian solution

$$y = \exp\left(-\int a_0\right)\int b\eta \exp\left(\int a_0\right).$$

Now assume that the proposition is true for equations of order $<n$. If $L(y) = b\eta$ has a Liouvillian solution, then Theorem 1 implies that either $L(y) = b\eta$ has a solution in $k(\eta)$ or $L(y) = 0$ has a solution w such that w'/w is algebraic over k. Let us first decide if $L(y) = b\eta$ has a solution in $k(\eta)$. If η is algebraic over k, Lemma 2 allows us to decide if $L(y) = b\eta$ has a solution in $k(\eta)$ (a finite algebraic extension of k) and find one if it does. If η is transcendental over k, Lemma 3 implies that if $L(y) = b\eta$ has a solution in $k(\eta)$, it has one of the form $B\eta$, where B is in k and satisfies a differential equation of the form $L_0(y) = b$ whose coefficients can be computed from $a_{n-1}, \ldots, a_0$ and u. We use Lemma 2 to decide if $L_0(y) = b$ has a solution and find one if it does. If B is such a solution, then $B\eta$ satisfies $L(y) = b\eta$. Therefore we can assume that $L(y) = b\eta$ has no solution in $k(\eta)$. Let us now decide if $L(y) = 0$ has a solution w such that w'/w is algebraic over k. Theorem 4.1 of Singer (1980) allows us to decide this question. If $L(y) = 0$ has no such solution, Theorem 1 implies that $L(y) = b$ has no Liouvillian solution. If $L(y) = 0$ has such a solution, then Theorem 4.1 of Singer (1980) allows us to find a v, algebraic over k such that any element ζ satisfying $\zeta'/\zeta = v$ also satisfies $L(y) = 0$. For any such ζ, we can define a new variable y_1 by $y = \zeta y_1$. We then have $L(\zeta y_1) = (L_1(y_1'))\zeta$ where L_1 is a new differential operator of order $n-1$ whose coefficients lie in $k(v)$ and can be determined from the coefficients of L and v. Furthermore, if y satisfies $L(y) = b\eta$, then y_1' will satisfy $L_1(y_1') = b\eta\zeta^{-1}$ (where $y_1 = y\zeta^{-1}$). Note that $\theta = \eta\zeta^{-1}$ satisfies $\theta'/\theta = u - v$. Conversely, if z is a Liouvillian solution of $L_1(z) = b\theta$ where θ is any solution of $\theta'/\theta = u - v$ and ζ is any element satisfying $\zeta'/\zeta = v$, then $y = \zeta\int z$ is a Liouvillian solution of $L(y) = b\eta$. Using techniques of Risch (1970) or Baldassarri & Dwork (1979, pp. 68–71), we can determine if $\theta'/\theta = u - v$ has an algebraic solution and find one if it does (by finding θ we mean we have determined its minimal equation). If this latter equation has an algebraic solution, denote this by θ. If it does not have an algebraic solution, we can formally adjoin a

transcendental solution, which we also refer to as θ. Note that $C(k(v, \theta))$ is at worst algebraic over $C(k)$. Let k_0 be the algebraic closure of k in $k(v, \theta)$, $\eta_0 = b\theta$ and $u_0 = u - v + b'/b$. We then apply the induction hypothesis to decide if $L_1(z) = \eta_0$ has a Liouvillian solution and produce one if it does. By the above discussion, this is enough to determine if $L(y) = b\eta$ has a Liouvillian solution. □

Note that our procedure does not increase the order of the differential equation and that all calculations can be performed in algebraic extensions of $Q(x)$.

The following example illustrates the method described in Proposition 1.

EXAMPLE 3. Let $k = Q(x)$, $H = \exp(x^2)$ and consider the equation

$$y'' - 2y' + y = (2x - 1)\,e^{x^2}.$$

The first step in the algorithm is to determine if this equation has a solution in $k(\eta)$. If it does, the solution will be of the form $B\eta$ where $B \in k$ and B satisfies

$$B'' + 2(x-1)B' + (3 + 4x^2)B = 2x - 1.$$

If $B = p/q$ where p and q are relatively prime elements of $Q[x]$, q monic, one sees (using partial fractions) that $q = 1$. Therefore $B \in Q[x]$. Comparing degrees we see that no such B can exist.

We now must determine if $y'' - 2y' + y = 0$ has a solution ζ such that ζ'/ζ is algebraic over $Q(x)$. An algorithm for this is outlined by Singer (1981). We shall only test to see if $y'' - 2y' + y = 0$ has a solution ζ such that ζ'/ζ is *in* $Q(x)$ and use the simpler algorithm referred to in Lemma 3.4 of Singer (1981) (since $y'' - 2y' + y = 0$ has constant coefficients we immediately see that e^x is such a solution, but using this algorithm will be instructive). Write $\zeta = \exp(\int R)$ where $R \in Q(x)$ to be determined. Since

$$\zeta'' - 2\zeta' + \zeta = (R' + R^2 - 2R + 1)\zeta = 0,$$

R satisfies $R' + R^2 - 2R + 1 = 0$. If $c \in \bar{Q}$ is a zero of the denominator of R, we may write

$$R = \frac{\alpha - n}{(x-c)^n} + \frac{\alpha - n + 1}{(x-c)^{n-1}} + \dots.$$

Substituting the expression in $R' + R^2 - 2R + 1 = 0$ and equating coefficients, we see that $n = 1$ and $\alpha_{-1}^2 - \alpha_1 = 0$ or $\alpha_{-1} = 1$. This implies that

$$R = \frac{1}{(x - c_1)} + \dots + \frac{1}{(x - c_m)} + p(x)$$

for some $p(x) \in Q[x]$ (of course, we have not yet determined the c_i or even how many of them there are). We now determine the order of R at infinity. Let

$$R = \alpha_n x^n + \alpha_{n-1} x^{n-1} + \dots + \alpha_0 + \alpha_{-1} x^{-1} + \dots.$$

Substituting this expression in $R' + R^2 - 2R + 1$ we see that $n = 0$, so $p(x) = c \in Q$. Furthermore, the x^0 term in $R' + R^2 - 2R + 1$ is $c^2 - 2c + 1$ so $c = 1$. Therefore

$$= \exp\left(\int \frac{1}{(x - c_1)} + \dots + \frac{1}{(x - c_m)} + 1\right)$$

$$= f(x)\,e^x,$$

where $f(x) = (x - c_1)\dots(x - c_m)$. Substituting this expression for ζ into $y'' - 2y' + y = 0$

yields $\eta''-2\eta'+\eta=f''e^x=0$, so $f(x)=c_0+c_1x$. Any ζ will serve our purposes, so let $c_0=1$, $c_1=0$ and $\zeta=e^x$.

Proceeding as described in Proposition 1, we let $y=e^x y_1$. We then have

$$y''=-2y'+y=e^x y_1=(2x-1)e^{x^2}.$$

Therefore $y_1''=(2x-1)e^{x^2-x}$. Rewriting this as $(y_1')'=(2x-1)e^{x^2-x}$, we have $y_1'=e^{x^2-x}$ so

$$y=e^x\int e^{x^2-x}.$$

3. Elementary Solutions of Linear Differential Equations

Although our aim is to prove Theorem 2, a statement about elementary solutions of differential equations, we will first have to take a closer look at Liouvillian solutions. We shall need some facts from the Galois theory of differential equations. Let k be a differential field of characteristic 0 with algebraically closed subfield of constants. Let

$$L(y)=y^{(n)}+a_{n-1}y^{(n-1)}+\ldots+a_0y=0$$

be a linear differential equation with coefficients in k. We say E is the *Picard–Vessiot (P.V.) extension of* k corresponding to $L(y)=0$ if: (1) $E=k\langle y_1,\ldots,y_n\rangle$ (i.e. E is generated over k by $y_1,\ldots,y_n$ and all their derivatives) where $y_1,\ldots,y_n$ are solutions of $L(y)=0$, linearly independent over $C(k)$, and (2) $C(k)=C(E)$. It is known that given k and $L(y)$ as above, the corresponding P.V. extension exists and is unique up to isomorphism (Kolchin, 1973, p. 412). Let $E=k\langle y_1,\ldots,y_n\rangle$ be a P.V. extension of k corresponding to $L(y)=0$ and let σ be a differential k-automorphism of E. Since $\sigma(y_i)$ satisfies $L(y)=0$, we have that

$$\sigma(y_i)=\sum_{j=1}^{n}c_{ij}y_j$$

for some constants c_{ij}. From this fact, we can deduce that the group of differential k-automorphisms of E is isomorphic to a group of invertible $n\times n$ matrices with entries in $C(k)$ via the map identifying σ with (c_{ij}). The group of differential k-automorphisms of E over k is called the Galois group of E over k and is denoted by $G(E/k)$. It is known that an element $u\in E$ is actually in k if and only if $\sigma u=u$ for all σ in $G(E/k)$ (Kolchin, 1973, p. 398).

A differential field K is said to be a *primitive* (*logarithmic*) extension of k if K is a Liouvillian (elementary) extension of k and the elements used in building the tower from k to K are either algebraic or integrals of elements in the previous field (logarithms of elements in the previous field). u is said to be primitive (logarithmic) over k if u lies in a primitive (logarithmic) extension of k.

LEMMA 4. *Let k be a differential field of characteristic* 0 *with an algebraically closed field of constants. Let $a_{n-1},\ldots,a_0$ be in k and let E be the P.V. extension of k corresponding to*

$$L(y)=y^{(n)}+a_{n-1}y^{(n-1)}+\ldots+a_0y=0.$$

(*a*) *If u is a solution of $L(y)=0$ and $C(k\langle u\rangle)=C(k)$ then there is a differential k-isomorphism of $k\langle u\rangle$ into E.*

(*b*) *If $L(y)=0$ has a non-zero Liouvillian* (*elementary, primitive, logarithmic*) *solution, then for some r, $1\leqslant r\leqslant n$, $L(y)=L_{n-r}(L_r(y))$, where L_{n-r} and L_r are linear differential* *operators of order* $n-r$ *and* r *respectively with coefficients in* k *and the P.V. extension of* k *corresponding to* $L_r(y)=0$ *lies in a Liouvillian (elementary, primitive, logarithmic) extension of* k.

PROOF. (a) Let E be a P.V. extension of k associated with $L(y)=0$ and let $k\langle u\rangle\langle z_1,\ldots,z_n\rangle$ be the P.V. extension of $k\langle u\rangle$ associated with $L(y)=0$. Since $z_1,\ldots,z_n$ form a basis for the set of solutions of $L(y)=0$, we have that

$$u=\sum_{i=1}^{n} c_i z_i$$

for some $c_i\in C(k\langle u\rangle)=C(k)$. Therefore $u\in k\langle z_1,\ldots,z_n\rangle$. Since P.V. extensions are unique, there exists a k-isomorphism σ mapping $k\langle z_1,\ldots,z_n\rangle$ onto E. σ restricts to an isomorphism of $k\langle u\rangle$ into E.

(b) We shall prove this in the Liouvillian case, the other case following in a similar manner. Let E be the P.V. extension of k corresponding to $L(y)=0$. If $L(y)=0$ has a Liouvillian solution then, by part (a), we can assume $L(y)=0$ has a Liouvillian solution in E. Let V be the $C(k)$-vector space of such solutions and let $y_1,\ldots,y_r$ be a basis for this space. Let

$$L_r(y)=\frac{Wr(y,y_1,\ldots,y_r)}{Wr(y_1,\ldots,y_r)},$$

where $Wr(y_1,\ldots,y_r)$ is the Wronskian determinant. We claim that $L_r(y)$ has coefficients in k. First note that any $\sigma\in G(E/k)$ will leave V invariant. Therefore σ restricted to V can be represented by an $r\times r$ matrix (c_{ij}) with entries from $C(k)$. If we apply σ to the coefficients of $L_r(y)$ and denote this by $L_r^\sigma(y)$, we have

$$\begin{aligned} L_r^\sigma(y) &= \frac{Wr(y,\sigma y_1,\ldots,\sigma y_r)}{Wr(\sigma y_1,\ldots,\sigma y_r)} \\ &= \frac{\det(c_{ij})\cdot Wr(y,y_1,\ldots,y_r)}{\det(c_{ij})\cdot Wr(y_1,\ldots,y_r)} \\ &= L_r(y). \end{aligned}$$

Therefore, the coefficients of $L_r(y)$ are left fixed by all σ in $G(E/k)$ and so lie in k. We can find $b_{n-r-1},\ldots,b_0$ in k so that the operator

$$\begin{aligned} L_s(y) &= L(y)-(L_r(y))^{(n-r)}-b_{n-r-1}(L_r(y))^{(n-r-1)}-\ldots-b_0(L_r(y)) \\ &= L(y)-L_{n-r}(L_r(y)) \end{aligned}$$

has order equal to $s<r$. $L_s(y)=0$ will have r linearly independent solutions $y_1,\ldots,y_r$ so it must be identically zero. Therefore $L(y)=L_{n-r}(L_r(y))$. The P.V. extension of k corresponding to $L_r(y)=0$ is isomorphic to $k\langle y_1,\ldots,y_r\rangle$. Since each y_i is Liouvillian, this P.V. extension lies in a Liouvillian extension of k. □

LEMMA 5. *Let* k *be a differential field of characteristic* 0 *and let* $a_{n-1},\ldots,a_0$ *be elements of* k. *If*

$$L(y)=y^{(n)}+a_{n-1}y^{(n-1)}+\ldots+a_0y=0$$

has a non-zero elementary solution, then $L(y)=0$ *has a non-zero solution* u *such that*

$$u'/u=u_0'+\sum c_iu_i'/u_i$$

where the u_i *are algebraic over* k *and the* c_i *are constants algebraic over* k.

PROOF. We may assume that k has an algebraically closed field of constants. By part (b) of Lemma 4, we may write $L(y)=L_{n-r}(L_r(y))$ for some r, $1 \leqslant r \leqslant n$, where L_{n-r} and L_r are linear operators with coefficients in k and where the P.V. extensions E of k associated with $L_r(y)=0$ lies in an elementary extension of k. By part (a) of Lemma 1, $L_r(y)=0$ has a solution u such that u'/u is algebraic over k and by part (a) of Lemma 4, we may assume $u \in E$. Therefore u is elementary and so $\int u'/u = \log u$ is elementary. By Liouville's theorem (Theorem 3 of Rosenlicht (1976), we have that $u'/u = u_0' + \sum c_i u_i'/u_i$ for some u_i algebraic over k and constants c_i algebraic over k. □

LEMMA 6. *Let k be a differential field of characteristic 0 and let $a_{n-1}, \ldots, a_0, b$ be elements in k. If*

$$L(y) = y^{(n)} + a_{n-1} y^{(n-1)} + \ldots + a_0 y = b$$

has a non-zero solution in a primitive extension of k, then either:

(i) $L(y)=0$ has a non-zero solution in an algebraic extension of k, or
(ii) $L(y)=b$ has a solution in k.

PROOF. Note that we do not exclude the case $b=0$. Let K be a primitive extension of k that contains an element w such that $L(w)=b$. We proceed by induction on the transcendence degree of K over k.

Assume that the transcendence degree of K over k is 0. If $b=0$, we are done. If $b \neq 0$, let Tr denote the trace function from K to k. We then have $L(\mathrm{Tr}(w)) = Nb$ for some integer N. Therefore $L((1/N)\mathrm{Tr}(w)-w)=0$. Either $(1/N)\mathrm{Tr}(w)-w=0$, in which case w is in k, or $(1/N)\mathrm{Tr}(w)-w \neq 0$, in which case $L(y)=0$ has a non-zero solution in an algebraic extension of k.

We now assume that K is algebraic over $F(\theta)$ where $\theta' \in F$ and where the transcendence degree of F over k is less than the transcendence degree of K over k. We again treat separately the cases $b \neq 0$ and $b=0$.

If $b \neq 0$, we expand w in fractional powers of θ^{-1} and write

$$w = \sum_{i \geqslant p} \alpha_i \theta^{-\frac{i}{m}}$$

with the a_i algebraic over F. Since $\theta' \in F$, we have $w' = \alpha_p' \theta^{-\frac{p}{m}} +$ terms involving higher powers of $\theta^{-\frac{1}{m}}$. If $p \neq 0$, we compare coefficients of $\theta^{-\frac{p}{m}}$ in $L(w)=b$ to conclude that $L(\alpha_p)=0$. Therefore $L(y)=0$ has a non-zero solution algebraic over F and by induction, this equation will have a non-zero solution algebraic over k. If $p=0$, we have $L(a_p)=b$ and so, again by induction, we can conclude that $L(y)=b$ has a solution in k or $L(y)=0$ has a solution algebraic over k.

If $b=0$, we may assume k is algebraically closed and use part (b) of Lemma 4 to conclude that $L(y)=L_{n-r}(L_r(y))$ where L_{n-r} and L_r are linear operators with coefficients in k and where the P.V. extension E of k corresponding to $L_r(y)=0$ lies in a primitive extension of k. Parts (a) of Lemmas 1 and 4 imply that $L_r(y)=0$ has a solution u in E where u'/u is in k. Therefore $L(y)=0$ has a solution u in an extension $k(\theta_1, \ldots, \theta_m)$ of k, having the same constants as k, where (i) u'/u is in over k and (ii) for each i, $1 \leqslant i \leqslant m$, either θ_i is algebraic over $k(\theta_1, \ldots, \theta_{i-1})$ or θ_i' is in $k(\theta_1, \ldots, \theta_{i-1})$. Since θ_m' and u'/u are in $k(\theta_1, \ldots, \theta_{m-1})$, Theorem 2 of Rosenlicht (1976) allows us to conclude that u is algebraic over $k(\theta_1, \ldots, \theta_{m-1})$. Repeating this argument for $\theta_{m-1}, \ldots, \theta_1$ we finally conclude that u

is algebraic over (and therefore a number of) k. Since u is a non-zero solution of $L_r(y)=0$, we have $L(u)=0$. □

Parts of the next lemma appear implicitly in Ostrowski's 1946 paper. Given a differential field k of characteristic 0 and an element $u\in k$, $u\neq 0$, either there is an element $v\in k$ such that $v'=u'/u$ or $C(k(t))=C(k)$, where t is transcendental over k and $t'=u'/u$ (Risch, 1969, p. 172). Using this fact, we can construct a field, denoted by $k(\log)$, that has the following properties:

(i) $C(k(\log))=C(k)$.
(ii) $k(\log)$ is a purely transcendental extension of k generated by a set of algebraically independent elements $\{t_i\}$, where for each i, there exists a $u_i\in k$ such that $t_i'=u_i'/u_i$.
(iii) If $u\in k$, $u\neq 0$, there exists a $t\in k(\log)$ such that $t'=u'/u$.

We let $k[\log]$ denote the differential ring generated over k by the t_i. If $t\in k(\log)$ and satisfies $t'\in k$, then the Kolchin–Ostrowski theorem (Corollary of Rosenlicht (1976)) implies that $t=\sum c_it_i+v$ where each c_i is a constant, the sum is finite and $v\in k$. In particular, if $\{\tilde{t}_i\}$ is some other set of elements satisfying (ii), then $k[\{\tilde{t}_i\}]=k[\{t_i\}]$. We shall fix, once and for all, a set $\{t_i\}$ satisfying (ii).

Lemma 7. *Let k be a differential field of characteristic zero and let $k(\log)$ be as above. Let b be an element of $k[\log]$ that is a polynomial in the t_i of total degree at most n and let $a\in k$.*

(*a*) *If $y'=b$ has an elementary solution, then it has a solution g in $k[\log]$, where g is a polynomial in the t_i of total degree at most $n+1$. Furthermore, the terms of degree $n+1$ in g have constant coefficients.*

(*b*) *If $y'+ay=b$ has a solution in $k(\log)$, then it has a solution g in $k[\log]$, where g is a polynomial in the t_i of total degree at most $n+1$.*

Proof. The proof is by induction on r, the number of t_i that appear in b.

(i) $r=0$. Note that in this case b has total degree 0 in the t_i.

(a) From Liouville's theorem we can conclude that $y'=b$ has a solution of the form $y=u_0+\sum c_i\log u_i+d$ where the c_i, d are constant, $u_i\in k$ and the $\log u_i$ are in $k(\log)$. As already noted, the Kolchin–Ostrowski theorem implies that each $\log u_i$ may be written as a linear polynomial in the t_i, so $y'=b$.

(b) Let $w\in k(\log)$ be a solution of $y'+ay=b$. If $y'+ay=0$ has a solution u in k, let $w=uv$. v satisfies $(uv)'+auv=b$ so $v'=b/u$. Since v is elementary, Liouville's theorem implies that v is a polynomial of degree at most 1 in the t_i, so $w=uv$ satisfies the conclusion of the Lemma. Therefore we can assume that $y'+ay=0$ has no solution in k. We now claim that $y'+ay=0$ has no solution in $k(\log)$. If $u\in k(\log)$ were a solution of $y'+ay=0$, then $u\in k(t_1,\ldots,t_n)$ for some n. Since $u'/u\in k$ and $t_i'\in k$ for $i=1,\ldots,n$, the Corollary in Rosenlicht (1976) implies that u would be algebraic over k. Since $k(t_1,\ldots,t_n)$ is a purely transcendental extension of k, we would have $u\in k$, a contradiction. Now, let $w\in k(t_1,\ldots,t_n)$ for some n. We shall now show by induction on n that $w\in k$ (an even stronger conclusion than our Lemma). If $n=0$, we are done. Let $K=k(t_1,\ldots,t_{n-1})$ and let $t=t_n$. We first claim that $w\in K[t]$. If not, then some irreducible polynomial p in t divides the denominator of w and we may write

$$w=\frac{A_\alpha}{p^\alpha}+\frac{A_{\alpha-1}}{p^{\alpha-1}}+\ldots.$$

Substituting this expression into $y'+ay=b$ we have

$$\frac{-\alpha p' A_\alpha}{p^{\alpha+1}}+\ldots+\frac{aA_\alpha}{p^\alpha}+\ldots=b.$$

Since $p \nmid p'A_\alpha$, we have a contradiction. Therefore $w \in K[t]$ and we may write $w=A_m t^m+\ldots+A_0$. Substituting into $y'+ay=b$, we have

$$A'_m t^m+(mA_m t'+A'_{m-1})t^{m-1}+\ldots+aA_m t^m+\ldots=b.$$

If $m>0$, we have $A'_m+aA_m=0$, contradicting the fact that $y'+ay=0$ has no solutions in $k(\log)$. Therefore, $w \in K$ and by induction $w \in k$.

(ii) $r>0$. Let t be one of the t_i appearing in b and let $b=b_n t^n+\ldots+b_0$ where the b_i are polynomials in $r-1$ of the t_i, say $t_1,\ldots,t_{r-1}$, of degree at most $n-i$. Let $K=k(t_1,\ldots,t_{r-1})$.

(a) If $y'=b$ has a solution in an elementary extension of k, then $b=w'_0+\sum c_i w'_i/w_i$ for some $w_i \in K(t)$ and constants c_i. Let

$$w_0=B_m t^m+\ldots+B_0+\sum_i\sum_j \frac{B_{ij}}{p_i^j}$$

be the partial fraction decomposition of w_0 where the $p_i \in K[t]$ are monic and irreducible and let $w_i=d_i\prod p_j^{n_j}$, $d_j \in K$ and n_j natural numbers. We then have

$$\begin{aligned} b_n t^n+\ldots+b_0 &= B'_m t^m+mB_m t^{m-1}(u'/u)+\ldots+B'_0 \\ &\quad+\left(\sum_i\sum_j \frac{B_{ij}}{p_j^i}\right)'+\sum_i c_i\sum_j \frac{n_j p'_j}{p_j}+\sum_i c_i\frac{d'_i}{d_i} \end{aligned}$$

where $t'=u'/u$. Comparing degrees we have that $m=n+1$ and $B'_m=0$. Furthermore,

$$\left(\sum_i\sum_j \frac{B_{ij}}{p_j^i}\right)'=0 \quad\text{and}\quad \sum_i c_i\sum_j \frac{n_j p'_j}{p_j}=0.$$

Therefore, the B_i and d_i satisfy

$$\begin{aligned} B'_{n+1} &= 0 \\ b_n &= (n+1)B_{n+1}(u'/u)+B'_n \\ b_{n-1} &= nB_n(u'/u)+B'_{n-1} \\ &\vdots \\ b_1 &= 2B_2(u'/u)+B'_1 \\ b_0 &= B_1(u'/u)+B'_0+\sum c_i\frac{d'_i}{d_i}. \end{aligned}$$

Since B_{n+1} is a constant and B_n satisfies $B'_n=b_n-(n+1)B_{n+1}(u'/u)$, the induction hypothesis implies that B_n is a polynomial of degree at most 1 in $k[t_1,\ldots,t_{r-1}]$, with the terms of degree 1 having constant coefficients. Since B_{n-1} satisfies $B'_{n-1}=b_{n-1}-nB_n(u'/u)$, B_{n-1} is a polynomial of degree at most 2 in $k[t_1,\ldots,t_{r-1}]$ with terms of degree 2 having constant coefficients. In this way, we have for $i=1,\ldots,n$, B_i is a polynomial of degree at most $n-i+1$ in $k[t_1,\ldots,t_{r-1}]$ with terms of degree $n-i+1$ having constant coefficients. $B_0+\sum c_i\log d_i$ satisfies $y'=b_0+B_1(u'/u)$. Therefore, by the induction hypothesis, this latter equation has a solution B that is a polynomial in $k[\log]$ of degree at most $n+1$ with the terms of degree $n+1$ having constant coefficients. Furthermore, we have $B_0+\sum c_i\log d_i=B+c$ where c is a constant. We would be done if we knew that the d_i

were in k (at this point we only know they are in K). We have

$$\sum c_i d_i' + (B_0 - B)' = 0.$$

Using a standard trick (Rosenlicht, 1976) we may assume that the c_i are linearly independent over the rationals. Since B_0, B and the d_i are in $k(\log)$, we may apply Theorem 2 of Rosenlicht (1976) to conclude that the d_i are algebraic over k. Since $k(\log)$ is a purely transcendental extension of k, we have that the d_i are in k. Therefore $B_0 = B - \sum c_i \log d_i + c$. As noted before, the Kolchin–Ostrowski theorem implies that each $\log d_i$ is a linear polynomial in the t_i with constant coefficients, so the right-hand side is a polynomial of degree at most $n+1$ in the t_i, whose terms of degree $n+1$ have constant coefficients.

(b) Let $w \in k(\log)$ be a solution of $y' + ay = b$. If $y' + ay = 0$ has a solution u in k, let $w = uv$. v satisfies $(uv)' + auv = b$, so $v' = b/u$. Since b/u is a polynomial of degree n in the t_i, part (a) implies that v is a polynomial of degree at most $n+1$ in the t_i. So $w = uv$ will satisfy the conclusion of the Lemma. Therefore, we may assume $y' + ay = 0$ has no solution in k. This implies as before, that $y' + ay = 0$ has no solution in $k(\log)$. We now will show that $w \in K[t]$. If not, some irreducible polynomial p in t divides the denominator of w and we may write

$$w = \frac{A_\alpha}{p^\alpha} + \frac{A_{\alpha-1}}{p^{\alpha-1}} + \dots.$$

Substituting into $y' + ay = b$, we have

$$\frac{-\alpha p' A_\alpha}{p^{\alpha+1}} + \dots + \frac{aA_\alpha}{p^\alpha} + \dots = b.$$

Since $p \nmid p'A_\alpha$ and $b \in K[t]$, we would have a contradiction. Therefore, we have $w = B_m t^m + \dots + B_0$. Substituting, we have

$$B_m' t^m + (mB_m t' + B_{m-1}') t^{m-1} + \dots + aB_m t^m + \dots = b_n t^n + \dots.$$

Since $y' + ay = 0$ has no solution in $k(\log)$, we have $m = n$ and $B_n' + aB_n = b_n$. Since we can have at most one solution of $y' + ay = b_n$ in $k(\log)$, we can apply the induction hypothesis to conclude that B_n is a polynomial in the t_i of degree at most 1. Furthermore, since $B_n \in k[t_1, \dots, t_{r-1}]$ we conclude that B_n is a polynomial in $t_1, \dots, t_{r-1}$ of degree at most 1. B_{n-1} satisfies $y' + ay = b_{n-1} - nB_n t'$. Since the right-hand side of this equation is a polynomial of degree at most 1 in $t_1, \dots, t_{r-1}$ we can again apply the induction hypothesis to conclude that B_{n-1} is a polynomial in $t_1, \dots, t_{r-1}$ of degree at most 2. In this way we see that each B_i is a polynomial of degree at most $n-i+1$ in $t_1, \dots, t_{n-1}$. Therefore, w is a polynomial of degree at most $n+1$ in $t_1, \dots, t_r$. □

The next lemma shows that the space of elementary solutions of a homogeneous linear differential equation has a basis of a very special form.

LEMMA 8. *Let k be an algebraically closed differential field of characteristic 0 and let $L(y) = 0$ be an nth order homogeneous linear differential equation with coefficients in k. Then there exists a differential field K such that $C(K) = C(k)$ and elements $u_i, \theta_i, i = 1, \dots, m$, in K such that*

(*i*) *Each $u_i \in k[\log]$ is a polynomial of degree at most $n-1$.*

(*ii*) *Each* θ_i *satisfies*

$$\theta_i'/\theta_i = v_{i0}' + \sum c_{ij} \frac{v_{ij}'}{v_{ij}}$$

for some $v_{ij} \in k$ *and constants* c_{ij}.

(*iii*) $\{u_1\theta_1, \ldots, u_m\theta_m\}$ *forms a basis for the space of elementary solutions of* $L(y) = 0$.

PROOF. Note we do not conclude that the θ_i are transcendental over k. The proof proceeds by induction on n. By Lemma 5, if $L(y) = 0$ has an elementary solution, it has a solution $\theta \neq 0$ such that $\theta'/\theta = v_0' + \sum c_i(v_i'/v_i)$ for some $v_i \in k$ and constants c_i. If $n = 1$, then $\{\theta\}$ is the desired basis. Now assume the lemma true for equations of order less than n. Let $y = \theta Y$ and substitute into $L(y) = 0$. We then have that y satisfies $L(y) = 0$ if and only if Y satisfies $\bar{L}(Y') = 0$, where $\bar{L}$ is a linear operator of order $n-1$ with coefficients in k. Therefore, the map $\phi(y) = (y/\theta)'$ is a linear map of the space of elementary solutions of $L(y) = 0$ onto the space of elementary solutions w of $\bar{L}(y) = 0$ such that w has an elementary anti-derivative. We wish to construct a good basis for the image of ϕ. To do this, let $\{u_1\theta_1, \ldots, u_m\theta_m\}$ be a basis for the space V of all elementary solutions of $\bar{L}(y) = 0$ where the u_i, θ_i satisfy (i) and (ii) above. Let S be the set of elements z of V satisfying (a) $z = uv$ where $u \in k[\log]$ is a polynomial of degree at most $n-2$ and $v'/v = w_0' + \sum c_i(w_i'/w_i)$ for some $w_i \in k$ and constants c_i and (b) z has an elementary anti-derivative. We claim that S spans the image of ϕ. Let w be in the image of ϕ, i.e. w is a solution of $\bar{L}(y) = 0$ and w has an elementary anti-derivative. We may write $w = \sum c_i u_i \theta_i$. By combining terms where $\theta_i/\theta_j \in k$ for $i \neq j$, we may write $w = \sum z_i \theta_i$ where each $z_i \in k[\log]$ is a polynomial of degree at most $n-2$ and θ_i are as before and satisfy $\theta_i/\theta_j \notin k$ for $i \neq j$. Since θ_i/θ_j satisfies $(\theta_i/\theta_j)'/(\theta_i/\theta_j) \in k$ and $\theta_i/\theta_j \notin k$, repeated application of Theorem 2 of Rosenlicht (1976) shows that $\theta_i/\theta_j \notin k(\log)$. Since w has an elementary anti-derivative, Theorem 2 of Rosenlicht (1975) implies that each $z_i\theta_i$ has an elementary anti-derivative and so is in S. Therefore S spans the image of ϕ. Let $\{u_1v_1, \ldots, u_rv_r\}$ be a maximal linearly independent subset of S. We then have that $\theta, \theta\int u_1v_1, \ldots, \theta\int u_rv_r$ is a basis for the space of elementary solutions of $L(y) = 0$. We will show that it is of the desired form. For each v_i that is not algebraic over $k(\log)$, Theorem 2 of Rosenlicht (1975) implies that u_iv_i will have an anti-derivative of the form U_iv_i for some U_i in $k(\log)$. Since U_i satisfies $y' + (v_i'/v_i)y = u_i$, part (b) of Lemma 7 implies that U_i may be chosen to be in $k[\log]$ and to be a polynomial of degree at most $n-1$. For each v_i that is algebraic over $k(\log)$, we again have that v_i must be in k. Therefore, $u_iv_i \in k[\log]$ is a polynomial of degree at most $n-2$, so by part (a) of Lemma 7, $\int u_iv_i \in k[\log]$ is a polynomial of degree at most $n-1$. □

LEMMA 9. *Let* k *be an algebraically closed differential field of characteristic* 0 *and let* $a_{n-1}, \ldots, a_0, b$ *be elements of* k. *Let* θ *satisfy* $\theta'/\theta \in k$ *and* $C(k(\theta)) = C(k)$. *If*

$$L(y) = y^{(n)} + a_{n-1}y^{(n-1)} + \ldots + a_0 y = b\theta$$

has a solution elementary over k, *then* $L(y) = b\theta$ *has a solution of the form* $B\theta$, *where* $B \in k[\log]$ *is a polynomial in the* t_i *of degree at most* n.

PROOF. We proceed by induction on n. Let w be an elementary solution of $L(y) = b\theta$. If $n = 0$, then $w = (b/a_0)\theta$. Now assume that the result is true for equations of order less than n. If $L(y) = 0$ has no elementary solution, then part (c) of Lemma 1 implies $w \in k(\theta)$. If θ is in k, we may write $w = (w\theta^{-1})\theta$ to satisfy the conclusion of the Lemma. If θ is transcendental over k, then Lemma 3 implies that $L(y) = b\theta$ has a solution of the form $B\theta$

with B in k, so we again can satisfy the conclusion of the lemma. Therefore we may assume that $L(y)=0$ has an elementary solution. Lemma 5 implies that $L(y)=0$ has a solution of the form η with $\eta'/\eta = w_0' + \sum c_i(w_i'/w_i)$, where the $w_i \in k$ and the c_i are constants. Since k is algebraically closed, we may select such an η that also satisfies $C(k(\theta, \eta)) = C(k)$ (cf. Risch, 1969). Letting $w = \eta v$ we see that v satisfies $L_{n-1}(v') = b\eta\theta^{-1}$ where L_{n-1} is an operator of order $n-1$ with coefficients in k. Since v is elementary, we may apply the induction hypothesis and conclude that $L_{n-1}(y) = b\theta\eta^{-1}$ has a solution of the form $B\theta\eta^{-1}$ where $B \in k[\log]$ is a polynomial in the t_i of degree at most $n-1$. Since v' and $B\theta\eta^{-1}$ satisfy the same inhomogeneous linear differential equation, we have that $v' = B\theta\eta^{-1} + \sum u_i\theta_i$ where $\{u_i\theta_i\}$ forms a basis for the elementary solutions of $L_{n-1}(y)=0$ as in Lemma 8. Letting $\theta_0 = \theta\eta^{-1}$ and combining terms, we may write $v' = u_0\theta_0 + \sum u_i\theta_i$ where $\theta_i/\theta_j \notin k$ for $i \neq j$. Since v is elementary, we may apply Theorem 2 of Rosenlicht (1975) and conclude that each $u_i\theta_i$ has an elementary anti-derivative. If θ_i is transcendental over k, then $u_i\theta_i$ will have an anti-derivative of the form $U_i\theta_i$ where U_i satisfies $U_i' + (\theta_i'/\theta_i)U_i = u_i$, so by part (b) of Lemma 7, U_i may be taken to be a polynomial in $k[\log]$ of degree at most n. If θ_i is algebraic over k (and therefore in k), then $u_i\theta_i$ is a polynomial in $k[\log]$ of degree at most $n-1$. Part (a) of Lemma 7 implies that it has an anti-derivative in $k[\log]$ of degree at most n. This polynomial may be written as $U_i\theta_i$ where U_i is a polynomial of degree at most n. Since $v = U_0\theta_0 + \sum u_i\theta_i$, we have $y = v\eta = U_0\theta_0\eta + \sum u_i\theta_i\eta$. Since $\sum U_i\theta_i\eta$ is a solution of $L(y)=0$ and $\theta_0 = \theta\eta^{-1}$ we have that $U_0\theta$ is a solution of $L(y) = b\theta$ of the desired form. □

We can finally give the

PROOF OF THEOREM 2. If we let $\theta = 1$ in Lemma 9, we get the first part of the conclusion of Theorem 2. Now assume $a_0 = 0$. If $L(y) = b$ has an elementary solution w, w' satisfies

$$L_{n-1}(y) = y^{(n-1)} + a_{n-1}y^{(n-2)} + \ldots + a_1 y = b.$$

Using the first part of Theorem 2, we can conclude that $L_{n-1}(y) = b$ has a solution $P_0 \in k[\log]$ that is a polynomial of degree at most $n-1$ in the t_i. Since w' and P_0 satisfy the same inhomogeneous linear equation, we may write $w' = P_0 + \sum c_i u_i\theta_i$, where $\{u_i\theta_i\}$ is a basis for the elementary solutions of $L_{n-1}(y)=0$ as in Lemma 8. Combining terms if necessary, we may write $w' = w_0\theta_0 + \sum w_i\theta_i$ where $\theta_0 = 1$, $\theta_i/\theta_j \notin k(\log)$ for $i \neq j$ and each $w_i \in k[\log]$ is a polynomial of degree $\leqslant n-1$ in the t_i. Since w is elementary each $w_i\theta_i$ will have an elementary anti-derivative. In particular, Lemma 7 implies that $w_0\theta_0 = w_0$ has an elementary anti-derivative $P \in k[\log]$ that is a polynomial in the t_i of degree at most n, whose terms of degree n have constant coefficients. Since $\int \sum w_i\theta_i$ is a solution of $L(y) = 0$, we have $L(P) = b$. □

The following lemma shows that the P above may be of degree n and that when $a_0 \neq 0$, the coefficient of the highest degree term need not be constant.

LEMMA 10. *If m and n are integers, with $n \geqslant 0$, and c is a constant, then $y = (c/x^{m)}) \log^n x$ satisfies a linear differential equation of order n with coefficients in $Q(x)$.*

PROOF. We proceed by induction on n. When $n = 0$, y satisfies $y = c/x^m$. For $n > 0$, let $z = x^m y$. We then have $z' = (nc/x)\log^{n-1} x$, so by induction, z' satisfies a linear differential equation of order $n-1$. Therefore z satisfies an nth order linear differential equation $L(z) = b$. If we replace z by $x^m y$ and rearrange terms, we see that y satisfies a linear differential equation of order n. □

Liouville's theorem on integration in finite terms, Theorem 3 of Rosenlicht (1976) says that if an element v of a differential field k has an elementary anti-derivative, then $v = u_0' + \sum c_i(u_i'/u_i)$, where each $u_i \in k$ and each c_i is constant. If we apply our Theorem 2 to the differential equation $y' = v$, we may conclude that v has an anti-derivative of the form $v_0 + \sum d_i \log v_i$, where the d_i are constant and the v_i are only algebraic over k. We need an additional argument to show that the v_i are actually in k. We may write $v = v_0' + \sum c_i(v_i'/v_i)$ with the $v_i \in k_0$ an algebraic extension of k. Letting Tr denote the trace function from k_0 to k, we have $nv = \mathrm{Tr}\, v = (\mathrm{Tr}\, v_0)' + \sum c_i(Nv_i)'/Nv_i$, where Nv_i is the norm of v_i and n is some integer. Letting $u_0 = (1/n)\mathrm{Tr}\, v_0$, $u_i = Nv_i$ and $c_i = (1/n)d_i$, gives the conclusion of Liouville's theorem. This raises the question of whether or not we can improve Theorem 2 to conclude that the u_i are actually in k. The following example shows that we cannot.

EXAMPLE 4. Let $t = e^{2x} + 1$ and consider the differential field $k = C(x, t)$. The differential equation

$$y' + \left(\frac{t'}{2t}\right) y = \left(\frac{t+1}{t(t-1)}\right) t'$$

has an elementary solution. In fact, all solutions are of the form

$$y = 2 + \frac{2}{\sqrt{t}} \log\left(\frac{\sqrt{t}-1}{\sqrt{t}+1}\right) + \frac{c}{\sqrt{t}},$$

where c is a constant. We claim that we cannot write any such y as $y = \sum u_i \log v_i + w$ where u_i, v_i and w are in k. Assuming the claim, we can conclude that the first part of Theorem 2 cannot be improved. To prove the claim assume $y = \sum u_i \log v_i + w$ as above. We then have that

$$\theta = \log\left(\frac{\sqrt{t}-1}{\sqrt{t}+1}\right)$$

and the $\log u_i$ are algebraically dependent over $k(\sqrt{t})$. The Kolchin–Ostrowski theorem implies that there are constants c_i and a, b in k such that

$$\theta + \sum c_i \log v_i = a + b\sqrt{t}.$$

Differentiating, we have

$$\frac{t'}{t(t-1)}\sqrt{t} + \sum c_i \frac{v_i'}{v_i} = a' + \left(b' + \frac{1}{2}\frac{t'}{t} b\right)\sqrt{t}.$$

Note that $b \neq 0$, since $\sqrt{t} \notin k$. Since $t' = 2(t-1)$, we have

$$\frac{t'}{t(t-1)} = \frac{2}{t} = b' + \frac{1}{2}\frac{t'}{t} b.$$

If we write $b = p/q$ where $p, q \in C(x)[e^{2x}]$, p and q relatively prime and q monic, then comparing the partial fraction expansions of both sides of the last equation allows us to conclude that $q = 1$. We therefore have

$$2 = (e^{2x} + 1)b' + e^{2x} b,$$

where $b = b_m(e^{2x})^m + \ldots + b_0$, $b_i \in C(x)$. Comparing highest powers of e^{2x} we get $0 = b_m' + (2m+1)b_m$. This yields a contradiction, since this latter equation has no non-zero solution in $C(x)$.

We can also show that the second part of Theorem 2 cannot be improved. The function y above has an elementary integral given by

$$w = \int y = 2x + \frac{1}{2}\log^2\left(\frac{\sqrt{t}-1}{\sqrt{t}+1}\right) + c\log\left(\frac{\sqrt{t}-1}{\sqrt{t}+1}\right).$$

Therefore all solutions of

$$y'' + \left(\frac{t'}{2t}\right)y' = \left(\frac{t+1}{t(t-1)}\right) + 1$$

are of the form

$$2x + \frac{1}{2}\log^2\left(\frac{\sqrt{t}-1}{\sqrt{t}+1}\right) + c_1\log\left(\frac{\sqrt{t}-1}{\sqrt{t}+1}\right) + c_2,$$

where c_1 and c_2 are constants.

To show that the second part of Theorem 2 cannot be improved, we must show that such an element cannot be written as $w + \sum u_i \log v_i$ with w, u_i, v_i algebraic over k. If it could, we would then have

$$\theta^2 = w + \sum u_i \log v_i.$$

We may assume that the $\log v_i$ that appear with non-zero coefficients in this latter expression are algebraically independent over k. The Kolchin–Ostrowski theorem allows us to conclude that $\theta = \sum c_i \log v_i$ for some constants c_i. Substituting this expression in the above formula and using the fact that the $\log v_i$ are algebraically independent over k, we have that all the u_i and c_i are 0. This implies that θ is algebraic over k. The Kolchin–Ostrowski theorem implies that θ would then be an element of $k(\sqrt{t})$, i.e. $\theta = a + b\sqrt{t}$ for some $a, b \in k$. The discussion in the preceding paragraph shows that this is impossible.

We now come to the

PROOF OF THEOREM 3. By Theorem 2, $L(y) = b$ has a solution in k[log]. By Lemma 6, either $L(y) = 0$ has a solution algebraic over k or $L(y) = b$ has a solution in k. □

The following example from Davenport (1986) shows that Theorem 3 cannot be improved.

EXAMPLE 5. The equation

$$y' + \left(\frac{1}{2x}\right)y = \frac{x+1}{x(x-1)}$$

has the elementary solution

$$y = 2 + \frac{2}{\sqrt{x}}\log\left(\frac{\sqrt{x}-1}{\sqrt{x}+1}\right).$$

Using partial fraction decompositions, one can show that $y' + y/2x = (x+1)/x(x-1)$ nor $y' + y/2x = 0$ have solutions in $C(x)$, yet $y' + y/2x = 0$ does have the solution $y = 1/\sqrt{x}$, which is algebraic over $C(x)$.

We end this section with a discussion of the problem of deciding if $L(y) = b$ has an elementary solution. In what follows, $\bar{Q}$ will denote the algebraic closure of the rationals.

LEMMA 11. *Let k be an algebraic extension of $\bar{Q}(x)$ and $L(y) = 0$ a linear differential equation with coefficients in k with associated Picard–Vessiot extension K. One can effectively find elements $u_1, \ldots, u_m$, algebraic over k such that $y_1 = \exp(\int u_1), \ldots,$*

$y_m = \exp(\int u_m)$ satisfy $L(y) = 0$ and any z in K that satisfies $L(y) = 0$ with z'/z algebraic over k lies in the $\bar{Q}$ span of $y_1, \ldots, y_m$.

PROOF. In 1981 Singer gave a procedure to decide if $L(y) = 0$ has a solution y_1 such that $y_1'/y_1 = u_1$ is algebraic over k and to find such an element if one exists. If no such element exists we are done. Otherwise, let $y = y_1 Y$ and substitute in $L(y) = 0$. Y' will then satisfy a homogeneous linear differential equation $\bar{L}(y) = 0$ of order lower than $L(y) = 0$. By induction, find $v_1, \ldots, v_r$, algebraic over k such that $z_1 = \exp(\int v_1), \ldots, z_r = \exp(\int v_r)$ satisfies the conclusion of the lemma for $\bar{L}(y)$. Since k is algebraically closed, $z_1, \ldots, z_r$ can furthermore be chosen so that $C(k(z_1, \ldots, z_r, v_1, \ldots, v_r)) = C(k)$. One easily sees that any solution z of $L(y) = 0$ with z'/z algebraic over k, lies in the $\bar{Q}$ span of $w_1 = y_1$, $w_2 = y_1 \int z_1, \ldots, w_{r+1} = y_1 \int z_r$. If each of the w_i had algebraic logarithmic derivative we would be done. In general, we must find a maximal linearly independent set of vectors $(c_1, \ldots, c_{r+1})$ in $(\bar{Q})^{r+1}$ such that $c_1 w_1 + \ldots + c_{r+1} w_{r+1}$ has logarithmic derivative algebraic over k. Let $c_1 w_1 + \ldots + c_{r+1} w_{r+1}$ have logarithmic derivative algebraic over k. Dividing by y_1 and differentiating, we see that $z = c_2 z_1 + \ldots + c_{r+1} z_r$ has an anti-derivative whose logarithmic derivative is algebraic over k. Combining those z_i such that z_i/z_j are algebraic over k and renumbering, we have $z = p_1 z_1 + \ldots + p_m z_m$, where z_i/z_j is not algebraic over k and each p_i is of the form $\sum c_j h_{ij}$ where h_{ij} is algebraic over k. Theorem 1 of Rosenlicht (1975) implies that $z = p_i z_i$ for some i. (To effectively test if z_i/z_j is algebraic, note that $(z_i/z_j)'/(z_i/z_j) = v_i - v_j$. $w'/w = v_i - v_j$ has a solution w algebraic over k with $C(k(w)) = C(k)$ if and only if all solutions w of this equation with $C(k(w)) = C(k)$ are algebraic over k (Risch, 1969). The techniques of Risch (1970) allow one to effectively decide this question.) Therefore, for each i, we must find a maximal linearly independent set of $(c_2, \ldots, c_{r+1})$ such that $(\sum c_j h_{ij}) z_i$ has an anti-derivative z with z'/z is algebraic over k or not. If z_i is not algebraic over k and if $(\sum c_j h_{ij}) z_i$ has an anti-derivative z such that z'/z is algebraic over k, then Theorem 2 of Rosenlicht (1975) implies that it has an anti-derivative of the form $a z_i$ for some a in $k(\{h_{ij}\}, v_i)$. Furthermore, a satisfies $a' + v_i a = \sum c_j h_{ij}$. Using the Main Theorem of Risch (1968, p. 7), we can find a maximal linearly independent set of $(c_2, \ldots, c_{r+1})$ such that this equation has such a solution a. (An alternative approach would be to generalise the techniques developed in Lemma 2 to handle the case where B contains parameters.) Now assume z_i is algebraic over k. If $(\sum c_j h_{ij}) z_i$ has an anti-derivative z such that z'/z is also algebraic over k, then z is algebraic over k (Rosenlicht, 1975). In this case $a' = (\sum c_j h_{ij}) z_i$ has a solution algebraic over $k(\{h_{ij}\}, z_i)$ and so, by taking traces, must have a solution in this latter field. We can again use the Main Theorem of Risch (1970) to find a maximal linearly independent set of $(c_2, \ldots, c_{r+1})$ such that this equation has a solution a. □

The following is an example of the method described in Lemma 11.

EXAMPLE 6. Let $k = \bar{Q}(x)$ and

$$L(y) = y''' - (4x+3)y'' + (4x^2 + 8x + 1)y' - (4x^2 + 4x - 1)y = 0.$$

We must determine if this equation has a solution y such that y'/y is algebraic over k. We use the algorithm of Lemma 3.4 of Singer (1981) (as in Example 3) to determine if $L(y) = 0$ has a solution y such that y'/y is in k and find that $y = e^x$ is such a solution. Letting $y = e^x Y$, we see that Y' satisfies

$$L(y) = y'' - 4xy' + (4x^2 - 2)y = 0.$$

Inductively (and omitting the details) we find that

$$z_1 = \exp(x^2) = \exp(\int x/2) \quad \text{and} \quad z_2 = x\exp(x^2) = \exp(\int (x/2) + (1/x))$$

satisfy the conclusion of the lemma for $I(y) = 0$.

We must now check to see if $z_2/z_1 \in k$. Since $(z_2/z_1)'/(z_2/z_1) = 1/x$, we must decide if $w'/w = 1/x$ has a solution algebraic over k. We shall only test to see if this equation has a solution in k (since it does). Partial fractions and a degree argument allow us to conclude that $w = x$ is a solution. Therefore $z_2/z_1 \in k$.

The algorithm now has us determine those constants c_1 and c_2 such that $(c_1 + c_2 x)z_1$ has an anti-derivative whose logarithmic derivative is algebraic over k. Since z_1 is not algebraic over k (i.e. $w' - 2xw = 0$ has no non-zero solutions algebraic over k), $(c_1 + c_2 x)z_1$ will have such an anti-derivative if and only if it has one of the form az_1 for some $a \in k$. Therefore we must find those constants c_1 and c_2 such that $a' + 2xa = c_1 + c_1 x$ has a solution a in k. We follow the procedure for this as given by Risch (1969). Let $a = p/q$ with q monic and p and q relatively prime elements of $\bar{Q}[x]$. A partial fraction argument shows that $q = 1$. Comparing degrees, one can show that $p \in \bar{Q}$. We then have $p' + 2xp = 2xp = c_1 + c_2 x$, so $c_1 = 0$ and c_2 is arbitrary. Therefore $(c_1 + c_2 x)z$ has an anti-derivative z such that z'/z is algebraic over k if and only if $c_1 = 0$. We can therefore conclude that

$$y_1 = e^x, \qquad y_2 = e^x \int xz_1 = \tfrac{1}{2}e^{x^2 - x}$$

satisfy the conclusion of Lemma 11.

LEMMA 12. *Let k be an algebraic extension of $\bar{Q}(x)$ and $L(y) = 0$ a linear differential equation with coefficients in k.*

(i) *One can decide in a finite number of steps if $L(y) = 0$ has a non-zero elementary solution and if so find $v_0, \ldots, v_n$ algebraic over k and $c_1, \ldots, c_n$ in $\bar{Q}$ such that any solution of $y'/y = v_0 + \sum c_j v_j'/v_j$ is a solution of $L(y) = 0$.*

(ii) *One can decide in a finite number of steps if $L(y) = 0$ has a non-zero algebraic solution and, if so, find one.*

PROOF. (i) Let $y_1, \ldots, y_m$ be as in the conclusion of Lemma 11. We claim that $L(y) = 0$ has a non-zero elementary solution if and only if, for some i, y_i is elementary over k. The sufficient condition is obvious. To prove the necessary condition, we note that if $L(y) = 0$ has a non-zero elementary solution, Lemma 5 implies that $L(y) = 0$ has a non-zero solution y, elementary over k, such that y'/y is algebraic over k. We may write, after possibly renumbering the y_i's, $y = d_1 y_1 + \ldots + d_m y_m$, for some d_i in $\bar{Q}$ with $\prod d_i \neq 0$. If we differentiate this $m-1$ times, we get

$$y^{(j)} = d_1 p_j(u_j) y_1 + \ldots + d_m p_j(u_m) y_m$$

for $j = 0, \ldots, m-1$, where each p_j is a differential polynomial with constant coefficients. This gives a system of linear equations for the y_i. The determinant of the coefficient matrix is

$$(\prod d_i)(\prod y_i)^{-1} Wr(y_1, \ldots, y_m) \neq 0.$$

Therefore, we can solve for the y_i's in terms of the $y^{(j)}$, $0 \leqslant j \leqslant s-1$, and the $p_j(u_i)$, $0 \leqslant j \leqslant m-1$, $1 \leqslant i \leqslant m$. Therefore, each y_i, $1 \leqslant i \leqslant s$ is elementary. Note that y_i elementary implies that y_i'/y_i has the form described in the conclusion of this lemma. Therefore, to decide if $L(y) = 0$ has a non-zero elementary solution, we find $u_1, \ldots, u_m$ as in Lemma 11 and, using the results of Risch (1970), decide if any of these has an elementary anti-derivative.

(ii) Let $y_1, \ldots, y_m$ be as in Lemma 11. Note that any algebraic solution of $L(y) = 0$ must lie in the $\bar{Q}$ span of these elements. Therefore the same proof as in (i) shows that $L(y) = 0$ has a non-zero algebraic solution if and only if at least one of the y_i is algebraic. One can decide this using results from Risch (1970). □

Let F be a differential field of characteristic 0 with constant subfield C. We say that we can solve the *problem of parameterised integration in finite terms* for F if:

(a) For any elements $f_1, \ldots, f_n$ of F, one can determine in a finite number of steps a system $\mathbf{L}$ of linear equations in N variables with coefficients in C such that $d_1 f_1 + \ldots + d_N f_N$ has an integral in an elementary extension of F for $d_1, \ldots, d_N$ in the algebraic closure $\bar{C}$ of C if and only if $(d_1, \ldots, d_N)$ satisfies $\mathbf{L}$. For each $(d_1, \ldots, d_N)$ in $\bar{C}^N$ satisfying $\mathbf{L}$ one can find, in a finite number of steps v_0, in F, v_i in $\bar{C}F$ and c_i in $\bar{C}$ for $i = 1, \ldots, m$ such that

$$d_1 f_1 + \ldots + d_N f_N = v_0' + c_1 v_1'/v_1 + \ldots + c_m v_m'/v_m.$$

(b) Let f, g_i, $i = 1, \ldots, m$ be elements of F. One can find, in a finite number of steps, $h_1, \ldots, h_r$ in F and a set $\mathbf{L}$ of linear equations in $m+r$ variables with coefficients in C, such that $y' + fy = \sum c_i g_i$ holds for y in F and c_i in $\bar{C}$ if and only if $y = \sum y_i h_i$ where the y_i are elements of $\bar{C}$ and $(c_1, \ldots, c_m, y_1, \ldots, y_r)$ satisfy $\mathbf{L}$.

In 1976, Mack showed that the problem of parameterised integration in finite terms can be solved in any purely transcendental elementary extension of $C(x)$, C a finitely generated extension of Q. This result was extended in the appendix of Singer *et al.* (1986) to include regular log-explicit extensions of $C(x)$. We need to use the fact that if we can solve the problem of parameterised integration in finite terms for a field F, then we can solve this problem for any purely transcendental elementary extension of F. The proofs appearing in Mack (1976) or Singer *et al.* (1986) immediately yield this result.

PROPOSITION 2. *Let k be an algebraic extension of $\bar{Q}(x)$. Assume that one can solve the problem of parameterised integration in finite terms for all algebraic extensions of k. Let $L(y) = b$ be a linear differential equation with coefficients in k.*

(i) *One can find, in a finite number of steps, a basis $y_1, \ldots, y_m$ for the space of elementary solutions of $L(y) = 0$. Furthermore, these elements may be chosen to lie in a purely transcendental extension of an algebraic extension of k.*
(ii) *One can decide in a finite number of steps if $L(y) = b$ has an elementary solution and, if so, find one that lies in a purely transcendental elementary extension of an algebraic extension of k.*

PROOF. (i) Use Lemma 12 (i) to decide if $L(y) = 0$ has an elementary solution and, if so, find one, say y_1, such that y_1'/y_1 is algebraic over k. Substituting $y = y_1 Y$ into $L(y) = 0$, we get a homogeneous linear differential equation $\bar{L}(y) = 0$ that is satisfied by Y'. Arguing by induction, we can find $\bar{y}_1, \ldots, \bar{y}_r$ that lie in a purely transcendental elementary extension of an algebraic extension of k. Using the remarks immediately preceding this proposition, we can find a system $\mathbf{L}$ of linear equations with coefficients in $\bar{Q}$ such that $d_1 \bar{y}_1 + \ldots + d_r \bar{y}_r$ has an elementary anti-derivative if and only if $(d_1, \ldots, d_r)$ satisfies $\mathbf{L}$. From $\mathbf{L}$ we can find a basis $y_2, \ldots, y_m$ for the space of $d_1 \bar{y}_1 + \ldots + d_r \bar{y}_r$ that have elementary anti-derivatives. These anti-derivatives can be chosen to lie in a purely transcendental elementary extension of an algebraic extension of k. Therefore $y_1, y_1 \int y_2, \ldots, y_1 \int y_m$, forms a basis of the desired type for $L(y) = 0$.

(ii) Using (i) we could argue as we did in the Liouvillian case in the paragraph immediately following Example 1. This forces us to consider linear differential equations of order one more than the equation we want to look at. Using Theorem 3 we can avoid this. To decide if $L(y) = b$ has an elementary solution, we first use Lemma 2 to decide if $L(y) = b$ has a solution in k. If it does we are done. If not, use Lemma 11 (ii) to decide if $L(y) = 0$ has a non-zero algebraic solution. If not, then $L(y) = b$ does not have an elementary solution. Otherwise, let y_1 be such a solution. Substituting $y = y_1 Y$ into $L(y) = b$, we see that Y' will satisfy a linear differential equation $\bar{L}(y) = \bar{b}$ with coefficients algebraic over k and order one less than the order of L. Arguing inductively, we decide if $\bar{L}(y) = \bar{b}$ has an elementary solution $\bar{y}$ of the prescribed form. If not, then the original equation will have no such solution. Otherwise we must decide if there exist constants $d_2, \ldots, d_m$ in $\bar{Q}$ such that $\bar{y} + d_2 \bar{y}_2 + \ldots + d_m \bar{y}_m$ has an elementary anti-derivative where $\bar{y}, \bar{y}_2, \ldots, \bar{y}_m$ forms a basis for the space of elementary solutions of $\bar{L}(y) = 0$. If such constants exist, then $y_1 \int (\bar{y} + d_2 \bar{y}_2 + \ldots + d_m \bar{y}_m)$ is a non-zero elementary solution of $L(y) = b$. If not, then $L(y) = b$ has no such solution.

4. Final Comments

In Section 3 we reduced the problem of deciding if $L(y) = b$ has an elementary solution (where this equation has coefficients in an algebraic extension of $Q(x)$) to showing that the problem of parameterised integration in finite terms can be solved for algebraic extensions of $Q(x)$. We know that part (b) of this problem can be solved for such fields (cf. Risch, 1968). We feel that a solution of this problem would be of independent interest (cf. Davenport, 1984b, Problem 6).

Another open problem is to extend the results of Singer (1981) to include equations $L(y) = 0$ that have coefficients in a given Liouvillian or elementary extension of $Q(x)$. Even the following problem is open: Let K be an elementary extension of $Q(x)$ and let $L(y) = 0$ be a homogeneous linear differential equation with coefficients in K. Decide if $L(y) = 0$ has a solution in K. When the order of L is one, Risch (1968, 1969, 1970) solved this problem. A solution of this problem for arbitrary n would be a starting point for generalising the results of Singer (1980) as well as being useful in giving a simplified presentation of the general Risch (1968) algorithm.

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