Some structural results on D^n-finite functions[⋆]

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Abstract

D-finite (or holonomic) functions satisfy linear differential equations with polynomial coefficients. They form a large class of functions that appear in many applications in Mathematics or Physics. It is well-known that these functions are closed under certain operations and these closure properties can be executed algorithmically. Recently, the notion of D-finite functions has been generalized to differentially definable or D^n-finite functions. Also these functions are closed under operations such as forming (anti)derivative, addition or multiplication and, again, these can be implemented. In this paper we investigate how D^n-finite functions behave under composition and how they are related to algebraic and differentially algebraic functions.

Keywords: Holonomic functions, closure properties, differential Galois theory, algorithms

1. Introduction

A formal power series $f(x) = \sum_{k \geq 0} a_k x^k$ is called D-finite, if it satisfies a linear differential equation with polynomial coefficients [11, 19, 20]. The most commonly used special functions [1, 4, 17] are of this type as well as many generating functions of combinatorial sequences. D-finite functions are not only closed under certain operations, but these closure properties can be executed algorithmically. A key is the finite description of D-finite functions in terms of the polynomial coefficients and sufficiently many initial values. Given such D-finite representations, the defining differential equation for the antiderivative, the derivative, addition, multiplication, algebraic substitution, etc. as well as sufficiently many initial values can be computed algorithmically. This has been implemented in several computer algebra systems [2, 10, 13, 15, 18]. These implementations can be used to automatically prove and derive results on holonomic functions [9].

Given a differential ring R, the set of *differentially definable* functions over R, denoted by $D(R)$, is defined as those (formal) power series satisfying linear differential equations with coefficients in R. In this notation, the classical D-finite functions are $D(K[x])$ (for some field K of characteristic zero). It has been shown [6, 7] that these functions are also closed under forming antiderivative, derivative, addition, and multiplication and that also these results are algorithmic. Since $D(K[x])$ is again a differential ring, the construction can be

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iterated, giving rise to DD-finite functions $\mathrm{D}(\mathrm{D}(K[x])) = \mathrm{D}^2(K[x])$ or more generally D^n-finite functions $\mathrm{D}(\mathrm{D}^{n-1}(K[x])) = \mathrm{D}^n(K[x])$. Simple examples of DD-finite functions are the tangent or the iterated exponentials, neither of which is D-finite. Once more, these functions can be represented by the coefficients of the defining differential equation plus initial values.

It is well-known that algebraic functions are D-finite and that the composition of D-finite with algebraic functions is again D-finite. In this paper, we prove the analogous results for D^n-finite functions. In general, the composition of two D-finite functions is not D-finite. However, having a whole scale of differentially definable functions at our disposal, it is possible to consider the composition of D^n-finite with D^m-finite functions. Furthermore, we show that the D^n-finite functions form an increasing set that does not exceed the differentially algebraic functions. All results presented are algorithmic and have been implemented in the open source computer algebra software SAGE [21] in the package dd_functions [5].

2. Composition of D^n-finite functions

Before we introduce new properties of differentially definable functions, we recall the concepts and some results developed in [6]. From now on, we fix the following notation: K is a field of characteristic zero, $K[[x]]$ denotes the ring of formal power series over K, ∂ the standard derivation in $K[[x]]$ and $\langle S\rangle_K$ the K-vector space generated by the set S. We also frequently use the notations $f' = \partial(f)$ and $f^{(i)} = \partial^i(f)$ for higher order derivatives.

Definition 1. Let R be a non-trivial differential subring of $K[[x]]$ and $R[\partial]$ the ring of linear differential operators over R. We call $f \in K[[x]]$ *differentially definable over* R if there is a non-zero operator $\mathcal{A} \in R[\partial]$ that annihilates f, i.e., $\mathcal{A} \cdot f = 0$. By $\mathrm{D}(R)$ we denote the set of all $f \in K[[x]]$ that are differentially definable over R. We define the *order of* f *w.r.t.* R as the minimal order of the operators that annihilate f (i.e., the minimal ∂-degree of $\mathcal{A} \in R[\partial]$ such that $\mathcal{A} \cdot f = 0$).

Note that $R \subset R[\partial]$ and hence for non-trivial subrings of $K[[x]]$ the set of differentially definable functions is never empty. In our notation, the classical D-finite functions are $\mathrm{D}(K[x])$. It is well known [11] that $\mathrm{D}(K[x])$ is closed under derivation, addition, and multiplication, i.e., they form a differential subring of $K[[x]]$. Hence $\mathrm{D}(\mathrm{D}(K[x])) = \mathrm{D}^2(K[x])$ is well defined and we refer to it as the set of *DD-finite functions*.

Example 2. $c_0(x) = \exp(x) \in K[[x]]$ is D-finite satisfying the linear differential equation $c_0'(x) - c_0(x) = 0, c_0(0) = 1$, and so is the constant function $c_1(x) = 1 \in K[[x]]$. Hence $f(x) = \exp(\exp(x) - 1) \in K[[x]]$ is DD-finite as solution to $c_1(x)f'(x) - c_0(x)f(x) = 0,\ f(0) = 1$. The coefficients in the defining differential equation for $f(x)$ can be represented in turn using their respective defining (in)homogeneous differential equations.

In analogy to D-finite functions, differentially definable functions have equivalent characterizations in terms of inhomogeneous differential equations or finite dimensional vector spaces.

Theorem 3. *Let R be a differential subring of $K[[x]]$, $R[\partial]$ the ring of linear differential operators over R, and $F = Q(R)$ be the field of fractions of R. Let $f \in K[[x]]$. Then the following are equivalent:*

1. $f \in \mathrm{D}(R)$

2. $\exists \mathcal{A} \in R[\partial]\, \exists g \in \mathrm{D}(R) \colon \mathcal{A} \cdot f = g$
3. $\dim\left(\langle f^{(i)} \mid i \in \mathbb{N}\rangle_F\right) < \infty$

Proof. See [6]. □

Several closure properties satisfied by D-finite functions have been shown to hold also for differentially definable functions. The proofs are very similar to the classical case and they are also constructive and have been implemented in the open source computer algebra system SAGE [5, 7]. Next we recall some of these closure properties.

Theorem 4. *Let R be a non-trivial differential subring of $K[[x]]$ and $f(x), g(x) \in \mathrm{D}(R)$ with orders d_1 and d_2, respectively, and $r(x) \in R$. Then:*

1. *$f'(x) \in \mathrm{D}(R)$ with order at most d_1.*
2. *Any antiderivative of $f(x)$ is in $\mathrm{D}(R)$ with order at most $d_1 + 1$.*
3. *$f(x) + g(x) \in \mathrm{D}(R)$ with order at most $d_1 + d_2$.*
4. *$f(x)g(x) \in \mathrm{D}(R)$ with order at most $d_1 d_2$.*
5. *If $r(0) \neq 0$, then its multiplicative inverse $1/r(x)$ in $K[[x]]$ is in $\mathrm{D}(R)$ with order 1.*

Proof. See [6]. □

By Theorem 4 we have that given a differential subring R of $K[[x]]$, $\mathrm{D}(R)$ is again a differential subring of $K[[x]]$. Hence the construction can be iterated with closure properties holding at each level. As we refer to $\mathrm{D}^2(K[x])$ as DD-finite functions, we call $\mathrm{D}^n(K[x])$ the set of D^n-finite functions.

Another property that has an immediate generalization from D-finite to D^n-finite is the structural relation that any function algebraic over the quotient field $K(x)$ is D-finite [11]. We first recall the classical result and its proof.

Theorem 5. *Let $a(x) \in K[[x]]$ be algebraic over $K(x)$. Then $a(x)$ is D-finite.*

Proof. Let $a(x)$ be algebraic over $K(x)$ and $m(x, y) \in K(x)[y]$ be its (monic, irreducible) minimal polynomial. Then we have that $m(x, a(x)) = 0$ and differentiating this equality gives

$$a'(x)\partial_y(m)(x, a(x)) + \kappa_\partial(m)(x, a(x)) = 0, \tag{1}$$

where κ_∂ denotes the coefficient-wise derivation over $K(x)[y]$ and ∂_y the standard derivation with respect to y in $K(x)[y]$.

Then for the polynomial $\partial_y(m)$, clearly $\deg_y(\partial_y(m)) < \deg_y(m)$ and by the irreducibility of the minimal polynomial, m and its derivative are coprime. Hence there exist polynomials $r(x, y), s(x, y) \in K(x)[y]$ such that

$$r(x, y)m(x, y) + s(x, y)\partial_y(m)(x, y) = 1.$$

Plugging in $y = a(x)$ gives the equality $s(x, a(x))\partial_y(m)(x, a(x)) = 1$. Multiplying (1) by $s(x, a(x))$ and using this identity, we obtain a polynomial expression for $a'(x)$,

$$a'(x) = -s(x, a(x))\kappa_\partial(m)(x, a(x)).$$

Hence for any polynomial $p(x,y) \in K(x)[y]$, the derivative of $p(x,a(x))$ can be expressed again as a polynomial in $a(x)$, i.e.,

$$\begin{aligned}\partial(p(x,a(x))) &= a'(x)\partial_y(p)(x,a(x)) + \kappa_\partial(p)(x,a(x)) \\ &= \big(\kappa_\partial(p) - s\partial_y(p)\kappa_\partial(m)\big)(x,a(x))\end{aligned}$$

Then by induction it follows that

$$\dim_{K(x)}(\langle a(x), a'(x), a''(x), \dots\rangle) \le \dim_{K(x)}(\langle 1, a(x), a^2(x), \dots\rangle) = \deg_y(m),$$

where $\dim_{K(x)}$ is the dimension as a $K(x)$-vector space. Thus, by Theorem 3, $a(x)$ is D-finite with order at most $\deg_y(m)$. Therefore there is a linear relation among $a(x), \dots, a^{(m-1)}(x)$ amd so $a(x)$ is D-finite of order at most $\deg_y(m)$. □

Note, that in this proof, the fact that $a(x)$ is algebraic over $K(x)$ in particular has never been used. If instead, we consider a differential integral domain R and its field of fractions, the proof carries over immediately to the following generalization.

Proposition 6. *Let R be a differential subring of $K[[x]]$ and F its field of fractions. If $a(x) \in K[[x]]$ is algebraic over F then $a(x) \in \mathrm{D}(R)$.*

This means in particular that functions that are algebraic over the ring $\mathrm{D}^k(R)$, are in the next level in the hierarchy, i.e., in $\mathrm{D}^{k+1}(R)$. The proof of Theorem 5 is constructive and gives a way to compute a D^{n+1}-finite equation given an algebraic equation with D^n-finite function coefficients as we illustrate in the next example for $n = 1$. For more details how the procedure is currently implemented in SAGE [5] we refer to the appendix.

Example 7. Let $a(x)$ be given by the algebraic equation

$$\tfrac{1}{2}\cos(x)a(x)^2 - 2a(x) + \cos^2(x) = 0$$

with D-finite coefficients. Then we have for the defining polynomial φ and its derivative w.r.t. y,

$$\varphi(x,y) = \tfrac{1}{2}\cos(x)y^2 - 2y + \cos^2(x), \quad \text{and} \quad \varphi_y(x,y) = \cos(x)y - 2.$$

Applying the extended Euclidean algorithm gives

$$r(x,y)\varphi(x,y) + s(x,y)\varphi_y(x,y) = t(x)$$

with

$$r(x,y) = 2\cos(x), \quad s(x,y) = 2 - y\cos(x), \quad t(x) = 2\cos^3(x) - 4.$$

With this we obtain the following non-homogeneous DD-finite linear differential equation for $a(x)$,

$$2\cos(x)\big(\cos^3(x) - 2\big)a'(x) + \sin(x)\big(\cos^3(x) + 4\big)a(x) - 6\sin(x)\cos^2(x) = 0.$$

Another important classical closure property states that the composition of D-finite functions with algebraic functions (if well defined) is again D-finite [11]. Also this result extends to the more general setting. Here, and in the following, we denote by $F_n(x)$ the field of fractions of $\mathrm{D}^n(K[x])$.

Theorem 8. *Let $f \in \mathrm{D}^n(K[x])$ for some $n \geq 1$ and $a(x)$ be algebraic over $F_m(x)$ for some $m \geq 0$ with $a(0) = 0$. Then $g(x) = (f \circ a)(x)$ is in $\mathrm{D}^{n+m}(K[x])$.*

Proof. Let $f \in \mathrm{D}^n(K[x])$ be of order d and $a(x)$ be algebraic of degree p over $F_m(x)$.

We proceed by induction on n. The base case $n = 1$ (i.e., f is D-finite) can be proven using the same ideas as in [19, Theorem 2.7], which corresponds to the case $m = 0$. Using the chain rule, we can write $g^{(k)}(x)$ as a linear combination of $f^{(l)}(a(x))$ with coefficients in $K[a, a', a'', \ldots]$. Since $a(x)$ is algebraic over $F_m(x)$, this can be rewritten as

$$g^{(k)}(x) = \sum_{l=1}^{k} Q_{k,l}(x, a(x)) f^{(l)}(a)(x)),$$

where $Q_{k,l}(x, a(x)) \in F_m(x)(a) = F_m(x)[a, a^2, \ldots, a^{p-1}]$.

On the other hand, since f is D-finite, there are polynomials $p_0(x), \ldots, p_d(x) \in K[x]$ such that

$$p_d(x) f^{(d)}(x) + \ldots + p_0(x) f(x) = 0.$$

Composing this equality with $a(x)$ gives

$$\dim\left(\langle f(a(x)), f'(a(x)), f''(a(x)), \ldots\rangle_{F_m(x)(a)}\right) \leq d.$$

Altogether, we have

$$\langle g(x), g'(x), \ldots\rangle_{F_m(x)} \subset \langle f(a(x)), f'(a(x)), \ldots, f^{(d-1)}(a)(x))\rangle_{F_m(x)(a)}.$$

As $F_m(x)(a)$ is finite over $F_m(x)$, the vector space spanned by $g(x)$ and its derivatives over $F_m(x)$ has finite dimension (in fact, at most dp). Hence, $g(x) \in \mathrm{D}^{m+1}(K[x])$.

Now let $n > 1$. Since $f(x) \in \mathrm{D}^n(K[x])$, there are $r_0(x), \ldots, r_d(x) \in \mathrm{D}^{n-1}(K[x])$ such that

$$r_d(x) f^{(d)}(x) + \ldots + r_0(x) f(x) = 0.$$

By the induction hypothesis, we have that $(r_d \circ a)(x) \in \mathrm{D}^{n+m-1}$. Thus

$$\dim\left(\langle f(a(x)), f'(a(x)), f''(a(x)), \ldots\rangle_{F_{n+m-1}(x)}\right) \leq d.$$

On the other hand, since $n > 1$, we have that $n + m - 1 > m$. Hence

$$\dim\left(\langle 1, a(x), a(x)^2, \ldots\rangle_{F_{n+m-1}(x)}\right) \leq p.$$

Analogously to the base case $n = 1$, using the chain rule and the algebraicity of $a(x)$ over $F_m(x)$, we can express $g^{(i)}(x)$ as a linear combination of products of $f^{(j)}(a(x))$ and $a(x)^l$, showing that

$$\dim\left(\langle g(x), g'(x), g''(x), \ldots\rangle_{F_{n+m-1}(x)}\right) \leq dp,$$

yielding $g(x) \in \mathrm{D}^{n+m}(K[x])$. $\square$

Let us give a simple example of algebraic substitution in a DD-finite function by composing the double exponential, see Example 2, and the square root. How the Theorem is realized in the current version of the SAGE-implementation [5] is described in the appendix below.

Example 9. Let $f(x) = \exp(\exp(x) - 1)$, which is DD-finite satisfying the differential equation $f'(x) - \exp(x)f(x) = 0$, and let $a(x)$ be the algebraic function such that $a(x)^2 + 2a(x) - x = 0$ with $a(0) = 0$. Using the classical embedding Theorem 5, we find that $2(x+1)a'(x) - a(x) = 1$.

Let $h(x) = f(a(x))$. Since the order of f is $d = 1$ and the degree of the algebraic equation for a is $p = 2$, we get as an upper bound for the DD-finite equation $dp = 2$. By the chain rule we have that

$$h'(x) = f'(a(x))a'(x), \quad \text{and} \quad h''(x) = f''(a(x))a'(x)^2 + f'(a(x))a''(x).$$

The iterated differential equations for $a(x)$ and $f(x)$, respectively, can be plugged into these equations and, using elimination, we find that

$$2(x+1)h'(x) - \exp(a(x))(a(x)+1)h(x) = 0.$$

The coefficients in this equation are obviously D-finite.

In the classical setting of holonomic functions, there is no general result about composition of D-finite functions. The next result shows that the composition of two D^n-finite functions (if well-defined) stays within the chain of rings $K[x] \subset \mathrm{D}(K[x]) \subset \mathrm{D}^2(K[x]) \subset \ldots \subset \mathrm{D}^n(K[x]) \subset \ldots$.

Theorem 10. *Let $f(x) \in \mathrm{D}^n(K[x])$ of order d and $g(x) \in \mathrm{D}^m(K[x])$ with $g(0) = 0$, then $(f \circ g)(x) \in \mathrm{D}^{n+m}(K[x])$ with order at most d.*

Proof. We proceed by induction on n. For the base case $n = 0$, i.e., $f \in K[x]$, we have that $(f \circ g)(x) = f(g(x)) \in \mathrm{D}^m(K[x])$ by the closure properties addition and multiplication (Theorem 4), even when $g(0) \neq 0$.

Now, suppose that $n > 0$ and assume that for any $k < n$, if $h(x) \in \mathrm{D}^k(K[x])$ then $(h \circ g)(x) \in \mathrm{D}^{k+m}(K[x])$.

In order to show $(f \circ g) \in \mathrm{D}^{n+m}(K[x])$ we need to prove that the vector space generated by $(f \circ g), (f \circ g)', \ldots$ over $F_{n+m-1}(x)$, the field of fractions of $\mathrm{D}^{n+m-1}(K[x])$, has finite dimension. Let $V_{n+m-1}(f \circ g)$ denote this vector space.

Again using the chain rule, we have that,

$$(f \circ g)^{(k)}(x) = \sum_{l=1}^{k} (f^{(l)} \circ g)(x) B_{k,l}(g'(x), \ldots, g^{(k-l+1)}(x)).$$

As $n > 0$, also $n + m - 1 \geq m$ and so $g^{(j)}(x) \in F_{n+m-1}(x)$ for any $j \in \mathbb{N}$. Hence,

$$V_{n+m-1}(f \circ g) \subset \langle f \circ g, f' \circ g, f'' \circ g, \ldots \rangle_{F_{n+m-1}(x)}.$$

Since f has order d in D^n, using Theorem 3 it follows that for any $p \geq d$ there are elements $r_{p,i}(x) \in \mathrm{D}^{n-1}(K[x])$, for $i = 0, \ldots, d$ such that,

$$r_{p,d}(x) f^{(p)}(x) = r_{p,0}(x) f(x) + \ldots + r_{p,d-1}(x) f^{(d-1)}(x).$$

By the induction hypothesis, the composition $(r_{p,i} \circ g) \in \mathrm{D}^{n+m-1}(K[x])$ for any $i = 0, \ldots, d$, hence

$$r_{p,d}(g(x)) f^{(p)}(g(x)) = r_{p,0}(g(x)) f(g(x)) + \ldots + r_{p,d-1}(g(x)) f^{(d-1)}(g(x)).$$

Thus, we may conclude that

$$\dim(V_{n+m-1}(f\circ g)) \leq \dim(\langle f\circ g, f'\circ g, f''\circ g, \dots\rangle_{F_{n+m-1}(x)}) \leq d.$$

$\square$

Next we give some simple examples for this result. In the appendix we describe how this closure property can be implemented and provide computational details on the following examples.

Example 11. Let $f(x) = \exp(x)$ and $g(x) = \sin(x)$. Both are D-finite functions with annihilating operators $\partial - 1$ and $\partial^2 + 1$, respectively. Their composition $h(x) = (f \circ g)(x) = \exp(\sin(x))$ is DD-finite satisfying the linear differential equation $h'(x) - \cos(x)h(x) = 0$ with D-finite function coefficients.

Example 12. Let $f(x) = \log(x + 1)$ and $g(x) = \exp(x) - 1$. Both are D-finite functions with annihilating operators $(x+1)\partial^2 + \partial$ and $\partial^2 - \partial$, respectively. Their composition $h(x) = (f\circ g)(x) = \log(\exp(x))$ is DD-finite. The execution of the procedure described in the appendix yields the differential equation $h''(x) = 0$ with initial values $h(0) = 0, h'(0) = 1$. In this case it is possible to find that $h(x) = x$, i.e., that it is even polynomial. This is not always the case in general.

Example 13. Let $f(x) = \sin(x)$ which is D-finite with annihilating operator $\partial^2 + 1$. Then $g(x) = (f \circ f)(x) = \sin(\sin(x))$ is DD-finite satisfying the linear differential equation

$$\cos(x)g''(x) + \sin(x)g'(x) + \cos^3(x)g(x) = 0.$$

We can continue this example by iterating the composition with the sine once more and thus obtain a D^3-finite function.

Example 14. Let $f(x) = \sin(x)$ and $g(x) = f(f(x)) = \sin(\sin(x))$ as in the previous example. Then $h(x) = (f \circ g)(x) = \sin(\sin(\sin(x)))$ is $\mathrm{D}^3 - finite$ satisfying the linear differential equation

$$g'(x)h''(x) - g''(x)h'(x) + g'(x)^3h(x) = 0.$$

Note that even though in this example $(f \circ g)(x) = (g \circ f)(x)$, computationally it makes a difference on how the annihilating operator for $h(x)$ is computed, see also Appendix A.

By Theorem 6, we have that any function $a(x)$ algebraic over $K(x)$ is D-finite, hence for $f \in \mathrm{D}^n(K[x])$, $(f \circ a)(x) \in \mathrm{D}^{n+1}(K[x])$ (if $a(0) = 0$). However, by Theorem 8 for $m = 0$, even $(f \circ a)(x) \in \mathrm{D}^n(K[x])$ holds. The following lemma underlines that the condition on algebraicity is essential for this reduction in layers.

Lemma 15. *Let $g(x) \in \mathrm{D}^n(K[x])$, $n \geq 1$, of order 1 and consider the power series $f(x) = \exp(g(x) - g(0))$. If $f(x) \in \mathrm{D}^n(K[x])$, then $g(x)$ is algebraic over $F_{n-1}(x)$.*

Proof. Let $f(x) \in \mathrm{D}^n(K[x])$ and $\mathcal{A} \in \mathrm{D}^{n-1}(K[x])[\partial]$ be an operator that annihilates f, i.e., $\mathcal{A} \cdot f = 0$, written as $\mathcal{A} = r_d(x)\partial^d + \ldots + r_0(x)$. By the chain rule, we have for any $k > 0$,

$$f^{(k)}(x) = \sum_{l=1}^{k} \exp^{(l)}(g(x) - g(0))B_{k,l}(g'(x), g''(x), \dots, g^{(k-l+1)}(x)).$$

Since $\exp'(x) = \exp(x)$ and $g(x)$ is in $\mathrm{D}^n(K[x])$ with order 1, this can be reduced to,

$$f^{(k)}(x) = f(x)q_k(g(x)),$$

for some polynomials $q_k(y) \in F_{n-1}(x)[y]$. Hence,

$$(\mathcal{A} \cdot f)(x) = 0 \quad \Leftrightarrow \quad \sum_{k=0}^{d} r_k(x)q_k(g(x)) = 0,$$

i.e., $g(x)$ is algebraic over $F_{n-1}(x)$. $\square$

3. Iterated exponentials

Theorem 10 provides an upper bound on where the composition of two formal power series may belong depending on their respective positions in the D^n–hierarchy. We have shown earlier that additional properties (such as, e.g., algebraicity) can reduce this upper bound.

Naturally the question arises: is the bound provided by Theorem 10 tight, i.e., are there examples of D^n-finite and D^m-finite functions such that their composition belongs to $\mathrm{D}^{n+m}(K[x])$ but not to $\mathrm{D}^{n+m-1}(K[x])$? In this section we give an affirmative answer to this question by showing that the iterated exponentials are functions of this type. In order to do so we need to recall some background on Differential Galois Theory that we detail below. We recall now the basic definitions of Differential Algebra. For further reading we refer to [3, 14, 22].

Recall that for a field K, a mapping $\partial : K \to K$ is called a derivation, if $\partial(f + g) = \partial(f) + \partial(g)$ and $\partial(fg) = \partial(f)g + f\partial(g)$ for all $f, g \in K$. The pair (K, ∂) is called a differential field and we denote the field of constants by $C_K = \{f \in K : \partial(f) = 0\}$. Moreover, if $E \supset K$ is a field extension and $(E, \tilde{\partial})$ is a differential field such that $\tilde{\partial}|_K \equiv \partial$ we say that $(E, \tilde{\partial})$ is a differential extension of (K, ∂). If it is clear from the context, we use ∂ also to refer to the derivation in a differential extension and the shorthand notation $f' = \partial(f)$.

Lemma 16. *Let (K, ∂) be a differential field and $L \in K[\partial]$ a linear differential operator. Let E be a differential field extension of K, C_E denote the set of constants, and $S_L(E)$ be the set of all solutions to $Ly = 0$ in E. Then $S_L(E)$ is a C_E-vector space.*

Proof. As L is a linear differential operator, we have that $L(f + g) = L(f) + L(g)$, hence if $f, g \in S_L(E)$, clearly $f+g \in S_L(E)$. Moreover, if $c \in C_E$, then $(cf)' = cf'$ for all $f \in E$. Hence $L(cf) = cL(f)$, yielding that, if $f \in S_L(E)$, then for any constant $c \in C_E$, $cf \in S_L(E)$. $\square$

Based on this idea, a type of field extension can be defined that includes all the linearly independent solutions but does not add any constants.

Definition 17. [3, 22] Let (K, ∂) be a differential field of characteristic zero with field of constants C_K and let $L \in K[\partial]$ be of order n. We call E a *Picard-Vessiot* extension of K for L, if there are n elements $y_1, \ldots, y_n$ that are C-linearly independent such that $L(y_i) = 0$ and $E = K(y_1, y_1', \ldots, y_1^{(n-1)}, y_2, y_2', \ldots, y_n^{(n-1)})$ and $C_E = C_K$.

Proposition 18. *Let (K, ∂) be a differential field with algebraically closed field of constants C of characteristic zero and let $L \in K[\partial]$ be of order n. Then:*

- *There exists a Picard-Vessiot extension E for L.*
- *The field E is unique up to K-∂-isomorphisms.*

These Picard-Vessiot extensions (also denoted as *PV-extensions*) are the basic ones considered for differential Galois theory. They are equivalent to algebraic extensions, where all solutions to a given polynomial equation are added to the original field. In the differential case, it is necessary to keep the field of constants fixed. Following the same idea as for building a Galois group for a field extension, the differential group for a differential field extension can be defined as those automorphisms that fix the small field and commute with the derivation,

$$G_{[E:K]} = \{\sigma \mid \sigma \text{ is a } K\text{-automorphism of } E \text{ and } \sigma(z') = (\sigma(z))' \text{ for all } z \in E\}.$$

This Galois group is particularly interesting when considering PV-extensions. If the action of the elements of the Galois group is restricted to the solution space of the linear differential equation defining the PV-extension, it is a linear action.

Lemma 19. *Let (K, ∂) be a differential field with field of constants C and let E be a PV-extension of K for an operator $L \in K[\partial]$. Let G be the differential Galois group of E over K. Then for any $\sigma \in G$:*

- $\sigma(S_L(E)) \subset S_L(E)$,
- $\sigma|_{S_L(E)}$ *is a C-linear map.*

Proof. Let $0 = L(y) = a_n y^{(n)} + \cdots + a_1 y' + a_0 y$ for some $y \in E$. Then σ can be applied to both sides. Using the fact that σ fixes K and commutes with the derivation, we have

$$0 = \sigma(a_n y^{(n)} + \cdots + a_1 y' + a_0 y) = a_n \sigma(y)^{(n)} + \cdots + a_1 \sigma(y)' + a_0 \sigma(y) = L(\sigma(y)).$$

Hence $\sigma(S_L(E)) \subset S_L(E)$. In addition, since $C \subset K$ and σ fixes K, we have that σ is a C-linear map over E, so in particular it is C-linear over $S_L(E)$. $\square$

Moreover, as σ is an automorphism, $\sigma|_{S_L(E)}$ is a linear bijection. Additionally, as PV-extensions have finite dimensional solution spaces for L (in fact, $\dim_C(S_L(F)) = \deg(L)$, see [22, Lemma 1.7]), one can associate each element of the Galois group to an invertible square matrix of size $\deg(L)$, having an embedding to the group $\mathrm{GL}_{\deg(L)}(C)$.

In particular, if we consider first order differential equations (i.e., $y' = ay$ for $a \in K$), then the PV-extension is of the form $K(z)$ for some z and the Galois group is included in C^*. If σ_c is the mapping associated with the constant $c \neq 0$, then $\sigma_c(z) = cz$. The next lemma summarizes the results for the Galois group of this type of PV-extensions. The three cases correspond to characterizing behaviors of the Galois groups.

Lemma 20. *Let (K, ∂) be a differential field with algebraically closed field of constants C, $E = K(z)$ its PV-extension for the equation $y' = ay$ and G the differential Galois group of E over K. Then*

- $z \in K$ *if and only if* $G = \{\sigma_1 = id_K\}$.
- z *is algebraic over K if and only if there is $n \in \mathbb{N}$ such that*

$$G = \{\sigma_\alpha \mid \alpha^n = 1\}.$$

In this case $z^n \in K$.

- *z is transcendental over K if and only if $G = \{\sigma_c \mid c \in C^*\}$.*

Proof. For a solution z satisfying $z' = az$ we have

- $z \in K$: in this case $E = K(z) = K$, hence the only element in G is the identity map $G = \{\sigma_1\}$.
- z is transcendental over K: in this case it easy to see that

$$\sigma_c \colon K(z) \to K(z), \qquad \sigma_c|_K \equiv id_K, \qquad \sigma_c(z) = cz$$

  is an automorphism if and only if $c \neq 0$. As different constants yield different mappings, $G = \{\sigma_c \mid c \in C^*\}$.

- z is algebraic over K: since any automorphism of $K(z)$ is determined by the image of z and these are all roots of the same polynomial, G must be finite. The finite subgroups of C^* are all of the form $\{c \mid c^n = 1\}$. If this is the Galois group then z^n is fixed by this group and so, by the usual Galois correspondence must lie in K.

□

The analogies with the algebraic case do not end here. There is also a differential closure of a field using linear differential equations and it is called *Picard-Vessiot closure*. To define it properly, we need a small result of skew-polynomials.

Lemma 21. *Let (K, ∂) be a differential field and $K[\partial]$ its ring of linear differential operators. Let I be a left ideal and J be a right ideal. Then there are $L_1, L_2 \in K[\partial]$ such that*

$$I = K[\partial]L_1, \qquad J = L_2K[\partial].$$

The proof of this lemma can be found in [22], and it is based on the Euclidean division of skew-polynomials. This Lemma allows to define the concept of greatest common right-divisor and least common left-multiple.

Definition 22. Let (K, ∂) be a differential field and $K[\partial]$ its ring of linear differential operators. Let $L_1, L_2 \in K[\partial]$ be two operators.

- We define their *least common left-multiple* (LCLM) as the generator of the following ideal $K[\partial]L_1 \cap K[\partial]L_2$.
- We define their *greatest common right-divisor* (GCRD) as the generator of the ideal $K[\partial]L_1 + K[\partial]L_2$.

These two objects have nice properties regarding the solution spaces of the given operators.

- If $L_1(a) = 0$ or $L_2(a) = 0$, then $\mathrm{LCLM}(L_1, L_2)(a) = 0$. So it captures the union of the solutions.
- If $\mathrm{GCRD}(L_1, L_2)(a) = 0$, then $L_1(a) = L_2(a) = 0$. So it captures the common solutions.

Let K be a differential field and denote by K_L the PV-extension for $L \in K[\partial]$. Then,

- if L_1 is a left-multiple of L_2, then $K_{L_2} \subset K_{L_1}$;

- given K_{L_1}, K_{L_2} let $L_3 = LCLM(L_1, L_2)$, then $K_{L_1}, K_{L_2} \subset K_{L_3}$.

Hence, since all PV-extensions form a directed system w.r.t. the inclusion, we can define their direct limit. This direct limit is a differential field extension of K with the same field of constants C, where all linear differential equations with coefficients in K have as many C-linearly independent solutions as the order of the equation. We denote this direct limit by $\overline{K}^{\partial}$.

Setting $K_0 = K$, and $K_{i+1} = \overline{K_i}^{\partial}$ yields a chain of differential extensions. We define the *Picard-Vessiot closure* as the limit of those fields, i.e.,

$$K_{PV} = \bigcup_{i=0}^{\infty} K_i.$$

This PV-closure has some important properties that are the analogues to the properties that the algebraic closure of a field have, but in the differential context.

Lemma 23. *Let (K, ∂) be a differential field with an algebraically closed field of constants C and K_{PV} its PV-closure. Then the field of constants of K_{PV} is again C and for any $L \in K_{PV}[\partial]$, $S_L(K_{PV})$ is a C-vector space of dimension* $\deg(L)$.

Proof. Let $L = r_0 + r_1\partial + \ldots + r_d\partial^d \in K_PV[\partial]$, then there is M such that $r_i \in K_M$ for all $i = 0, \ldots, d$. Hence $S_L(K_{M+1})$ is a C-vector space of dimension $\deg(L)$. Since that is the highest dimension the solution space can get, we have $S_L(K_{PV})$ is a C-vector space of dimension $\deg(L)$. □

Before answering the main question of this section, we need one more result of differential algebra that translates some differential properties of hyperexponential elements to algebraic properties.

Proposition 24. *Let (K, ∂) be a differential field with algebraically closed field of constants C. Let E be a PV-extension of K. Let $u, v \in E \setminus \{0\}$ such that:*

$$\frac{u'}{u} = a \in K, \qquad \frac{v'}{v} = u,$$

then u is algebraic over K.

Proof. Assume that u is transcendental over K. As $u'/u \in K$, we have that $K(u)$ is the PV-extension of K associated with an equation of the form $u' - au = 0$. Using Lemma 20 we then have that the differential Galois group of $K(u)$ over K is $G = \{\sigma_c \mid c \in C^*\}$, where $\sigma_c|K = id_K$ and $\sigma_c(u) = cu$. For any extension of σ_c to E, we have,

$$\frac{\sigma_c(v)'}{\sigma_c(v)} = \frac{\sigma_c(v')}{\sigma_c(v)} = \sigma_c(\frac{v'}{v}) = cu.$$

Since E is a PV-extension of K it is finitely generated. In particular, it has a finite transcendence degree m. Let $c_0, \ldots, c_m$ be $m + 1$ $\mathbb{Q}$-linearly independent constants and consider the elements $v_i = \sigma_{c_i}(v)$. Then they must be algebraically dependent over $K(u)$.

Using a consequence of the Kolchin-Ostrowsky Theorem [12, Ch.VI, §5, Ex. 4], we have that there are integers $n_0, \ldots, n_m$ (not all zero) such that,

$$z := \prod_{i=0}^{m} v_i^{n_i} \in K(u).$$

It is easy to see then that for $\beta = (\sum_{i=0}^{m} n_i c_i)\, u$ we have $z' = \beta z$. We have that z is not algebraic over K since otherwise $u = z'/((\sum_{i=0}^{m} n_i c_i)\, z)$ would be algebraic over K. We may factor the numerator and denominator of z as

$$z = \gamma u^{r_0} \prod_{j=1}^{t} (u - \alpha_j)^{r_j},$$

where $\gamma, \alpha_i \in \overline{K}$, the algebraic closure of K, $\alpha_j \neq 0$ for $j > 0$, and the r_i are (positive or negative) integers. For the logarithmic derivative of z we obtain,

$$\frac{z'}{z} = \frac{\gamma'}{\gamma} + r_0 a + \sum_{j=1}^{t} r_j \frac{u' - \alpha_j'}{u - \alpha_j}.$$

Expanding the quotients in the sum, we find that $(u' - \alpha_j')/(u - \alpha_j) = a + (a\alpha_j - \alpha_j')/(u - \alpha_j)$. But we know that $z' = \beta z$, hence

$$\left(\sum_{i=0}^{m} n_i c_i\right) u = \beta = \frac{\gamma'}{\gamma} + a \sum_{j=0}^{t} r_j + \sum_{j=1}^{t} r_j \frac{a\alpha_j - \alpha_j'}{u - \alpha_j}.$$

If $a\alpha_j - \alpha_j' = 0$ then $(\alpha_j/u)' = 0$ so $\alpha_j = cu$ for some non-zero constant c contradicting the fact that u is transcendental over K. Therefore each $r_i = 0$ for $j > 0$. We then have

$$\left(\sum_{i=0}^{m} n_i c_i\right) u = \frac{\gamma'}{\gamma} + r_0 a$$

again contradicting the fact that u is assumed to be transcendental over K. Therefore this assumption is incorrect and u is algebraic over K. □

Now we can finally address the main question of this section. Consider an algebraically closed field C and the differential ring $C[x]$, where C is the field of constants and $x' = 1$. Let $F_0 = K_0 = C(x)$ and we start building the differentially definable functions over $C[x]$. We define $F_i = Fr(\mathrm{D}^i(C[x]))$, the field of fractions of $\mathrm{D}^i(C[x])$, and K_i as the ith step in the PV-closure of $C(x)$. We then have the following inclusions:

$$\begin{array}{ccccccccccc}
C[x] & \subset & \mathrm{D}(C[x]) & \subset & \dots & \subset & \mathrm{D}^n(C[x]) & \subset & \dots & \subset & C[[x]] \\
\cap & & \cap & & \ddots & & \cap & & \ddots & & \\
F_0 & \subset & F_1 & \subset & \dots & \subset & F_n & \subset & \dots & \subset & C((x)) \\
\cap & & \cap & & \ddots & & \cap & & \ddots & & \\
K_0 & \subset & K_1 & \subset & \dots & \subset & K_n & \subset & \dots & \subset & K_{PV}
\end{array}$$

Next we introduce the following set of recursively defined functions:

- $e_0 = 1 \in C[x]$.
- For $i \geq 0$, assume $e_i \in \mathrm{D}^i(C[x])$. Let $\hat{e}_i$ be the antiderivative of e_i with $\hat{e}_i(0) = 0$. By closure properties, $\hat{e}_i$ is again in $\mathrm{D}^i(C[x])$. Define $e_{i+1} = \exp(\hat{e}_i)$.

Clearly, $e_{i+1}' - e_i e_{i+1} = 0$ and, as $\hat{e}_i(0) = 0$, also $e_{i+1} \in C[[x]]$. Hence $e_{i+1} \in \mathrm{D}^{i+1}(C[x])$ but, as we show in the remainder of this section, $e_{i+1} \notin \mathrm{D}^i(C[x])$. To do so, we are going to prove an even stronger statement.

Proposition 25. *Let $c \in C^*$ and $n \in \mathbb{N} \setminus \{0\}$. Then $e_n^c = \exp(c\hat{e}_{n-1}) \notin K_{n-1}$.*

Proof. The proof proceeds by induction on n. For the base case $n = 1$, we need to show that $e^{cx} \notin C(x)$ for any constant $c \neq 0$. But it is a well known result that the exponential is not algebraic over $C(x)$, so, in particular, is not a rational function.

Now let $n > 1$ and assume the hypothesis holds for all $1 \leq i < n$. Suppose there exists a $c \in C^*$ such that $e_n^c \in K_{n-1}$. By the construction of the field K_{n-1} and since $e_{n-1} \in \mathrm{D}^{n-1}(C[x]) \subset K_{n-1}$, there is a PV-extension E of K_{n-2} where both e_n^c and e_{n-1} lie.

Let $u = ce_{n-1}$ and $v = e_n^c$. Then from Proposition 24 follows directly that ce_{n-1} is algebraic over K_{n-2}. Hence, by Lemma 20, there is a non-zero m such that $(ce_{n-1})^m \in K_{n-2}$, and so in particular, $e_{n-1}^m \in K_{n-2}$ which is a contradiction to the induction hypothesis (as $m \in \mathbb{Z} \subset C^*$). Thus $e_n^c \notin K_{n-1}$ for any $c \in C^*$. $\square$

Corollary 26. *For any $n \in \mathbb{N} \setminus \{0\}$, $e_n \notin \mathrm{D}^{n-1}(C[x])$.*

As a consequence, we obtain that the iterated exponentials are an instance of functions $f \in \mathrm{D}^n(C[x])$ and $g \in \mathrm{D}^m(C[x])$ with the property that $(f \circ g) \in \mathrm{D}^{n+m}(C[x]) \setminus \mathrm{D}^{n+m-1}(C[x])$.

4. Relation to differentially algebraic functions

Throughout this paper, the chain of differential rings constructed with linear differential equations over the polynomial ring is considered. We have shown how composition and algebraic substitution of functions move along the chain,

$$K[x] \subset \mathrm{D}(K[x]) \subset \mathrm{D}^2(K[x]) \subset \cdots \subset \mathrm{D}^n(K[x]) \subset \cdots.$$

We define the *limit ring* as the union

$$\mathrm{D}^\infty(K[x]) = \bigcup_{n \geq 0} \mathrm{D}^n(K[x]).$$

As simple consequence of the properties of differentially definable functions shown earlier, this limit ring is closed under division and composition, i.e.,

- If $f \in \mathrm{D}^\infty(K[x])$ and $f(0) \neq 0$ then $1/f \in \mathrm{D}^\infty(K[x])$.
- If $f, g \in \mathrm{D}^\infty(K[x])$ and $g(0) = 0$, then $(f \circ g) \in \mathrm{D}^\infty(K[x])$.

The question remains, how big this limit ring is. In this section we show that indeed it does not exceed the differentially algebraic functions. Let us recall their definition.

Definition 27. Let $f(x) \in K[[x]]$ and $R \subset K[[x]]$. We say that f is *differentially algebraic* over R if there are $n \in \mathbb{N}$ and a polynomial $P(y_0, \ldots, y_n) \in R[y_0, \ldots, y_n]$ in $n+1$ variables such that f satisfies:

$$P(f(x), f'(x), \ldots, f^{(n)}(x)) = 0.$$

Clearly, any D-finite function $f \in \mathrm{D}(K[x])$ is differentially algebraic over $K[x]$ with a linear polynomial P. Now let $f \in \mathrm{D}^2(K[x])$ be of order d satisfying

$$r_0(x)f(x) + \cdots + r_d(x)f^{(d)}(x) = 0, \tag{2}$$

with $r_i \in \mathrm{D}(K[x])$ of orders s_i. Let $S = \sum_{i=0}^{d} s_i$. Let us see that f is differentially algebraic over $K[x]$ with a polynomial $P(y_0, \ldots, y_{S+d-1})$ of total degree at most S. First note that, with (2), also any derivative of the left hand side vanishes entirely, yielding

$$\partial^n \left(\sum_{i=0}^{d} r_i(x) f^{(i)}(x) \right) = \sum_{i=0}^{d} \sum_{k=0}^{n} \binom{n}{k} r_i^{(k)}(x) f^{(n-k+i)}(x) = 0, \quad n \geq 0.$$

As all the r_i are D-finite, their defining equations can be used to reduce the derivatives above to

$$\sum_{i=0}^{d} \sum_{k=0}^{s_i - 1} \left(\sum_{j=0}^{n} p_{i,k,j}(x) f^{(j+i)}(x) \right) r_i^{(k)}(x) = 0,$$

for some polynomials $p_{i,k,j}(x)$. This can be rewritten in terms of a matrix-vector multiplication $M \cdot (r_0, \ldots, r_d^{(s_d-1)})^T = 0$. Hence the right nullspace of M is not trivial and, as M is a square matrix of size S, we have that $\det(M) = 0$. This determinant is a polynomial in f and its derivatives up to order $S + d$ of total degree at most S and with coefficients in $K(x)$. In order to obtain an algebraic differential equation, we can clear the denominators of x.

A similar argument can be used to derive the following result leading to the main conclusion that any D^n-finite function is differentially algebraic.

Theorem 28. *Let $f(x) \in K[[x]]$ be differentially algebraic over $\mathrm{D}^n(K[x])$. Then f is differentially algebraic over $\mathrm{D}^{n-1}(K[x])$.*

Proof. Let f be differentially algebraic over $\mathrm{D}^n(K[x])$. Then there exists a polynomial

$$P(y_0, \ldots, y_m) = \sum_{\alpha \in \Lambda} p_\alpha(x) (y_0, \ldots, y_m)^\alpha,$$

with coefficients $p_\alpha(x)$ in $\mathrm{D}^n(K[x])$ for $\alpha \in \Lambda \subset \mathbb{N}^{m+1}$, such that

$$P(f(x), f'(x), \ldots, f^{(m)}(x)) = 0. \tag{3}$$

Also any derivative of the left hand side of (3) vanishes. Hence, using the notation $\mathbf{f}(x) = (f(x), f'(x), \ldots, f^{(m)}(x))$, we have

$$\partial^n P(\mathbf{f}(x)) = \sum_{\alpha \in \Lambda} \sum_{k=0}^{n} \binom{n}{k} p_\alpha^{(k)}(x) \partial^{n-k} \mathbf{f}(x)^\alpha = 0$$

Since the coefficients $p_\alpha(x)$ are in $\mathrm{D}^n(K[x])$, each of them satisfies a linear differential equation of order d_α with coefficients in $\mathrm{D}^{n-1}(K[x])$ for $\alpha \in \Lambda$. These can be used to reduce the equation above to

$$\sum_{\alpha \in \Lambda} \sum_{j=0}^{d_\alpha - 1} \left(\sum_{k=0}^{n} r_{\alpha,k,j}(x) \partial^k \mathbf{f}(x)^\alpha \right) p_\alpha^{(j)} = 0,$$

for some $r_{\alpha,k,j} \in \mathrm{D}^{n-1}(K[x])$. Let $S = \sum_{\alpha \in \Lambda} d_\alpha$. With $\mathbf{p}(x) = (p_\alpha^{(j)}(x) \mid \alpha \in \Lambda \wedge 0 \leq j \leq d_\alpha)$ this can be rewritten as a matrix-vector multiplication $M \cdot \mathbf{p}(x)^T = 0$, where M is an $S \times S$ matrix. Hence the matrix M has zero determinant. This determinant is a polynomial in f and its derivatives up to order at most $m + S$ with coefficients in $\mathrm{D}^{n-1}(K[x])$. It also has maximal degree $S \deg(p)$. This concludes the proof that f is differentially algebraic over $\mathrm{D}^{n-1}(K[x])$. $\square$

Note that Definition 27 is equivalent to the statement that the transcendence degree $\operatorname{tr.deg.}_F(F(f(x), f'(x), \ldots, f^{(n)}, \ldots))$ is finite, where F is the quotient field of R. Theorem 28 then follows from the fact that if $K_0 \subset K_1 \subset K_2$ are fields then $\operatorname{tr.deg.}_{K_0}(K_2) = \operatorname{tr.deg.}_{K_0}(K_1) + \operatorname{tr.deg.}_{K_1}(K_2)$.

The inclusion in the set of differentially algebraic functions is a simple consequence of this theorem.

Corollary 29. *Let $f(x) \in \mathrm{D}^n(K[x])$. Then f is differentially algebraic over $K(x)$.*

Proof. Let $f \in \mathrm{D}^n(K[x])$. Clearly it is differentially algebraic over $\mathrm{D}^{n-1}(K[x])$ with a linear polynomial P. Repeated application of Theorem 28 yields the result. □

The implementation of these results is described in Appendix A including more details on the computation of the following examples that conclude this section.

Example 30. Let $f(x) = \exp(\exp(x) - 1)$, which is DD-finite with annihilating operator $\partial - \exp(x)$. Using Corollary 29 we get that $f(x)$ satisfies the non-linear equation

$$f''(x)f(x) - f'(x)^2 - f'(x)f(x) = 0.$$

Example 31. Let $g(x) = e_3(x) = \exp(\int_0^x f(x)dx)$ for $f(x)$ as in the previous example. We know that $g(x)$ is D^3-finite with annihilating operator $\partial - f$. Then, repeated application of Theorem 28 shows that $g(x)$ satisfies the non-linear equation

$$\begin{aligned} g'''(x)g'(x)g(x)^2 - g''(x)^2 g(x)^2 - g''(x)g'(x)^2 g(x) \\ -g''(x)g'(x)g(x)^2 + g'(x)^4 + g'(x)^3 g(x) = 0. \end{aligned}$$

Example 32. Let the power series $f(x)$ be defined by

$$f(x) = \cos(x/2)^2 \, e^{\tan(x/2)}.$$

Although $f(x)$ seems to be D^3-finite (since $\tan(x)$ is D^2-finite), it can be easily checked that $f(x)$ is actually D^2-finite satisfying the linear differential equation

$$(\cos(x) + 1)f'(x) + (\sin(x) - 1)f(x) = 0.$$

By Corollary 29 we get that $f(x)$ satisfies the non-linear differential equation

$$\begin{aligned} f'''(x)f'(x)f(x) + f'''(x)f(x)^2 - 2f''(x)^2 f(x) + f''(x)f'(x)^2 - \\ 6f''(x)f'(x)f(x) + 3f''(x)f(x)^2 + 6f'(x)^3 - 4f'(x)^2 f(x) + \\ f'(x)f(x)^2 - f(x)^3 = 0. \end{aligned}$$

5. Conclusions

In this paper, we have added more structural information on D^n-finite functions and with these results also enlarged the existing tool-box [5]. Among the open problems on a more practical side, is the speeding-up of the implementation. Having to work with differential equations whose coefficients are defined recursively makes the computations very costly.

A non-trivial example of a family of DD-finite functions are Mathieu's functions that also have been discussed in [7]. Mathieu's equation in its standard form is given by [4, 16]

$$w'' + (a - 2q\cos(2x))w = 0, \tag{4}$$

for some parameters a and q. It is well known that the composition of $w(x)$ with $\arcsin(x)$ satisfies the D-finite differential equation

$$(1 - x^2)h''(x) - xh'(x) + (a - 2q(1 - x^2))h(x) = 0. \tag{5}$$

Since the arcsine is D-finite, we know that $h(x) = w(\arcsin(x))$ is D^3-finite and the computations in the appendix, Example 39, yield

$$\arcsin(x)'h''(x) - \arcsin(x)''h'(x) + (\arcsin(x)')^3(a - 2q\cos(2\arcsin(x)))h(x) = 0.$$

It is not obvious how to reduce this to the D-finite equation (5). It would be interesting to find a way to simplify differential equations or even to find substitutions (if existent) that allow to move to a lower level in the hierarchy.

Another open question is how to derive (or guess [8]) a D^n-finite equation for a given differentially algebraic function - even if not necessarily the optimal one. Besides these questions related to D^n-finite functions, it will also be interesting to study the analogous extension to holonomic sequences as well as the properties of coefficient sequences of D^n-finite functions.

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Appendix A. Implementation of the closure properties

Lastly, we want to address the issues related to implementing the operations described in this paper. The proofs of the closure properties all rely on the Characterization Theorem 3 which provides a bound on the order of the resulting operator. Given this bound, an ansatz for the homogeneous equation is set up. Equating the coefficients of the basis elements to zero yields a linear system. A nontrivial element of the nullspace for the system matrix can be computed. The implementation is along the lines described in [7] for the closure properties stated in Theorem 4 and we follow the same notation. For this, we first recall some basic results of differential algebra.

Definition 33. [12, 22] Let (K, ∂) be a differential field of characteristic zero and V a K-vector space. A map $\vec{\partial} : V \to V$ is a *derivation over* V *w.r.t.* ∂ if it satisfies

1. $\vec{\partial}(v + w) = \vec{\partial}(v) + \vec{\partial}(w)$ for all $v, w \in V$.
2. $\vec{\partial}(cv) = \partial(c)v + c\vec{\partial}(v)$ for all $c \in K$ and $v \in V$.

Then $(V, \vec{\partial})$ is a *differential vector space over* (K, ∂) and we denote by $\Delta_\partial(V)$ the set of all derivations over V w.r.t. ∂.

Given a vector of generators Φ, a derivation $\vec{\partial} \in \Delta_\partial(V)$ can be represented by a matrix. This matrix is uniquely defined if Φ is a basis. Since we do not need uniqueness and later it is computationally simpler to work with a vector of generators rather than having to determine a basis, we stick to this more general setting.

Definition 34. Let (K, ∂) be a differential field, $(V, \vec{\partial})$ be a differential vector space over (K, ∂), and $\Phi = (\phi_1, \ldots, \phi_n)$ be a vector of generators of V. We define a *derivation matrix of* $\vec{\partial}$ *w.r.t.* Φ as a matrix $M = (m_{ij})_{i,j=1}^n$ satisfying

$$\vec{\partial}(\phi_j) = m_{1j}\phi_1 + \cdots + m_{nj}\phi_n, \quad \text{for all } j = 1, \ldots, n.$$

In this setting, for any $v = v_1\phi_1 + \cdots + v_n\phi_n \in V$ we have

$$\begin{aligned}\vec{\partial}(v) &= \partial(v_1)\phi_1 + v_1\vec{\partial}(\phi_1) + \cdots + \partial(v_n)\phi_n + v_n\vec{\partial}(\phi_n)\\ &= \sum_{i=1}^{n}\left(\sum_{j=1}^{n} m_{ij}v_j + \partial(v_i)\right)\phi_i.\end{aligned}$$

In matrix-vector notation the coefficients in the representation of $\vec{\partial}(v) = \hat{v}_1\phi_1 + \cdots + \hat{v}_n\phi_n$ can thus be computed as

$$\begin{pmatrix}\hat{v}_1\\ \hat{v}_2\\ \vdots\\ \hat{v}_n\end{pmatrix} = M\begin{pmatrix}v_1\\ v_2\\ \vdots\\ v_n\end{pmatrix} + \begin{pmatrix}\partial(v_1)\\ \partial(v_2)\\ \vdots\\ \partial(v_n)\end{pmatrix}.$$

Summarizing, whenever there is a proof based on the Characterization Theorem 3, the algorithm follows these steps [7]:

1. Determine a finite dimensional vector space W containing the desired vector space.
2. Determine a vector of generators Φ for W and let n denote the number of generators.

3. Compute a derivation matrix C of W w.r.t. Φ.
4. Compute a vector $\mathbf{v}$ that represents the given function in W w.r.t. Φ.
5. Compute a matrix $M = (\mathbf{v}|\vec{\partial}(\mathbf{v})|\ldots|\vec{\partial}^n(\mathbf{v}))$.
6. Compute a non-trivial vector on the right nullspace of M.

The final steps 5 and 6 do not depend on the particular operation we are considering. In the following, we give the choices for the larger vector space W, the vector of generators Φ, the definition of the derivation matrix M, and the vector $\mathbf{v}$ for the implementation of the different properties discussed in Section 2.

We start by discussing an algorithm for the composition of differentially definable functions. Let $f(x) \in \mathrm{D}^n(K[x])$ of order d and $g(x) \in \mathrm{D}^m(K[x])$. Let $h(x) = f(g(x))$ and $V_h = \langle h(x), h'(x), \ldots\rangle_{F_{n+m-1}(x)}$. Then, using the results in Theorem 10, we have

1. $W_\circ(f,g) = \langle (f\circ g)(x), (f'\circ g)(x), \ldots, (f^{(d-1)}\circ g)(x)\rangle_{F_{n+m-1}(x)}$
2. $\Phi_\circ(f,g) = \big((f\circ g)(x), (f'\circ g)(x), \ldots, (f^{(d-1)}\circ g)(x)\big)$
3. A derivation matrix for $W_\circ(f,g)$ w.r.t. $\Phi_\circ(f,g)$ is given by
$$M_\circ(f,g) = g'\mathcal{C}_f(g),$$
where $\mathcal{C}_f(g)$ is the companion matrix of f, i.e., the matrix associated with a differential operator that annihilates f, with all entries composed with $g(x)$. This is the *recursive* step of the algorithm.
4. $\mathbf{v}_\circ = \mathbf{e}_{d,1} = (1,0,0,\ldots,0)$

Next we give the details for the examples 11–14 stated in Section 2.

Example 35. Let $f(x) = \exp(x)$ and $g(x) = \sin(x)$. Both are D-finite functions with annihilating operators $\partial - 1$ and ∂^2+1, respectively. Then their composition $h(x) = (f\circ g)(x) = \exp(\sin(x))$ is DD-finite. To compute the differential equation for $h(x)$ we work on the vector space $W_\circ(e^x, \sin(x)) = \langle e^{\sin(x)}\rangle_{F_1}$. We use the companion matrix of $f(x)$, compose each entry with $g(x)$ and multiply it by $g'(x)$, obtaining the derivation matrix
$$C = \big(g'(x)\big)\,.$$
Using this we compute the matrix $M = (1 \quad g'(x))$. An element of the nullspace is the vector $(-g'(x), 1)$ so an operator that annihilates $h(x)$ is
$$\partial - g'(x) = \partial - \cos(x).$$

Example 36. Let $f(x) = \log(x+1)$ and $g(x) = \exp(x) - 1$. Both are D-finite functions with annihilating operators $(x+1)\partial^2 + \partial$ and $\partial^2 - \partial$, respectively. Their composition $h(x) = (f\circ g)(x) = \log(\exp(x))$ is DD-finite.

To compute a differential equation for $h(x)$ we work on the vector space
$$W_\circ(\log(x+1), e^x - 1) = \langle \log(e^x), \frac{1}{e^x}\rangle_{F_1} = \langle x, \frac{1}{e^x}\rangle_{F_1}.$$
We use the companion matrix of $\log(x+1)$, compose each entry with $g(x)$ and then multiply everything by $g'(x)$. In this case we have
$$\mathcal{C}_{\log(x+1)} = \begin{pmatrix} 0 & 0 \\ 1 & -\frac{1}{1+x}\end{pmatrix},$$

then composing with $g(x) = \exp(x) - 1$ and multiplying by $g'(x) = \exp(x)$ we obtain

$$C = \begin{pmatrix} 0 & 0 \\ g'(x) & -1 \end{pmatrix},$$

which leads to the matrix

$$M = \begin{pmatrix} 1 & 0 & 0 \\ 0 & g'(x) & 0 \end{pmatrix},$$

which has in its nullspace the vector $(0, 0, 1)$. Hence, an annihilating operator for $h(x)$ is ∂^2. This means that $h(x)$ is a polynomial of degree at most 1, and computing the initial values from $f(x)$ and $g(x)$ we can see that $h(x) = x$ as it was expected.

Example 37. Let $f(x) = \sin(x)$ which is D-finite with annihilating operator $\partial^2 + 1$. Then $h(x) = (f \circ f)(x) = \sin(\sin(x))$ is DD-finite.

To compute the annihilating operator for $h(x)$, we need to use the companion matrix of $f(x)$, compose each entry with $f(x)$ and then multiply everything by $f'(x)$. In this case, since

$$\mathcal{C}_{\sin(x)} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix},$$

we obtain the derivation matrix

$$C = \begin{pmatrix} 0 & -f'(x) \\ f'(x) & 0 \end{pmatrix},$$

which leads to the matrix

$$M = \begin{pmatrix} 1 & 0 & -f'(x)^2 \\ 0 & f'(x) & f''(x) \end{pmatrix}.$$

We find that $(f'(x)^3, -f''(x), f'(x))$ is in the right nullspace of M, hence the annihilating operator for $h(x) = \sin(\sin(x))$ is:

$$f'(x)\partial^2 - f''(x)\partial + f'(x)^3 = \cos(x)\partial^2 + \sin(x)\partial + \cos(x)^3.$$

Example 38. Let $f(x) = \sin(x)$ and $g(x) = f(f(x)) = \sin(\sin(x))$ as in the previous example. Then $h(x) = (f \circ g)(x) = \sin(\sin(\sin(x)))$ is $\mathrm{D}^3 - finite$.

To compute the differential equation for $h(x)$ we need to use the companion matrix for $f(x)$, compose each entry with $g(x)$ and then multiply everything by $g'(x)$. These computations follow the same lines as in the previous example. Hence an annihilating operator for $h(x)$ is

$$g'(x)\partial^2 - g''(x)\partial + g'(x)^3.$$

In this example we have that $f(g(x)) = h(x) = g(f(x))$ and the algorithm could be executed either way. However, the most expensive step is the recursive call in step 3. While the composition of the companion matrix of $f(x) = \sin(x)$ with $g(x)$ is very cheap, it becomes very costly if we reverse the order. In fact for any C-finite function $f(x)$ the composition of the companion matrix is trivial no matter what the complexity of $g(x)$ is, while for $h(x) = (g \circ f)(x)$ the basic companion matrix is

$$\mathcal{C}_{\sin(\sin(x))} = \begin{pmatrix} 0 & -(\sin(x)')^2 \\ 1 & -\frac{\sin(x)''}{\sin(x)'} \end{pmatrix} = \begin{pmatrix} 0 & -\cos(x)^2 \\ 1 & \frac{\sin(x)}{\cos(x)} \end{pmatrix},$$

making the composition step very time consuming.

We conclude this set of examples with the Mathieu functions [4] that are also discussed in Section 5.

Example 39 (Mathieu). Let $f(x)$ be a Mathieu function for some parameters a, q, and $g(x) = \arcsin(x)$. These two functions are annihilated by the differential operators

$$\partial^2 + (a - 2q\cos(2x)), \quad \text{and} \quad (1-x^2)\partial^2 - x\partial,$$

respectively. From these equations we have that $f(x)$ DD-finite and that $g(x)$ is D-finite. Hence $h(x) = (f \circ g)(x)$ is D^3-finite.

To compute the differential equation for $h(x)$ we need to use the companion matrix of the Mathieu function $f(x)$ and then compose every entry of it with $g(x)$, and multiply them by $g'(x)$. This leads to the matrix

$$C = \begin{pmatrix} 0 & (2q\cos(2\arcsin(x)) - a)\arcsin(x)' \\ \arcsin(x)' & 0 \end{pmatrix}.$$

which leads to the matrix

$$M = \begin{pmatrix} 1 & 0 & (2q\cos(2\arcsin(x)) - a)\arcsin(x)'^2 \\ 0 & \arcsin(x)' & \arcsin(x)'' \end{pmatrix}.$$

Computing an element in the right nullspace gives the following annihilating operator for $h(x)$,

$$\arcsin(x)'\partial^2 - \arcsin(x)''\partial + (\arcsin(x)')^3(a - 2q\cos(2\arcsin(x))).$$

Next we discuss an algorithm for the results of Proposition 6 that any algebraic function is differentially definable. Let be $f(x)$ algebraic over the fraction field F of a differential ring R. Let $m(x,y) \in F[y]$ the minimal polynomial of $f(x)$, $p = \deg_y(m)$ and $V_f = \langle f(x), f'(x), \dots \rangle_F$. Then, following the proof of Theorem 5, we have

1. $W_a(f) = \langle 1, f(x), f(x)^2, \dots, f(x)^{p-1} \rangle_F$
2. $\Phi_a(f) = \left(1, f(x), f(x)^2, \dots, f(x)^{p-1}\right)$
3. Following the notation in Theorem 5, let $q(y) = -s(x,y)\kappa_\partial(m)(x,y)$. Then a derivation matrix $M_a(f)$ for $W_a(f)$ w.r.t. $\Phi_a(f)$ can be built with the formula
$$M_a(f) = \left(\operatorname{coeff}(jq(y)y^{j-1} \mod m(x,y), y^i)\right)_{i,j=0}^{p-1}.$$
In this matrix the ith column is the representation of $\partial(f(x)^i)$ in $\Phi_a(f)$. It can be computed using only polynomial operations.
4. $\mathbf{v}_a = \mathbf{e}_{p,1} = (1, 0, 0, \dots, 0)$

We have seen that the composition with algebraic functions behaves nicer than the composition with arbitrary D^n-finite functions. Let $F_n(x)$ be the field of fractions of D$^n(K[x])$ and $f(x) \in$ D$^n(K[x])$ be of order d and $a(x)$ algebraic over $F_m(x)$ for some $m < n$ with degree p. Let $h(x) = f(a(x))$ and $V_h = \langle h(x), h'(x), \dots \rangle_{F_{n+m-1}(x)}$. Then, using the results in Theorem 8, we have

1. $W(f,a) = W_\circ(f,a) \otimes W_a(a)$
2. $\Phi = \Phi_\circ(f,a) \otimes \Phi_a(a)$

3. A *derivation matrix* for $W(f,a)$ w.r.t. Φ is:

$$M = M_\circ(f,a) \otimes \mathcal{I}_p + \mathcal{I}_d \otimes M_a(a),$$

i.e., the Kronecker sum of the derivation matrices for composition and algebraic inclusion.

4. $\mathbf{v} = \mathbf{v}_\circ \otimes \mathbf{v}_a = \mathbf{e}_{pd,1} = (1, 0, 0, \ldots, 0)$

The main result of Section 4 is that any D^n-finite function is differentially algebraic over $K(x)$. This can be shown by the step-wise reduction given in Theorem 28. It states that any function that is differentially algebraic over $\mathrm{D}^n(K[x])$ is differentially algebraic over $\mathrm{D}^{n-1}(K[x])$. In this last part we discuss the implementation of this reduction step. The algorithm follows essentially the proof of Theorem 28 and we use the same notation. Let $f(x)$ be differentially algebraic over $\mathrm{D}^n(K[x])$, i.e., there exists a polynomial

$$P(y, \ldots, y_m) = \sum_{\alpha \in \Lambda} p_\alpha(x)(y, \ldots, y_m)^\alpha,$$

with coefficients $p_\alpha(x)$ in $\mathrm{D}^n(K[x])$ for $\alpha \in \Lambda \subset \mathbb{N}^{m+1}$, such that

$$P(f(x), f'(x), \ldots, f^{(m)}(x)) = 0.$$

The basic idea of the proof can be restated as follows: if F is a differential field and $\{v_1, \ldots, v_n\}$ generates a F-vector space and E is an extension of F, then

$$\dim_E(\langle v_1, \ldots, v_n \rangle_E) \leq \dim_F(\langle v_1, \ldots, v_n \rangle_F).$$

In the case of Theorem 28, we know that for each coefficient $p_\alpha(x)$, its corresponding vector space $V_{F_{n-1}(x)}(p_\alpha(x)) = \langle p_\alpha(x), p'_\alpha(x), \ldots \rangle_{F_{n-1}(x)}$ has finite dimension d_α. Hence, in particular, if we consider $E = F_{n-1}(x)\{y\}$ the field of fractions of the differential polynomials over $F_{n-1}(x)$, we have that for all $\alpha \in \Lambda$, the dimension generated by $p_\alpha(x)$ in E is at most d_α.

We consider the differential equation $P(y, y', \ldots, y^{(m)})$ that $f(x)$ satisfies as an element of the vector space

$$V = \bigoplus_{\alpha \in \Lambda} V_E(p_\alpha(x)),$$

where $V_E(f) = \langle f, f', f'', \ldots \rangle_E$. Then P and all its derivatives can be written in terms of a matrix-vector multiplication as in the proof of Theorem 28.

Earlier we discussed that it suffices to determine a vector of generators for the vector space V and a derivation matrix to compute all the rows of the matrix M whose determinant must vanish. The choice of these generators influences the size of the resulting differential equation quite drastically. The best option is choosing a basis $\{e_1, \ldots, e_k\}$ of V to end up with the smallest possible matrix. Building this basis would require heavy computations in $\mathrm{D}^{n-1}(K[x])$ which is not feasible. In the current implementation, we employ some cheap simplifications to reduce the number of generators. The main steps of the full algorithm are:

1. For each coefficient $p_\alpha(x)$ and for each $j = 0, \ldots, d_\alpha - 1$, compute all the derivatives $p_\alpha^{(j)}(x)$. The set of these functions are generators of V. In the worst case when $\dim_E(V) = \sum_{\alpha \in \Lambda} d_\alpha =: S$, all of them are needed.

2. Let $\Lambda_1 = \Lambda$. For some ordering on the multi-indices in Λ^2 compare pairwise for (α, β) if there is a linear relation
$$p_\alpha(x) = c_1^{(\alpha,\beta)} p_\beta^{(j)}(x) + c_2^{(\alpha,\beta)},$$
where $c_1^{(\alpha,\beta)}, c_2^{(\alpha,\beta)} \in K$ are constants. Every time a relation is found, replace $p_\alpha(x)$ by it in $P(y, ..., y_m)$ and set $\Lambda_1 = \Lambda \setminus \{\alpha\}$.
3. For all coefficients that are (detectably) in $\mathrm{D}^{n-1}(K[x])$, we introduce the additional index $(-1, 0, \dots, 0)$. If such coefficients exist, we update the support as
$$\Lambda_1 = (\Lambda_1 \setminus \{\alpha \mid p_\alpha(x) \in \mathrm{D}^{n-1}(K[x])\}) \cup \{(-1, 0, \dots, 0)\}$$
and set $d_{(-1,0,\dots,0)} = 1$. Then
$$V = \bigoplus_{\alpha \in \Lambda} V_E(p_\alpha(x)) = \bigoplus_{\alpha \in \Lambda_1} V_E(p_\alpha(x)),$$
which is a better bound for the dimension of V and thus yields a smaller dimension $S_1 = \sum_{\alpha \in \Lambda_1} d_\alpha$ for the matrix.
4. In the same way as described in [7] for the closure property addition, the derivation matrix C for V w.r.t. the small group of generators is the direct sum of the companion matrices of all the coefficients $p_\alpha(x)$ for $\alpha \in \Lambda_1$.
5. Set up a vector $\mathbf{v}$ that gives the representation of the differential polynomial P w.r.t. the vector of generators $\{p_\alpha(x) \mid \alpha \in \Lambda_1\}$. Using the derivation matrix C built in step 4, compute $S_1 - 1$ derivatives of $\mathbf{v}$.
6. Let $M = (\mathbf{v}| \dots |\vec{\partial}^{S_1 - 1}\mathbf{v})$ and note that $(p_\alpha(x))_{\alpha \in \Lambda_1} M = \mathbf{0}$. Return the determinant of M.

We close by giving more details for the computations of Examples 30–32.

Example 40. Let $f(x) = \exp(\exp(x) - 1)$, which is DD-finite with annihilating operator $\partial - \exp(x)$. To obtain the non-linear equation we have to look into the coefficients of that operator, namely $\{1, \exp(x)\}$. In this case, steps 1-3 make no simplification, so we represent the differential equation as a vector product:
$$0 = f'(x) - \exp(x) f(x) = \begin{pmatrix} f'(x) & -f(x) \end{pmatrix} \begin{pmatrix} 1 \\ \exp(x) \end{pmatrix}.$$

Computing the derivative of $(y', -y)$ w.r.t. $(1, \exp(x))$, we end up with the matrix:
$$M = \begin{pmatrix} y' & y'' \\ -y & -y' - y \end{pmatrix},$$
which leads to the determinant $(y''y - y'^2 - y'y)$, showing that $f(x)$ satisfies:
$$f''(x) f(x) - f'(x)^2 - f'(x) f(x) = 0.$$

Example 41. Let $g(x) = e_3(x) = \exp(\int_0^x f(x)dx)$ for $f(x)$ as in the previous example. We know that $g(x)$ is D^3-finite with annihilating operator $\partial - f(x)$. To obtain the non-linear equation with coefficients over $K(x)$ we need to apply twice the reduction algorithm.

In the first application, we look into the coefficients of the operator, namely $\{1, f(x)\}$. As it happened with the previous example, steps 1-3 make no simplification, so we start with the vector $\mathbf{v} = (y', -y)$ and, computing the derivative w.r.t. $(1, f(x))$, we obtain the matrix

$$M = \begin{pmatrix} y' & y'' \\ -y & -y' - e^x y \end{pmatrix},$$

which leads to the determinant $(y''y - y'^2 - e^x y'y)$, showing that $g(x)$ satisfies the equation

$$g''(x)g(x) - g'(x)^2 - e^x g'(g)g(x) = 0.$$

If we apply the same algorithm to this new equation, we have to look into the new coefficients that in this case are $(1, 1, \exp(x))$. Steps 1-3 now reduce this list of generators to $(1, \exp(x))$. We then have that $\mathbf{v} = (y''y - y'^2, -y'y)$ and, computing its derivative w.r.t. the vector of generators, we obtain the matrix

$$M = \begin{pmatrix} y''y - y'^2 & y'''y - y''y' \\ -y'y & -y''y - y'^2 - y'y \end{pmatrix},$$

which leads to the determinant $(y'''y'y^2 - y''^2y^2 - y''y'^2y - y''y'y^2 + y'^4 + y'^3y)$, showing that $g(x)$ satisfies the non-linear equation

$$\begin{aligned} g'''(x)g'(x)g(x)^2 - g''(x)^2g(x)^2 - g''(x)g'(x)^2g(x) \\ -g''(x)g'(x)g(x)^2 + g'(x)^4 + g'(x)^3g(x) = 0. \end{aligned}$$

Example 42. Let the power series $h(x)$ be defined by

$$h(x) = \cos(x/2)^2 e^{\tan(x/2)}.$$

Although $h(x)$ seems to be D^3-finite (since $\tan(x)$ is D^2-finite), it can be easily checked that $h(x)$ is D^2-finite satisfying

$$(\cos(x) + 1)h'(x) + (\sin(x) - 1)h(x) = 0.$$

To compute a non-linear differential equation we have to look into the coefficients, namely $(\cos(x) + 1, \sin(x) - 1)$. As $\partial^3 + \partial$ annihilates $(\cos(x) + 1) =: k(x)$, $\sin(x) - 1 = -k'(x) - 1$ and $k''(x) = -k(x) + 1$, steps 1-3 change the list of generators to $(1, k(x), k'(x))$. We compute then the vector $\mathbf{v} = (-y, y', -y)$ and computing its derivatives w.r.t. $(1, k(x), k'(x))$ leads to the matrix

$$M = \begin{pmatrix} -y & -y' - y & -y'' - y' \\ y' & y'' + y & y''' + y \\ -y & 0 & y'' + y \end{pmatrix},$$

which determinant is $(y'''y'y + y'''y^2 - 2y''^2y + y''y'^2 - 3y''y^2 + 2y'^2y + y'y^2 - y^3)$, showing that $h(x)$ satisfies the differential equation

$$\begin{aligned} h'''(x)h'(x)h(x) + h'''(x)h(x)^2 - 2h''(x)^2h(x) + h''(x)h'(x)^2 \\ -3h''(x)h(x)^2 + 2h'(x)^2h(x) + h'(x)h(x)^2 - h(x)^3 = 0. \end{aligned}$$