Mahler equations and rationality

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Abstract
We give another proof of a result of Adamczewski and Bell [1] concerning Mahler equations: A formal power series satisfying a \( p \)- and a \( q \)-Mahler equation over \( \mathbb{C}(x) \) with multiplicatively independent positive integers \( p \) and \( q \) is a rational function. The proof presented here is self-contained and essentially a compilation of proofs contained in a recent preprint [9] of the authors.

Keywords. Linear difference equations, consistent systems, \( q \)-difference equation, Mahler equation

We consider two Mahler operators, i.e. two endomorphisms \( \sigma_j, j = 1, 2 \), on the field \( K = \mathbb{C}[[x]][x^{-1}] \) of formal Laurent series with complex coefficients defined by \( \sigma_1(f(x)) = f(x^p) \), \( \sigma_2(f(x)) = f(x^q) \) for any \( f(x) \in K \) where \( p \) and \( q \) are positive integers. Observe that \( \sigma_1 \) and \( \sigma_2 \) commute, i.e. \( \sigma_1 \sigma_2 = \sigma_2 \sigma_1 \). We consider the field \( \mathbb{C}(x) \) of rational functions with complex coefficients as a subfield of \( K \), the inclusion given by the expansion in a Laurent series at the origin. We want to prove the following theorem

**Theorem 1.** Assume that \( p \) and \( q \) are multiplicatively independent, i.e. there are no nonzero integers \( n_j \) such that \( q^{n_2} = p^{n_1} \). Suppose that the formal series \( f(x) \in K \) satisfies a system of two Mahler equations

\[
S_j(f(x)) = \sigma_j^{m_j}(f(x)) + b_{j,m_j-1}(x)\sigma_j^{m_j-1}(f(x)) + \ldots + b_{j,0}(x)f(x) = 0, \; j = 1, 2
\]

with \( b_{j,i}(x) \in \mathbb{C}(x) \).

Then \( f(x) \) is rational.

Remark: 1. This Theorem was recently proved by Adamczewski and Bell in [1]. Their tools include a local-global principle to reduce the problem to a similar problem over finite fields, Chebotarev’s Density Theorem, Cobham’s Theorem and some asymptotics - all very different from the techniques used in the present work.

2. [1] also provides background information about Mahler equations, in particular historical, many
Now we can uniquely write

Expanding the terms in the above equation in Laurent series we obtain the equations dependent, which means that there exist $a\in\mathbb{C}(x)$ and hence

We only prove the nontrivial implication. Suppose that $w_1(x^m),...,w_\ell(x^m)$ are $\mathbb{C}(x)$-linearly dependent, which means that there exist $a_k\in\mathbb{C}(x), k=1,...,\ell,$ not all zero, such that

$$a_1(x)w_1(x^m) + ... + a_\ell(x)w_\ell(x^m) = 0.$$  

Now we can uniquely write $a_k(x) = \sum_{j=0}^{m-1} x^j c_k^j(x^m), k=1,...,\ell,$ with rational functions $c_k^j(x).$ Expanding the terms in the above equation in Laurent series we obtain the equations

$$c_1^j(x^m)w_1(x^m) + ... + c_\ell^j(x^m)w_\ell(x^m) = 0, j=0,...,m-1,$$

and hence

$$c_1^j(x)w_1(x) + ... + c_\ell^j(x)w_\ell(x) = 0, j=0,...,m-1.$$  

Since at least one of them must be nontrivial we obtain the linear dependence of the $w_j, j=1,...,\ell.$

Proof. We may assume without loss of generality that $b_{j,0}(x) \neq 0, j=1,2.$ This follows from

Lemma 2. Consider $w_1(x),...,w_\ell(x) \in K$ and a positive integer $m.$ Then these series are $\mathbb{C}(x)$-linearly dependent if and only if $w_1(x^m),...,w_\ell(x^m)$ are.

Proof. We only prove the nontrivial implication. Suppose that $w_1(x^m),...,w_\ell(x^m)$ are $\mathbb{C}(x)$-linearly dependent, which means that there exist $a_k\in\mathbb{C}(x), k=1,...,\ell),$ not all zero, such that

$$a_1(x)w_1(x^m) + ... + a_\ell(x)w_\ell(x^m) = 0.$$  

Consider now the $\mathbb{C}(x)$-subspace $W$ of $K$ generated by $\sigma_1^m\sigma_2^r(f), m=0,...,m_1-1, r=0,...,m_2-1.$ By (1), $W$ is invariant under $\sigma_1$ and $\sigma_2,$ here the fact that the $\sigma_j$ commute is used.

Let $g_1,...,g_n$ be a $\mathbb{C}(x)$-basis of $W$ with $g_1=f$ and let $g=(g_1,...,g_n)^T.$ Then we have that

$$\sigma_1(g) = A(x)g, \sigma_2(g) = B(x)g,$$  

(2)

with $A,B \in \text{gl}_n(\mathbb{C}(x)).$ By Lemma 2, we actually have $A,B \in \text{GL}_n(\mathbb{C}(x))$ because the components of $\sigma_j(g)$ form again a basis of $W.$

Additionally, the coefficient matrices of (2) satisfy a certain consistency condition. Indeed, we have

$$0 = \sigma_1(\sigma_2(g)) - \sigma_2(\sigma_1(g)) = (\sigma_1(B)A - \sigma_2(A)B)g$$  

and as the components of $g$ form a basis we obtain

$$A(x^q)B(x) = B(x^p)A(x).$$  

(3)

Our statement then follows from

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Proposition 3. Consider a system

\[ y(x^p) = A(x)y(x), \quad y(x^q) = B(x)y(x) \]

with multiplicatively independent positive integers \( p \) and \( q \) and \( A(x), B(x) \in \text{GL}_n(\mathbb{C}(x)) \) satisfying the consistency condition (3). Suppose that \( g(x) \in (\mathbb{C}[x][x^{-1}])^n \) is a formal vectorial solution. Then \( g(x) \in \mathbb{C}(x)^n \).

Observe that we must actually have \( n = 1 \) in the proof of the Theorem because the components of \( g(x) \) are \( \mathbb{C}(x) \)-linearly independent.

The proof of Proposition 3 proceeds in three steps. We first prove that \( g(x) \) converges in a neighborhood of 0. In the second step (the heart of the proof) we show that \( g(x) \) can be extended analytically to a meromorphic function on \( \mathbb{C} \) with only finitely many poles. Finally we prove that \( g(x) \) has polynomial growth as \( |x| \to \infty \) and therefore must be in \( \mathbb{C}(x)^n \). We begin with the first step.

Lemma 4. The series \( g(x) \) is convergent in a neighborhood of 0.

Proof. This is a special case of [5], Theorem 1-2, and could also be deduced from [8], section 4. For the convenience of the reader, we provide a short proof. To do that, we truncate \( g(x) \) at a sufficiently high power of \( x \) to obtain \( h(x) \in (\mathbb{C}[x][x^{-1}])^n \) and introduce \( r(x) = h(x) - A(x)^{-1}h(x^p) \) and \( \tilde{g}(x) = g(x) - h(x) \). Then we have

\[ \tilde{g}(x) = A(x)^{-1}\tilde{g}(x^p) - r(x). \]

We denote the valuation of \( A(x)^{-1} \) at the origin by \( s \in \mathbb{Z} \) and introduce \( \tilde{A}(x) = x^{-s}A(x)^{-1} \) which is holomorphic at the origin.

First choose \( M \in \mathbb{N} \) such that \( pM+s > M \) and \( h(x) \) such that \( g(x)-h(x) \) has at least valuation \( M \). Then by (5), \( r(x) \) also has at least valuation \( M \). Now consider \( R > 0 \) such that \( \tilde{A}(x) \) is holomorphic and bounded on \( D(0, R) \). Then consider for positive \( \rho < \min(R, 1) \) the vector space \( E_\rho \) of all series \( F(x) = \sum_{m=-M}^{\infty} F_m x^m \) such that \( \sum_{m=-M}^{\infty} |F_m| \rho^m \) converges and define the norm \( |F(x)|_\rho \) as this sum. Then \( E_\rho \) equipped with \( | \cdot |_\rho \) is a Banach space and the existence of a unique solution of (5) in \( E_\rho \) for sufficiently small \( \rho > 0 \) follows from the Banach fixed-point theorem using that \( |x^s F(x^p)|_\rho \leq \rho^{Mp+s-M} |F(x)|_\rho \) for \( F(x) \in E_\rho \). Since any solution \( y(x) \in x^M \mathbb{C}[[x]] \) of \( y(x) = A(x)^{-1}y(x^p) \) must be zero, we have that \( \tilde{g}(x) \) coincides with the solution in \( E_\rho \). This proves the convergence of \( \tilde{g}(x) \) and hence of \( g(x) \).

We now turn to the task of showing that \( g(x) \) can be extended to a meromorphic function on \( \mathbb{C} \). By (4), rewritten \( g(x) = A(x)^{-1}g(x^p) \), the function \( g \) can only be extended analytically to a meromorphic function on the unit disk. As we want to extend it beyond the unit disk, we use the change of variables \( x = e^t, u(t) = y(e^t) \) and obtain a system of \( q \)-difference equations

\[ u(pt) = \tilde{A}(t)u(t), \quad u(qt) = \tilde{B}(t)u(t) \]

with \( \tilde{A}(t) = A(e^t), \tilde{B}(t) = B(e^t) \). It satisfies the consistency condition

\[ \tilde{A}(qt)\tilde{B}(t) = \tilde{B}(pt)\tilde{A}(t). \]
Observe that $\bar{A}(t), \bar{B}(t)$ are not rational in $t$, but rational in $e^t$.

The heart of the proof of Proposition 3 lies in understanding the behavior of solutions of (6). We do this by first showing in Lemma 5 that there is a formal gauge transformation $u = Gv, G \in \text{GL}_n(\mathbb{C}[t][t^{-1}])$, such that $v$ satisfies a system with constant coefficients. We then show in Lemma 6 that the transformation matrix $G(t)$ and its inverse can be continued analytically to meromorphic functions on the $t$-plane. The “quotient” function $d(t) = G(t)^{-1}g(e^t)$ then satisfies a system with constant coefficients which can be solved explicitly. In this way, we show in Lemma 7 that $d(t)$ can be extended analytically to an entire function on the Riemann surface of $\log(t)$. Using these three lemmas, we show in Lemma 8 that $g(x)$ can be continued analytically to a meromorphic function on the $x$-plane.

**Lemma 5.** There exists a convergent gauge transformation $u = G(t)v, G(t) \in \text{GL}_n(\mathbb{C}[t][t^{-1}])$, such that $v$ satisfies

$$v(pt) = A_1v(t), \ v/qt) = B_1v(t)$$

where $A_1, B_1 \in \text{GL}_n(\mathbb{C})$ commute.

**Proof.** Concerning the behavior at $t = 0$, it is known that there exists a formal gauge transformation $u = Gv, G \in \text{GL}_n(\mathbb{C}[[[t^{1/s}]][[t^{-1/s}]])$, $s \in \mathbb{N}^*$, that reduces $u(pt) = A(t)u(t)$ to a system $z(pt) = t^pA_1z(t)$, where $D$ is a diagonal matrix with entries in $\frac{1}{s}\mathbb{Z}$ and $A_1 \in \text{GL}_n(\mathbb{C})$ such that any eigenvalue $\lambda$ of $A_1$ satisfies $1 \leq |\lambda| < |p|^{1/s}$, moreover $D$ and $A_1$ commute. If we write $D = \text{diag}(d_1I_1, ..., d_rI_r)$ with distinct $d_j$ and $I_j$ identity matrices of an appropriate size, then $A_1 = \text{diag}(A_1^1, ..., A_1^r)$ with diagonal blocks $A_1^j$ of corresponding size. $D$ and $A_1$ are essentially unique, i.e. except for a permutation of the diagonal blocks and passage from some $A_1^j$ to a conjugate matrix. If $D$ happens to be 0, then $s$ can be chosen to be 1 and $G$ is convergent (see [7], ch. 12, [2], [6]).

Now by the consistency condition (7), the gauge transformation $v = B(t)u$ transforms $u(pt) = \bar{A}(t)u(t)$ to $v(pt) = \bar{A}(qt)v(t)$. The gauge transformation $v = G(qt)w$ then transforms this system to $w(pt) = (qt)^DA_1w(t)$. Now $(qt)^DA_1 = t^pDq^pA_1$ and there is a diagonal matrix $F$ with entries in $\frac{1}{s}\mathbb{Z}$ commuting with $D$ and $A_1$ such that the gauge transformation $w = t^F\bar{w}$ reduces the latter system to $\bar{w}(pt) = t^p\bar{A}_1\bar{w}(t)$, where $\bar{A}_1 = p^{-F}q^pA_1$ has again eigenvalues with modulus in $[1, |p|^{1/s}]$. Now we write $\bar{A}_1 = \text{diag}(\bar{A}_1^1, ..., \bar{A}_1^r)$ and fix some $j \in \{1, ..., r\}$. If $a_1^j, ..., a_r^j$ are the eigenvalues of $A_1^j$ then $p^{-l}q^{d_l}a_l^j, \ell = 1, ..., r$, are those of $\bar{A}_1^j$. By the uniqueness of the reduced form, the mapping $t \mapsto p^{-l}q^{d_l}t$ induces a permutation of the eigenvalues of $\bar{A}_1^j$. If we apply it several times, if necessary, we obtain the existence of some $\ell \in \{1, ..., r\}$ and of some positive integer $k$ such that $p^{-k}\bar{q}^{kd_l}a_l^j = a_l^j$. Due to our condition on $p$ and $q$, this is only possible if $d_l = 0$. Thus we have proved that $D = 0$ and $t = 0$ is a so-called regular singular point of $u(pt) = \bar{A}(t)u(t)$.

We therefore obtain a matrix $A_1$ with eigenvalues $\lambda$ in the annulus $1 \leq |\lambda| < p$ and $G(t) \in \text{GL}_n(\mathbb{C}[t][t^{-1}])$ such that $u = G(t)v$ reduces the first equation of (6) to $v(pt) = A_1v(t)$. This means

$$G(pt) = \bar{A}(t)G(t)A_1t^{-1} \text{ for small } t.$$  

Applying the same gauge transformation to the second equation of (6) yields an equation $v/qt) = \tilde{B}(t)v(t)$ with some $\tilde{B}(t) \in \text{GL}_n(\mathbb{C}[t][t^{-1}])$. It satisfies the consistency condition $A_1\tilde{B}(t) = \tilde{B}(pt)A_1$.

Now we expand $\tilde{B}(t) = \sum_{m=m_0}^{\infty} C_m t^m$. The coefficients satisfy $A_1C_m = C_m(p^mA_1), m \geq m_0$. As $A_1$ and $p^mA_1$ have no common eigenvalue unless $m = 0$, we obtain that $\tilde{B}(t) =: B_1$ is constant and commutes with $A_1$. We note the second equation satisfied by $G$

$$G(t) = \tilde{B}(t)G(t)B_1^{-1} \text{ for small } t.$$  

(10)
Lemma 6. The functions $G(t)^{\pm 1}$ can be continued analytically to meromorphic functions on $\mathbb{C}$ and there exists $\delta > 0$ such that both can be continued analytically to the sectors $\{ t \in \mathbb{C}^* \mid \delta < \arg(\pm t) < 2\delta \}$.

Proof. Let $\mathcal{M}$ be the set of poles of $\tilde{A}(t)^{\pm 1}$, i.e. the set of $t$ such that $e^t$ is a pole of $\tilde{A}(x)$ or $\tilde{A}(x)^{-1}$. Note that $\mathcal{M}$ is $2\pi i$-periodic, has no finite accumulation point and is contained in some vertical strip $\{ t \in \mathbb{C} \mid -D < \Re t < D \}$. By (9), $G(t)^{\pm 1}$ can be continued analytically to $\mathbb{C}^* \setminus (\mathcal{M} \cdot p^\mathbb{N})$ and thus to meromorphic functions on $\mathbb{C}$ which we denote by the same name. By construction, $G(t)^{\pm 1}$ are also analytic in some punctured neighborhood of the origin. By the properties of $\mathcal{M}$, the infimum of the $|\Re t_1|$ on the set of all $t_1 \in \mathcal{M}$ having nonzero real part is a positive number. As $\mathcal{M}$ is contained in some vertical strip there exist sectors $\{ t \in \mathbb{C}^* \mid \delta < \arg(\pm t) < 2\delta \}$ disjoint to $\mathcal{M}$ and hence to $\mathcal{M} \cdot p^\mathbb{N}$. Therefore $G(t)^{\pm 1}$ can be analytically continued to these sectors and the lemma is proved. 

Lemma 7. The function $d(t) = G(t)^{-1}g(e^t)$ can be continued analytically to the Riemann surface of $\log(t)$.

Proof. By Lemma 6 and because $g(x)$ is holomorphic in some punctured neighborhood of $x = 0$ by Lemma 4, $d(t)$ is defined and holomorphic for some sector $S = \{ t \in \mathbb{C} \mid |t| > K, \pi + \delta < \arg t < \pi + 2\delta \}$. By (4), (9), and (10) it satisfies

$$d(pt) = A_1 d(t), \quad d(qt) = B_1 d(t) \quad \text{for } t \in S.$$  

(11)

To solve (11), consider a matrix $L_1$ commuting with $B_1$ such that $p^{L_1} = A_1$. Put $F(t) = t^{-L_1} d(t)$. Then

$$F(pt) = F(t), \quad F(qt) = \tilde{B}_1 F(t) \quad \text{for } t \in S$$

(12)

where $\tilde{B}_1 = B_1 q^{-L_1}$. Thus $H(s) = F(e^s)$ is $\log(p)$-periodic on the half-strip $B = \{ s \in \mathbb{C} \mid \Re s > \log(K), \pi + \delta < \Im s < \pi + 2\delta \}$ and can be expanded in a Fourier series. This implies that

$$F(t) = \sum_{\ell = -\infty}^{\infty} F_\ell t^{\frac{2\pi i \ell}{\log(p)}} \quad \text{for } t \in S.$$  

(13)

The second equation of (12) yields conditions on the Fourier coefficients

$$F_\ell \exp \left( 2\pi i \frac{\log(q)}{\log(p)} \ell \right) = \tilde{B}_1 F_\ell \quad \text{for } \ell \in \mathbb{Z}.$$  

Therefore $F_\ell = 0$ unless $\exp \left( 2\pi i \frac{\log(q)}{\log(p)} \ell \right)$ is an eigenvalue of $\tilde{B}_1$. Since $p$ and $q$ are multiplicatively independent, the quotient $\frac{\log(q)}{\log(p)}$ is irrational and hence $\exp \left( 2\pi i \frac{\log(q)}{\log(p)} \ell \right)$ is not a root of unity. Therefore all the numbers $\exp \left( 2\pi i \frac{\log(q)}{\log(p)} \ell \right), \ell \in \mathbb{Z}$ are different and only finitely many of them can be eigenvalues of $\tilde{B}_1$. This shows that the Fourier series (13) has finitely many terms and thus $F(t)$ can be analytically continued to the whole Riemann surface $\hat{\mathbb{C}}$ of $\log(t)$. The same holds for $d(t) = t^{L_1} F(t)$. 


Lemma 8. The function $g(x)$ can be continued analytically to a meromorphic function on $\mathbb{C}$ with finitely many poles.

Remark: According to Theorem 4.2 of [8] (see also [4]), it is sufficient to show that $g(x)$ does not have the unit circle as a natural boundary and the rationality of $g(x)$ follows. We show how it follows naturally, in our context, that $g(x)$ can be continued analytically as a meromorphic function to all of $\mathbb{C}$ and, as well, that it has only finitely many poles. The rationality of $g(x)$ then follows as in [8] and [4] from a growth estimate (Lemma 9).

Proof. The function $h(t) = g(e^t)$ is holomorphic for $t$ with large negative real part by Lemma 4 and $2\pi i$-periodic. Using Lemma 6 we conclude that $h(t) = G(t)d(t)$ can be analytically continued to a meromorphic function on $\mathbb{C}$, in particular the point $t = 2\pi i$ is at most a pole of $h$. By its periodicity, this implies that $t = 0$ also is at most a pole of $h$ and that it can be continued analytically to a meromorphic function on $\mathbb{C}$ which we denote by the same name.

Since $h(t) = g(e^t)$ for $t$ with large negative real part, $h(t)$ is $2\pi i$-periodic for those values of $t$, hence also its analytic continuation to a meromorphic function on all of $\mathbb{C}$. This periodicity allows one to define a meromorphic function $\tilde{g}(x)$ on $\mathbb{C}\setminus\{0\}$ by $\tilde{g}(e^t) = h(t)$. As $\tilde{g}(x) = g(x)$ for small $|x| \neq 0$ by the construction of $h$, we have shown that $g(x)$ can be continued analytically to a meromorphic function on $\mathbb{C}$ which will again be denoted by the same name.

The formula $h(t) = G(t)d(t)$ and Lemma 6 also imply that $h$ is analytic in the sector $\tilde{S} = \{t \in \mathbb{C}^* | \delta < \arg t < 2\delta\}$. As this sector contains some half strip $\{t \in \mathbb{C} | \Re t > L, \mu \Re t < \Im t < \mu \Re t + 3\pi\}$ for some positive $L, \mu$ which has vertical width larger than $2\pi$ and $h$ is $2\pi i$-periodic, its poles are contained in some vertical strip $\{t \in \mathbb{C} | -L < \Re t < L\}$. This implies that $g(x)$ has only a finite number of poles.

The proof of Proposition 3 is completed once we have shown

Lemma 9. The function $g(x)$ has polynomial growth as $|x| \to \infty$.

Proof. This is shown in the proof of Theorem 4.2 in [8] (see also [4]). For the convenience of the reader, we reproduce it below.

Consider $r_0 > 1$ such that $g(x)$ and $A(x)$ are holomorphic on the annulus $|x| > r_0/2$. There are positive numbers $K, M$ such that $|A(x)| \leq K|x|^M$ for $|x| \geq r_0$. Consider now the annuli

$$A_j = \{x \in \mathbb{C} | r_0^p \leq |x| < r_0^{p+1}\}, \quad j = 0, 1, \ldots$$

covering the annulus $|x| \geq r_0$. Any $x \in A_j$ can be written $x = \xi^p$ with some $\xi \in A_0$. Then we estimate using (4) and the inequality for $|A(x)|$

$$|g(x)| = |g(\xi^p)| \leq K^j \left( |\xi|^{p-1} \cdots |\xi|^{p} |\xi| \right)^M \max_{r_0 \leq |\xi| \leq r_0^p} |g(\xi)|.$$

Hence there is a positive constant $L$ such that $|g(x)| \leq L K^j |x|^M$ for $x \in A_j$. Assuming $\log(r_0) \geq 1$ without loss in generality, we find that $j \leq \log(\log(|x|))/\log(p)$ for $x \in A_j$. Hence there exists $d > 0$ such that

$$|g(x)| \leq L (\log(|x|))^d |x|^M$$

for $|x| > r_0$.

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References


